THE ANALYTICAL THEORY OF LIGHT

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THE ANALYTICAL THEORY OF LIGHT

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PREFACE.

IN the following pages an attempt is made to give an account of physical optics without having recourse to any hypothesis respecting the nature of the influence that constitutes light or the character of the medium in which it is propagated. From a few simple experimental facts it is shown that a stream of light may be represented by a periodically varying vector transverse to the direction of the beam, and on this result, with an appeal where necessary to experimental facts, the treatment of the subject is based.

An abstract wave-theory cannot of course satisfy our requirements or be regarded as the last word of science on physical optics; but as it is the touchstone on which optical theories are tried, a thorough knowledge of its teachings is essential as a preparation for penetrating below the surface of ascertained facts into the domain of hypothesis. No one optical theory can at present be said to hold the field so completely as to render a consideration of others unnecessary, and so long as that is the case, much that is of value in preparing the ground for a solution of the problem may be learned from the various attempts that have been made to apply methods of ethereal physics to the explanation of the phenomena of light. The introduction of the salient points of these endeavours would have had the effect of veiling by wealth of material the main purpose of the book.

As the object kept in view has been to give an account of the analytical development of the wave-theory that might serve as an introduction to the study of higher optics, experimental methods and results have been introduced only with a sparing hand. Ample information on the descriptive side of the subject is to be found in books readily accessible to students, and it is for those that have already made an acquaintance with physical optics that the present work is intended. A detailed knowledge of instruments and of experimental methods can only be acquired in a physical laboratory.

In order to make the book as complete as possible within the limits adopted, I have not hesitated to levy contributions from those that have previously written on the subject. This is evidenced by the references given in the course of the work, but of published treatises I must in special mention the aid that I have received from Voigt's *Kompendium der theoretischen Physik*, the influence of which may be traced in all parts of the subject, from Liebisch's *Physikalische Krystallographie* and from Winkelmann's *Handbuch der Physik.*

I have to offer my best thanks to Mr I. O. Griffiths, B.A., Fellow of St John's College, Oxford, for his care and diligence in reading the proofs, and to my son for his help in the preparation of the diagrams. But above all I must express my indebtedness to Prof. Clifton, F.R.S., for the sympathetic kindness with which he has at all times been ready to render me assistance and advice.

Acknowledgement is also due to the Syndics of the Cambridge University Press for their kindness in undertaking to publish the book, and to the Officials of the Press for the accuracy and care with which they have executed the printing and corrections.

J W

OXFORD,
September, 1904.

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CHAPTER I.

GEOMETRICAL PROPOSITIONS OF THE WAVE THEORY.

1. THE science of Physical Optics may be regarded as comprising two fields of enquiry; the one includes the study of the physical properties of a stream of light, the other comprehends the investigation of the mechanism by means of which the stream is propagated. These two divisions may be termed respectively the kinematics and the dynamics of the subject.

The fundamental property of light, that forms the basis of physical optics, is its progressive movement. The fact that light travels through space with a finite, though very great velocity, was first deduced by Römer in 1676 from observations of the eclipses of Jupiter's satellites; it afforded Bradley in 1728 the explanation of astronomical aberration; and it was directly demonstrated by the experiments of Fizeau in 1849 and of Foucault in 1850. That it is energy that is radiated from a luminous body and is perceived by the eye as light is shown by the phenomena associated with a stream of light and by a consideration of the nature of the sources from which it is emitted. Now energy can be transmitted through space in either of two modes—by the transport of matter connected with the energy or by means of waves. Each of these methods of the transference of energy has in turn been applied to the explanation of the propagation of light.

The emission or corpuscular theory, adopted and expounded by Newton, attributes the sensation of light to the impact on the retina of particles ejected from a luminous body by the vibratory motions of its parts. The particles, according to Newton, must be assumed to be capable of exciting vibrations in an "ætherial medium" and it is to the waves thus set up that he ascribes the mutual dependence of reflection and refraction: he further suggests that the "bignesses" of the vibrations started by the corpuscles depend upon the colour or refrangibility of the light. Thus Newton to a certain extent adopted some of the features of the wave-theory, but it is to be noted that according to him the waves are the effect and not the cause of light. That these waves are not an essential adjunct of the emission theory has been shown by Boscovich and also by Biot, to whom several brilliant extensions of the theory are due. As thus developed, however, the emission theory is lacking in simplicity, and overcrowded with hypotheses; moreover

it contradicts the facts in an important particular, for it leads to the result that the propagational speed of light is greater in a dense medium, such as water, than it is in air, whereas direct experiments show that the reverse is the case.

The wave-theory, based on the second mode of the transport of energy, was first presented in an intelligible form by Huygens, but owes its recognition to the work of Fresnel. This theory regards light as consisting in vibrations of or in a medium, that is supposed to fill interstellar space and to pervade all ponderable media. When however we enquire into the character of the vibrations and the properties of the medium, we find that the wave-theory has assumed different forms: according to the dynamical theories the vibrations are assimilated to those of a medium, that has either intrinsic rigidity, or a quasi-rigidity imparted to it gyrostatically; while the electromagnetic theory applies to the problem the equations of an electromagnetic field and regards the ether as a dielectric medium subject to a rapidly periodic electric displacement. These two forms of the wave-theory must be regarded as distinct, until it is possible to form a conception of an ether that is competent to coordinate optical and electrical phenomena: on the other hand the explanation of the physical properties of a stream of light is independent of the particular idea that we may formulate respecting the nature of the vibrations in a train of luminous waves.

2. We owe to Huygens a very important principle, according to which the direction of propagation of a luminous disturbance within or at the confines of homogeneous media is made to depend upon the form and the orientation of a certain surface characteristic of each medium. This surface, which is called "the Wave-surface," is the locus of the points to which a disturbance emanating from a luminous point travels in unit time: in ordinary isotropic media it is a sphere; it is a double surface or a surface of two sheets in media such that in general two disturbances can be propagated in any direction with different speeds; and so on.

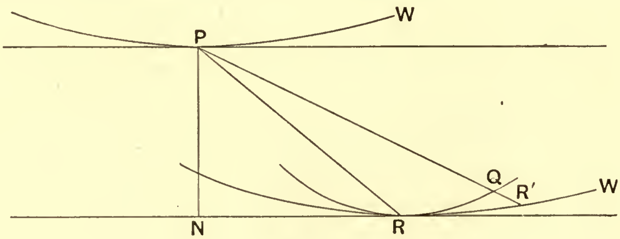


Fig. 1.

Huygens' principle, which follows at once from that of the superposition of small disturbances, lies at the very basis of the wave-theory and consists in this:—that each of the points of a wave W may be regarded as a centre of

disturbance, the wave at any subsequent time t being the envelope of secondary waves of form characteristic of the medium and proper to the wave W, described round the points of W as centres with dimensions corresponding to the time t.

It follows from this that the surfaces W and W' are so related that to a point P on the former there corresponds a point R on the latter, at which it is touched by the secondary wave that emanates from P, and it is clear that the time required for the disturbance to traverse all lines joining corresponding points is the same, being that in which the wave travels from W to W', and that this time is less than that in which it would traverse any other line connecting the surfaces; for any other line such as PR' cuts the secondary wave described round P in some point Q and the time along PQ is equal to that along PR.

Hence defining a ray as a line joining corresponding points on a wave in its successive positions, we arrive at Fermat's law that the time in which a disturbance is propagated along the rays from a wave-surface to its position at any subsequent time is the same and less than for any other path. This is expressed by saying that a ray is the course of earliest arrival*.

Fresnel introduced an important simplification into the study of the propagation of waves by recognising that, since a surface may be regarded as the envelope of its tangent planes, we may substitute for a wave of any form a system of plane waves coincident with the tangent planes of the wave-front at the given time. If now we consider a plane wave that touches the wave-front W at the point P, it follows from Huygens' principle that after a lapse of time t this wave will coincide with the tangent plane to the new position of the wave W' at the point R, and it becomes necessary to distinguish between the *ray-velocity* σ with which the disturbance traverses the ray and the *wave-velocity* ω with which the corresponding plane wave advances in the direction of its normal. These velocities are connected by the relation

$$\omega = \sigma \cos (NR)$$

where (NR) denotes the angle between the normal and the ray.

Another surface that is of fundamental importance in the study of waves of light is the pedal of the wave-surface. The physical significance of this surface arises from its being formed by the assemblage of points, that are obtained by taking on every radius-vector through some point a distance equal to the velocity of the plane wave that has its normal in this direction. It may therefore be called "the surface of wave-quickness."

* The ray is only a course of *earliest* arrival "for paths from P up to all points P' such that the successive wave-fronts between P and P' belonging to a radiant disturbance maintained at P do not develope any singularity along the course of the ray." Larmor, *Æther and Matter*, pp. 32 and 276.

3. Huygens' principle is of itself insufficient for the explanation of all the questions that arise in connection with the propagation of light, and the determination of its analytical expression, as well as the justification for its employment, must be reserved for future consideration. In the mean time however it will be convenient to consider it in its geometrical aspect for the purpose of obtaining some results that will be of service to us in the sequel.

When a wave meets the interface of two homogeneous media that have different optical properties, the waves in each medium at any subsequent time are by Huygens' principle the envelopes of the secondary waves characteristic of that medium, described in it round the points of the interface with dimensions corresponding to the time that elapses between the passage of the incident wave through these points and the instant under consideration.

In accordance with the simplification introduced by Fresnel, these reflected and refracted waves may be regarded as the envelopes of those that result from a system of plane waves, coincident with the tangent planes of the incident wave, and each reflected or refracted at a plane surface, separating two media identical with the given media and coincident with the tangent plane to the actual interface at the point, in which it is met by the corresponding ray of the incident wave. The problem is thus reduced to the consideration of the reflection and refraction of a plane wave at a plane surface, and in this case it is readily seen that the reflected and refracted waves are themselves also plane.

Now if the incident wave cut the interface of the media at times T and $T+t$ in the lines I and I', the reflected and refracted waves at time $T+t$ must by Huygens' principle pass through I' and also touch the wave-surfaces

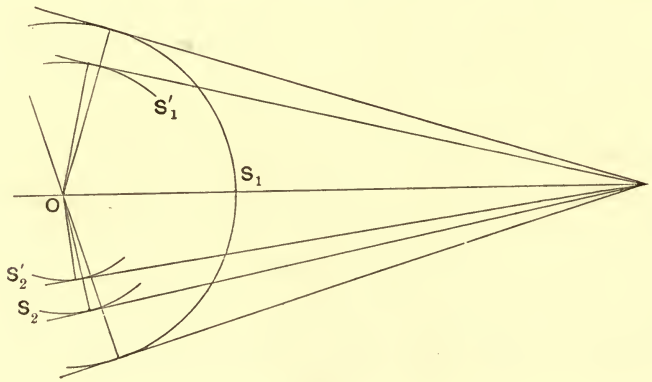


Fig. 2.

S_1, S_2 of the two media, described round any point of I with dimensions corresponding to the time t, and the position of the line I' is determined by the fact that if the second medium be identical with the first, the wave at

time $T+t$ would coincide with the tangent plane through I' to that sheet of S_1 which corresponds to the incident wave.

Hence we have the following construction:—round any point O of the line in which the incident wave cuts the interface at time T, describe the wave-surfaces S_1, S_2 characteristic of the media with dimensions corresponding to unit time: draw in the second medium a plane parallel to the incident wave to touch the corresponding sheet of S_1, and through the line in which this plane cuts the interface draw in the first medium tangent planes to the sheets of S_1 and in the second medium tangent planes to the sheets of S_2. These tangent planes represent respectively the reflected and the refracted waves: the vectors from O to the points of contact of the tangent planes to the wave-surfaces give the reflected and the refracted rays and the corresponding ray-velocities: the perpendiculars from O on the tangent planes give the wave-velocities.

It thus follows that the normals to the incident, the reflected and the refracted waves at any point of the interface separating two media lie in a plane perpendicular to this surface, and since the waves at any time intersect the interface in the same straight line, the sine of the angle between either wave and the surface bears to the corresponding wave-velocity a ratio that is the same for each of the waves.

4. For this construction we may substitute another, that will be found more convenient in theory and practice, though it is without the same physical significance. This is due to Sir William Hamilton* and is obtained from Huygens' construction by reciprocating with respect to a sphere of unit radius concentric with the wave-surfaces.

The polar reciprocal of any surface being the inverse of its first pedal, it follows that the surface required for the new construction is the inverse of the surface of wave-quickness: that is, the radius from the centre represents the wave-slowness, or the reciprocal of the propagational speed of a plane wave with its normal in that direction. On this account the surface is termed the surface of wave-slowness: obviously in an ordinary isotropic medium it is a sphere; and a double surface or a surface of two sheets in a doubly-refracting medium, having always a centre O round which it is symmetrical.

As an example of Hamilton's construction let us consider the case of the passage of light through a parallel plate of a doubly refracting crystal embedded in an ordinary isotropic substance, wherein the constant wave-velocity is Ω; supposing first of all that the crystal is more strongly refracting than the surrounding medium, so that Ω is greater than the wave-velocities within the crystal.

* *Trans. R. Irish Acad.* XVII. 141—144 (1833). Cf. also MacCullagh, *Collected Works*, p. 34.

Now in Huygens' construction the incident, reflected and refracted waves at any time intersect the surface separating the media in the same straight line: hence the corresponding points on the surfaces of wave-slowness of the media lie on a line perpendicular to the interface, and the reflected and the refracted wave-normals are determined by the following construction.

Round a point O of the line in which the incident wave cuts the interface at time t, describe the surfaces of wave-slowness of the media with dimensions corresponding to unit time: these will be a sphere for the outer medium, the radius of which represents the reciprocal of Ω and a double surface for the crystal entirely surrounding the sphere. Through the point E in which the incident wave-normal IO, produced into the plate, meets the sphere, draw EA perpendicular to the interface and produce it both ways to meet the sphere again in R and the surface of wave-slowness of the crystal in the points W_1, W_2 within the plate and the points W_1', W_2' without it.

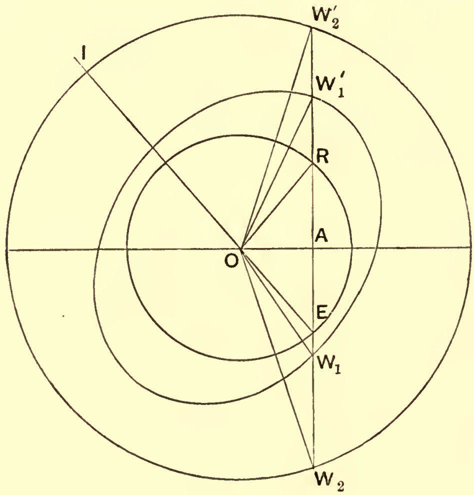


Fig. 3.

Then it is clear that OR is the normal of the wave given by ordinary reflection at the first surface of the plate: OW_1, OW_2 give the normals of the refracted waves and the slowness of these waves; and since the surface of wave-slowness is the polar reciprocal of the wave-surface, the perpendiculars from O on the tangent planes to the surface at W_1 and W_2 give the corresponding rays and the slowness of these rays. Each of the refracted waves on arrival at the second surface of the plate gives an emergent wave with its normal in the direction OE and two reflected waves with their normals parallel to OW_1' and OW_2' respectively, while each of these reflected waves is again divided at the first surface into an emergent wave and two reflected waves with their normals parallel to OR, OW_1, OW_2 respectively; and so on. Thus the normal to every wave within the plate is parallel to one of the

four lines OW_1, OW_2, OW_1', OW_2' and the corresponding wave-velocities are given by the reciprocals of the lengths of these lines. The directions of the rays are determined by the perpendiculars from O on the tangent planes to the surface of wave-slowness at the points W_1, W_2, W_1', W_2' and the ray-velocities are given by the reciprocals of the lengths of these perpendiculars.

In order to determine the directions of the waves within the plate analytically, let us assume that the first surface of the plate is the plane of xy and the plane of incidence that of xz, the positive quadrant of xz being that which contains the directions of propagation of the refracted waves. Referred to these axes, let the equation of the surface of wave-slowness be

$$F(x, y, z) = 0 \qquad (1).$$

If i be the angle of incidence, r the angle between the positive direction of the axis of z and the normal to any one of the waves within the plate, the coordinates of a point on the surface of wave-slowness are

$$x = \sin i/\Omega, \quad y = 0, \quad z = \sin i/(\Omega \tan r) \qquad (2).$$

Substituting these values in (1), we shall obtain an equation of the form

$$a_0 \tan^4 r + 4a_1 \tan^3 r + 6a_2 \tan^2 r + 4a_3 \tan r + a_4 = 0 \qquad (3),$$

in which the coefficients are functions of $\sin^2 i$ (since the surface is symmetrical with respect to its centre) and of the quantities that define the interface of the media and the plane of incidence in terms of axes fixed in the plate and dependent upon its structure.

The roots of (3) give the tangents of the four angles that the normals to the waves within the plate make with the positive direction of the axis of z—the normal to the first surface of the plate drawn inwards*.

5. In the case just considered, in which the surface of wave-slowness of the first medium lies entirely within that of the second, the line EA meets each sheet of the latter surface in two points, that lie one on each side of the interface, whatever the angle of incidence may be. If however the plate be less refracting than the outer medium and consequently the surface of wave-slowness for that medium lie without that of the plate, different cases occur as the angle of incidence increases.

When this angle is small, everything is as in the former case, but as the incidence increases, an angle is attained for which the line EA touches the inner sheet of the surface of wave-slowness of the plate, and for angles of incidence greater than this there is only one refracted wave†. This then is the critical angle of total reflection for the quicker wave and the corresponding ray is clearly in the plane of incidence and in the surface of the plate.

* Liebisch, *N. Jahrb. für Min.* (1885) II. 181. *Phys. Kryst.* p. 290.

† Anomalies however occur in the immediate neighbourhood of a singular point of the surface, see §156.

The same thing occurs with the outer sheet of the surface of wave-slowness.

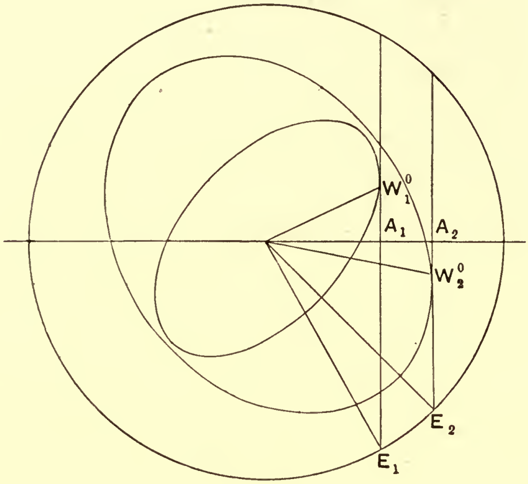


Fig. 4.

If then a tangent cylinder be drawn to the surface of wave-slowness of the second medium with its generating lines perpendicular to the interface of the media this cylinder will cut the surface of wave-slowness of the first medium in a curve, that will be the director curve of a cone having its vertex at the centre of the surfaces, the generating lines of which are the normals to the waves that are at the limit of total reflection at the interface of the media. Since the equation (3) has equal roots when the angle of incidence is the critical angle, the equation to the cone, obtained by equating the discriminant of (3) to zero, is

$$(a_0a_4 - 4a_1a_3 + 3a_2^2)^3 - 27\,(a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3)^2 = 0 \ldots (4),$$

in which $\sin i$ and the quantities defining the plane of incidence are the variables*.

6. It follows from the above investigations that we can find the directions of the reflected and refracted waves, due to the incidence of a plane wave on the interface of two homogeneous media, when the surfaces of wave-slowness, or which comes to the same thing the surfaces of wave-quickness, of the media are known. One of the chief methods of experimentally determining the form of these surfaces for any substance is by measures made with prisms, and we will now consider the case of a doubly refracting prism placed in a less refracting isotropic medium, in which the constant propagational speed of light is Ω.

With any point O of the edge of the prism as centre, describe a sphere

* Liebisch, *loc. cit.*

with radius $1/\Omega$, and the surface of wave-slowness of the prism: produce the incident wave-normal to meet the sphere in N and through N draw a line perpendicular to the face of entry of the prism OA, meeting the surface of wave-slowness in the points M and P on the same side of OA as the point

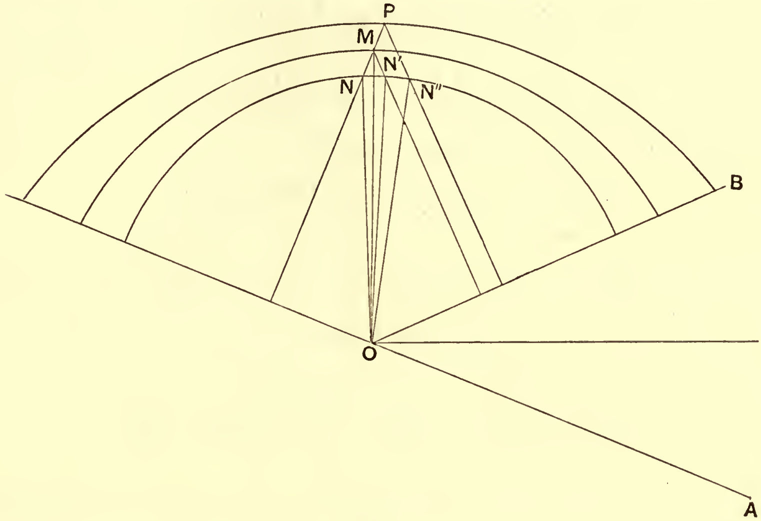


Fig. 5.

N: finally through M and P draw perpendiculars to the face of emergence OB meeting the sphere in the points N' and N'' respectively on the same side of OB as the points M and P. Then OM and OP are the refracted, ON' and ON'' the emergent wave-normals.

We will first suppose that the incident wave is parallel to the edge of the prism. Let

A be the angle of the prism, D the deviation of either of the waves,

i, i' the angles of incidence and emergence,

r, r' the angles that the refracted wave makes with the faces of the prism,

ψ the angle that the refracted wave-normal OM makes with the plane bisecting its angle.

Then if the figure represent the normal section of the prism, and $1/\omega$ be the length of OM or the slowness in that direction, we obtain at once

$$\sin i = (\Omega/\omega) \sin r \quad\text{.........(5)},$$

$$\sin i' = (\Omega/\omega) \sin r' \quad\text{.........(6)},$$

$$r + r' = A \quad\text{.........(7)},$$

$$i + i' = A + D \quad\text{.........(8)},$$

and
$$\psi + A/2 = \pi/2 + r, \quad \psi - A/2 = \pi/2 - r'$$

or
$$2\psi = \pi + r - r' \quad\text{.........(9)}.$$

From these equations we may eliminate r and r', angles that we are unable to measure, and one of the angles D, i or i' and thus obtain ψ and ω in terms of A and two other measurable angles.

First we have $$\sin i \pm \sin i' = (\Omega/\omega)(\sin r \pm \sin r')$$

or

$$\left.\begin{aligned} \sin\frac{i+i'}{2}\cos\frac{i-i'}{2} &= \frac{\Omega}{\omega}\sin\frac{r+r'}{2}\cos\frac{r-r'}{2} \\ \cos\frac{i+i'}{2}\sin\frac{i-i'}{2} &= \frac{\Omega}{\omega}\cos\frac{r+r'}{2}\sin\frac{r-r'}{2} \end{aligned}\right\} \text{............(10)},$$

and eliminating Ω/ω between these equations

$$\cot\frac{r-r'}{2}\tan\frac{r+r'}{2} = \cot\frac{i-i'}{2}\tan\frac{i+i'}{2},$$

whence $$\tan\psi = -\cot\frac{A}{2}\cot\left(i - \frac{A+D}{2}\right)\tan\frac{A+D}{2} \text{ ....(11)}.$$

Again eliminating $(i-i')/2$ between the equations (10), we find

$$\frac{\omega^2}{\Omega^2} = \frac{\cos^2\dfrac{r+r'}{2}}{\cos^2\dfrac{i+i'}{2}}\sin^2\frac{r-r'}{2} + \frac{\sin^2\dfrac{r+r'}{2}}{\sin^2\dfrac{i+i'}{2}}\cos^2\frac{r-r'}{2}$$

$$= C^{-2}\cos^2\psi + S^{-2}\sin^2\psi \text{ ....(12)},$$

where $$C = \frac{\cos\dfrac{A+D}{2}}{\cos\dfrac{A}{2}}, \quad S = \frac{\sin\dfrac{A+D}{2}}{\sin\dfrac{A}{2}} \text{ ....(13)}.$$

Thus by means of equations (11) and (12), it is possible to determine from measured quantities the points of the section of the surface of wave-quickness made by the normal section of the prism*.

It follows from (12) that the trace of the refracted wave on the normal section of the prism touches the ellipse

$$C^2x^2 + S^2y^2 = \Omega^2,$$

where the axes of x and y are taken along the internal and external bisectors of the angle between the lines OA and OB. With a given prism, the form of the ellipse depends upon the single parameter D and thus changes for each angle of incidence: but writing the equation in the form

$$\left(x^2/\cos^2\frac{A}{2} - \Omega^2\right) + \left(y^2/\sin^2\frac{A}{2} - \Omega^2\right)\tan^2\frac{A+D}{2} = 0,$$

we see that it is satisfied identically by $x = \pm\Omega\cos(A/2)$, $y = \pm\Omega\sin(A/2)$, and thus all the ellipses pass through the points in which the circle

$$x^2 + y^2 = \Omega^2$$

intersects the lines OA, OB.

* Stokes, *B. A. Report* (1862), p. 272.

In the case of minimum deviation $dD=0$, and the ellipse is unchanged by an infinitesimal variation in the angle of incidence. Moreover in this case the ellipse touches what may be called, for shortness, the line of the wave, that is, the line in which the normal section of the prism cuts a tangent cylinder to the wave-surface with its generating lines parallel to the edge of the prism. For in the case of a wave that undergoes minimum deviation, its intersection with a consecutive wave passes through a point on the ellipse, and the intersection of these two consecutive tangent planes to the wave-surface also passes through a point on the line of the wave: hence since the trace of the wave touches both the ellipse and the line of the wave, these two curves are tangents to one another and to the trace of the refracted wave. Thus in the case of minimum deviation, the ellipse (C^{-1}, S^{-1}) is a tangent to the refracted wave at the same point as the line of the wave, so that it has all the properties of this line and defines the wave-velocity and the projection of the refracted ray on the normal section of the prism. Hence the consideration of a complex line of the wave is replaced by that of an ellipse symmetrically placed with respect to the faces of the prism, with axes that are simple functions of the angle of the prism and of the deviation*.

7. Turning now to the refraction of plane waves that are not parallel to the edge of the prism, it is clear in the first place that the incident and emergent waves are inclined at the same angle χ to the edge. This follows at once from the construction given above, for the points N, N', N'' lie in a plane perpendicular to the edge, and therefore since ON, ON', ON'' are all equal, these lines make equal angles with the edge.

Secondly the law of sines applies to the traces of the waves on the normal section of the prism, provided that we take for the refractive index, not the true value $n=\Omega/\omega$, but the value m defined by

$$m=\sqrt{n^2+(n^2-1)\tan^2\chi}.$$

For if OM make an angle χ' with the normal section, we have, since M and N are at the same distance from it,

$$OM\sin\chi'=ON\sin\chi, \quad \therefore\ \sin\chi=n\sin\chi' \quad\ldots\ldots\ldots\ldots(14).$$

Also if M and N be projected on the normal section in the points μ and ν,

$$m=\frac{O\mu}{O\nu}=\frac{OM\cos\chi'}{ON\cos\chi}=n\,\frac{\cos\chi'}{\cos\chi}=\sqrt{n^2+(n^2-1)\tan^2\chi}\quad\ldots\ldots(15).$$

Let AOB (fig. 6) be the normal section of the prism: describe a sphere round O as centre and through O draw lines parallel respectively to the normals to the faces of the prism and to the incident, refracted and emergent waves to meet the sphere in the points N_1, N_2, S_1, S, S_2.

Let the internal and external bisectors of the angle AOB and the edge

* Cornu, *Ann. de l'école norm.* (2) III. 1 (1874).

of the prism cut the sphere in the points ξ, η, ζ and draw the great circles ζS_1, ζS, ζS_2 to cut the great circle $\xi\eta$ in the points σ_1, σ, σ_2.

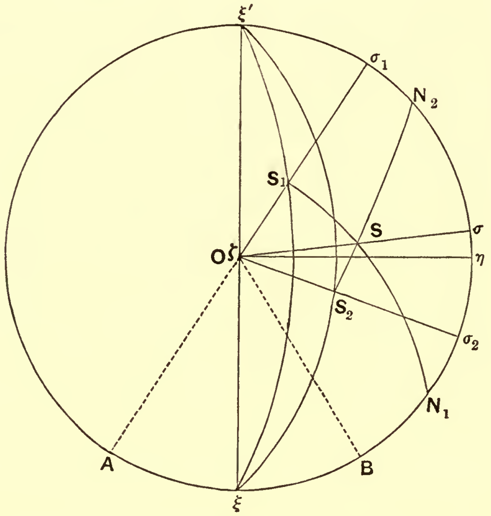


Fig. 6.

Then if D be the deviation S_1S_2 and D_0 the deviation $\sigma_1\sigma_2$ of the projections of the wave-normals on the normal section of the prism, we have from the triangle ζS_1S_2, in which $\zeta S_1 = \zeta S_2 = \pi/2 - \chi$ and the angle $S_1\zeta S_2$ is D_0,

$$\cos D = \sin^2\chi + \cos^2\chi \cos D_0$$

or

$$\sin\frac{D}{2} = \cos\chi \sin\frac{D_0}{2} \quad \text{.............................(16)},$$

so that the minimum value of D corresponds to the minimum value of D_0.

Let $\quad S_1N_1 = i, \quad S_2N_2 = i', \quad SN_1 = r, \quad SN_2 = r',$

$$S_1\xi = \lambda, \quad S_2\xi = \lambda', \quad S_1\xi\eta = \theta, \quad S_2\xi\eta = \theta',$$

then D depends upon the lateral deviation $\Delta\theta$ and the longitudinal deviation $\Delta\lambda$; but

$$\sin\chi = \sin\lambda \sin\theta = \sin\lambda' \sin\theta',$$

hence if the lateral deviation be zero, or $\theta = \theta'$, we have

$$\sin\lambda = \sin\lambda' \quad \text{or} \quad \lambda' = 180° - \lambda;$$

that is the arc S_1S_2 is bisected by the great circle $\eta\zeta$ and by symmetry the point S is on this great circle.

Let $\quad N_1\sigma_1 = i_0, \quad N_2\sigma_2 = i_0', \quad N_1\sigma = r_0, \quad N_2\sigma = r_0', \quad \xi\sigma = \psi,$

then

$$\left.\begin{array}{ll} \cos i = \cos i_0 \cos\chi, & \cos i' = \cos i_0' \cos\chi \\ \cos r = \cos r_0 \cos\chi', & \cos r' = \cos r_0' \cos\chi' \end{array}\right\} \quad \text{............(17)},$$

and since the projections of the wave-normals on the normal section of the prism obey the sine law with refractive index m, we have from (11) and (12)

$$\tan\psi = -\cot\frac{A}{2}\cot\left(i_0 - \frac{A+D_0}{2}\right)\tan\frac{A+D_0}{2} \quad\text{..........(18),}$$

and $$C_0^{-2}\cos^2\psi + S_0^{-2}\sin^2\psi = m^{-2} = \frac{\omega^2}{\Omega^2}\frac{\cos^2\chi}{\cos^2\chi'} = \frac{\tan^2\chi'}{\tan^2\chi} \quad\text{.........(19).}$$

It follows then that if we measure A, D, i, χ we can determine D_0, i_0, ψ, χ', ω in succession from (16), (17), (18), (19), (14) and hence find the direction and the propagational speed of the wave within the prism.

8. Let us take the normal section of the prism as the plane of xy, the internal and external bisectors of the angle between the traces of the faces on this plane being the axes of x and y respectively, and the positive quadrant containing the trace of the face of emergence. Then if l, m, n be the direction-cosines of the wave-normal, the equation of the refracted wave at unit time after passing through the origin is

$$lx + my + nz = \omega$$

or $$\xi x + \eta y + \zeta z = 1,$$

where ξ, η, ζ are the coordinates of the point M (fig. 5).

Now $\zeta = \sin\chi/\Omega$ and since the lines MN and MN' are perpendicular respectively to the faces of entry and emergence,

$$-\xi\cos\frac{A}{2} + \eta\sin\frac{A}{2} = ON\cos\chi\sin i_0 = \cos\chi\sin i_0/\Omega,$$

$$\xi\cos\frac{A}{2} + \eta\sin\frac{A}{2} = ON\cos\chi\sin i_0' = \cos\chi\sin i_0'/\Omega,$$

whence

$$\xi = \frac{\cos\chi(\sin i_0' - \sin i_0)\sin\frac{A}{2}}{\Omega\sin A}, \quad \eta = \frac{\cos\chi(\sin i_0' + \sin i_0)\cos\frac{A}{2}}{\Omega\sin A},$$

and the equation of the refracted wave becomes

$$\sin\frac{A}{2}(\sin i_0' - \sin i_0)x + \cos\frac{A}{2}(\sin i_0' + \sin i_0)y + \tan\chi\sin A z$$
$$- \Omega\sec\chi\sin A = 0 \quad\text{....................(20).}$$

If now x, y, z be the coordinates of the point in which this wave touches the wave-surface, they must satisfy equation (20) and that derived from it by giving infinitesimal variations to the angles i_0, i_0' and χ. Thus in the special case of a wave parallel to the edge of the prism we have, dropping the suffixes as no longer needed,

$$\sin\frac{A}{2}(\sin i' - \sin i)x + \cos\frac{A}{2}(\sin i' + \sin i)y - \Omega\sin A = 0 \quad\text{...(21)}$$

and

$$\left(x \sin \frac{A}{2} + y \cos \frac{A}{2}\right) \cos i' di' - \left(x \sin \frac{A}{2} - y \cos \frac{A}{2}\right) \cos i di + z \sin A\, d\chi = 0 \quad \text{.........(22).}$$

Let us suppose that we are looking in the plane of the normal section of the prism so that the eye receives the light incident upon the prism, and let OI be the normal to an incident wave parallel to the edge and inclined at an angle i to the face of entry, OI' the normal to a wave inclined at an angle $d\chi$ to the edge. Then if $i + di$ be the projection of the angle of incidence of this wave on the normal section, and if ϕ be the angle between the planes IOI' and IOZ, reckoned positive to the right of IOZ,

$$\tan \phi = -di/d\chi.$$

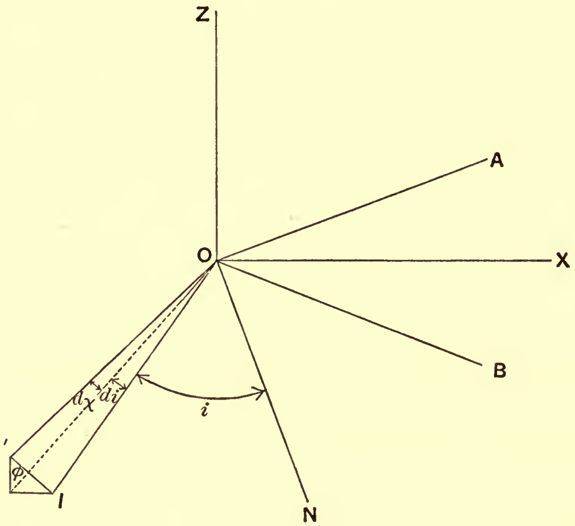


Fig. 7.

Similarly in the case of the emergent waves, the corresponding angle ϕ' is given by

$$\tan \phi' = di'/d\chi,$$

the change of sign arising from the fact that the increase in the angle i' is in the opposite direction to that of i.

Substituting these values, (22) becomes

$$\left(x \sin \frac{A}{2} + y \cos \frac{A}{2}\right) \cos i' \tan \phi'$$
$$+ \left(x \sin \frac{A}{2} - y \cos \frac{A}{2}\right) \cos i \tan \phi + z \sin A = 0 \quad \text{.........(23).}$$

It follows then that the image, seen through the prism, of the slit of a collimator placed parallel to the edge, will be turned through an angle ϕ_0' given by

$$\left(x \sin \frac{A}{2} + y \cos \frac{A}{2}\right) \cos i' \tan \phi_0' + z \sin A = 0 \quad \text{.........(24)},$$

and in order that the image may be parallel to the edge, the slit must be turned through an angle ϕ_0, where

$$\left(x \sin \frac{A}{2} - y \cos \frac{A}{2}\right) \cos i \tan \phi_0 + z \sin A = 0 \quad \text{.........(25)};$$

and if these angles be measured, the coordinates x, y, z may be determined from (24) and (25) together with the equation of the wave, which may be written

$$C \sin \frac{i-i'}{2}\, x - S \cos \frac{i-i'}{2}\, y + \Omega = 0 \quad \text{.............(26)},$$

where C, S have the meanings given above.

In the case of minimum deviation, $dD = di + di' = 0$, and we obtain by differentiating (26)

$$C \cos \frac{i-i'}{2}\, x + S \sin \frac{i-i'}{2}\, y = 0,$$

and from this equation and (26) we find

$$x = -C^{-1} \sin \frac{i-i'}{2}, \quad y = S^{-1} \cos \frac{i-i'}{2} \quad \text{.............(27)},$$

and then equations (24) and (25) give

$$\begin{aligned} z &= \operatorname{cosec}(i+i') \cos i \cos i' \tan \phi_0 \\ &= -\operatorname{cosec}(i+i') \cos i \cos i' \tan \phi_0' \quad \text{.............(28)}. \end{aligned}$$

Thus from equations (24), (25), (26) or in the simpler case of minimum deviation from (27) and (28) we can determine the ray corresponding to a wave within the prism that is parallel to its edge*.

* Cornu, *loc. cit.*

CHAPTER II.

ANALYTICAL EXPRESSION FOR A TRAIN OF PLANE WAVES.

9. THE analytical expression for a train of plane waves in an homogeneous transparent and isotropic medium is obtained by stating that the disturbance at a distance r from a fixed plane parallel to the wave-fronts at a time t is the same as that at a distance $r + \omega t'$ at the time $t + t'$, where ω is the propagational speed of the waves. It is hence given by one or more functions of the argument $\omega t - r$, since such functions alone have the special property of remaining unchanged in value when $t + t'$ is written for t and $r + \omega t'$ for r.

In the case however of an infinite train of plane waves of monochromatic light, it is possible to assign to these functions a more precise form, which may be deduced from the experimental fact that the state of things occurring at any instant in a given plane parallel to the wave-fronts is at the same instant exactly reproduced at certain definite intervals along the train of waves; from this it follows that the functions representing the train must be periodic with respect to r and hence also with respect to t.

This fact, which was first inferred by Newton, is shown very simply by the following experiment due to Michelson*, from which important deductions will be made later.

Light from a vacuum tube is analysed by prisms forming a spectrum from which any required radiation may be separated by passing through a slit S. The light from this slit is rendered approximately parallel by a collimating lens and then falls on a transparent film of silver on the surface of a thick parallel plate G_1. Here it divides, part being transmitted to a plane mirror M_1 and part being reflected to a mirror M_2. These mirrors return the light to the silvered surface, where the first part is reflected and the second is transmitted, so that both parts coincide and are received in a telescope T. A second plate G_2, of the same thickness as G_1 and parallel to it, is introduced to equalise the optical paths of the two streams. Now if the one mirror be parallel to the image of the other in the silvered surface and

* Michelson, *Phil. Mag.* (5) XIII. 236 (1882).

the telescope be focussed on infinity, there will be seen a series of concentric bright and dark rings, and on moving one mirror parallel to itself, so as to alter the distance traversed by one of the streams, the rings move in towards or out from the centre, which becomes alternately bright and dark, a given

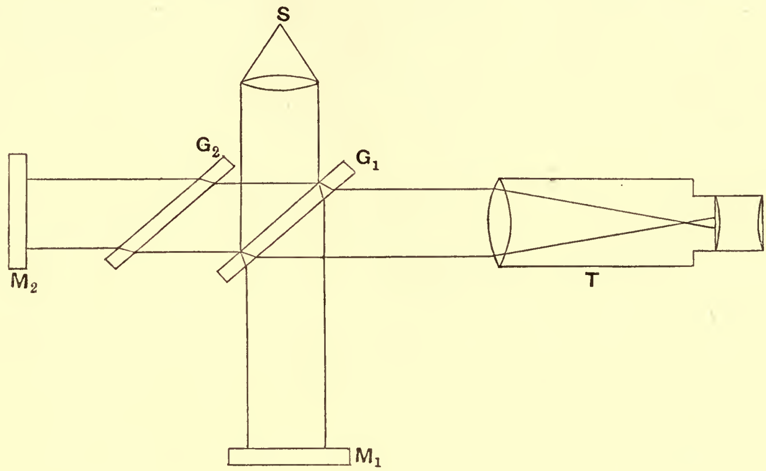


Fig. 8.

shift of the mirror always producing the same number of alternations. But in the case of the light that forms the centre of the pattern, it is clear that the motion of the mirror alters the path of one of the streams by an amount equal to twice the shift; and it thus follows that a given phenomenon is reproduced when the path of one of the streams is altered by any multiple of a given constant length λ and that the train of waves is periodic with respect to this length.

Hence the functions representing the train may be expanded in series of the form

$$\sum_1^\infty A_m \cos\left\{n\frac{2\pi}{\lambda}(\omega t - r) - a_n\right\} \quad \text{.....................(1)},$$

and since on repeating the above experiment with light from a different part of the spectrum, a result is obtained in all respects the same with the exception of the value that is to be assigned to λ, it follows that this quantity is characteristic of the colour of the light, and as (1) is an aggregate of terms in which the values of λ are different, we are led to retain only the first term of the series in the case of a train of waves of strictly monochromatic light.

Monochromatic light is thus said to consist in simple harmonic vibrations, of whatever nature these may be, the period of which is $\tau = \lambda/\omega$, where λ is the wave-length of the train of waves and ω is their propagational speed.

10. It is found however that as the motion of the mirror in Michelson's experiment is gradually increased, the distinctness of the system of rings varies, which would not be the case if the streams were of the simple harmonic type assumed above. It is thus necessary to suppose that in any actual case the light is not absolutely monochromatic and that the stream must be represented by a series of simple harmonic terms of periods that differ only very slightly from one another. It will be shown later how Michelson has been able to deduce the terms of this series in the case of light from different simple sources by means of determinations of the visibility of the system of rings.

A consideration of the state of things occurring in a luminous source, even of the simplest character, also leads to the result that the light emitted cannot be absolutely monochromatic*.

In the first place there is gradual loss of energy from communication to the ether†: thus, supposing that the vibration rises from zero to a maximum and then decreases again to zero, Fourier's theorem gives

$$e^{-k^2t^2}\cos nt = \frac{2}{\pi}\int_0^\infty \cos \alpha t \,.\, d\alpha \int_0^\infty e^{-k^2x^2}\cos nx \cos \alpha x \,.\, dx$$
$$= \frac{1}{2k\sqrt{\pi}}\int_0^\infty \left\{e^{-\left(\frac{\alpha-n}{2k}\right)^2} + e^{-\left(\frac{\alpha+n}{2k}\right)^2}\right\}\cos \alpha t \,.\, d\alpha,$$

and the second member represents an aggregate of trains of waves, each individual train being absolutely monochromatic. If the variation of the amplitude be slow, k is small compared with n and the second exponential may be neglected while the first is only sensible when α is very nearly equal to n.

In the next place there is departure from regularity due to abrupt changes of phase and amplitude. To illustrate this let us suppose that the vibrations in the source are given by

$$\psi(t) = \pm \sin 2\pi t/\tau,$$

wherein the positive sign applies from 0 to $m\tau$, $2m\tau$ to $3m\tau$, ..., and the negative sign from $m\tau$ to $2m\tau$, $3m\tau$ to $4m\tau$, ... Then since Fourier's theorem gives

$$m\tau \,.\, \psi(t) = \int_0^{m\tau} \sin\frac{2\pi}{\tau}z \,.\, dz + 2\sum_1^\infty \cos\frac{n\pi}{m\tau}t\int_0^{m\tau}\sin\frac{2\pi}{\tau}z \,.\, \cos\frac{n\pi}{m\tau}z \,.\, dz$$
$$= \sum_1^\infty \frac{\tau}{\pi} \,.\, \frac{1-\cos n\pi}{1-n^2/(4m^2)} \,.\, \cos\frac{n\pi}{m\tau}t,$$

* Lord Rayleigh, *Phil. Mag.* (5) XXXIV. 407 (1892); XXVII. 298 (1889).

† Jaumann, *Wied. Ann.* LIII. 832 (1894); LIV. 178 (1895). Galitzin, *ibid.* LVI. 78 (1895). Lommel, *ibid.* LVI. 741 (1895). Michelson, *Astrophys. J.* II. 251 (1895).

the light will consist of an aggregate of trains of waves given by

$$\Sigma \frac{2}{m\pi \{1 - n^2/(4m^2)\}} \cos \frac{n\pi}{m\tau} t,$$

the summation extending to all odd values 1, 3, 5, ... of n.

When n is nearly equal to $2m$, the terms of this series become relatively very great, the most important being

$$\cos \frac{2\pi}{\tau} \left(1 \pm \frac{1}{2m}\right) t, \quad \cos \frac{2\pi}{\tau} \left(1 \pm \frac{3}{2m}\right) t,$$

the train of waves with period τ not occurring at all.

Again there is the motion of the molecules as wholes to be considered, and the effect of this is twofold.

Firstly by Doppler's principle if ξ be the component of the velocity of a molecule in the direction of the line of sight and ω be the velocity of light, the natural wave-frequency N is changed by the motion into n, where

$$n = N(\omega + \xi)/\omega.$$

Now the number of molecules, for which the component velocity in the line of sight lies between ξ and $\xi + d\xi$, is proportional to $\exp\{-\beta\xi^2\} d\xi$; hence what would be a mathematical line is dilated in the spectrum into a band and the intensity of the part of the band corresponding to frequencies between n and $n + dn$ will be proportional to

$$e^{-\beta(n-N)^2\omega^2/N^2} dn,$$

or at a distance x from the centre in a spectrum formed on a scale of wave-frequencies to $\exp(-\alpha x^2)$*.

Secondly there is the motion of rotation to be considered. The effect of this will depend upon the law of radiation in various directions from a stationary molecule, but in any case it will in general cause the amplitude of the vibration emitted in a given direction to be a periodic function of the time, whence it follows that the light so radiated ceases to be monochromatic.

In the case in which the luminous source is a narrow band isolated from a spectrum, other considerations lead to the same result, and it will be shown in dealing with diffraction that the finiteness of the wave-length of light imposes a limit on the resolving power of a spectroscope and causes at each point of a spectrum a superposition of light of slightly different wave-lengths†.

* Ebert, *Wied. Ann.* xxxiv. 39 (1888); xxxvi. 466 (1889).

† The nature of white light and the origin of the periodicity introduced by dispersion into its constituents has been discussed by: Gouy, *J. de Phys.* (2) v. 354 (1886); *Ann. de Ch. et de Phys.* (6) xvi. 262 (1889); *C. R.* cxx. 915 (1895); cxxx. 241, 560 (1900). Schuster, *Phil. Mag.* (5) xxxvii. 509 (1894); *C. R.* cxx. 987 (1895). Poincaré, *C. R.* cxx. 757 (1895). Larmor, *Æther and Matter*, 239—251 (1900). Carvallo, *C. R.* cxxx. 79, 130, 401 (1900); *J. de Phys.* (3) ix. 138 (1900). Fabry, *C. R.* cxxx. 238 (1900). Corbino, *C. R.* cxxxiii. 402 (1901). Godfrey, *Phil. Trans.* cxcv. A. 329 (1901). Planck, *Drude's Ann.* vii. 390 (1902).

It thus follows that in any actual case a stream of light is complex in quality; but it is convenient in considering optical phenomena to assume that the light is monochromatic, and then to determine when necessary the modifications that are introduced by its departure from this simple character.

11. If two streams of light, coming initially from the same source, are made to cross one another at a small angle, interference phenomena may be observed in the region common to the two streams and at certain points the illumination is greater and at others it is less than that due to either of the beams alone. Beyond the region of crossing, however, each of the streams is found to have the same characteristics as if it alone existed, and we must therefore infer that the result of the superposition of two streams is merely a superposition of their effects without any permanent modification of the streams themselves. This being so, it must be possible so to choose the analytical expressions ϕ, χ, ψ, ... characterising a stream of light, that the result of the superposition of several streams is expressed by the sum of the corresponding functions ϕ_n, χ_n, ψ_n,... characteristic of the constituent streams, so that we have

$$\phi = \Sigma\phi_n, \quad \chi = \Sigma\chi_n, \quad \psi = \Sigma\psi_n, \ldots.$$

These equations are the analytical expressions of the principle of interference*.

12. The next step in the analytical specification of a train of plane waves of monochromatic light is afforded by the phenomenon of polarisation, discovered by Huygens in 1678 during the course of experiments on the double refraction in Iceland spar and published by him in 1690 in a book entitled "Traité de la Lumière."

Iceland spar, a crystal of calcium carbonate, cleaves very readily in three definite directions, so that a block may be obtained by cleavage in the form of a rhombohedron : the three obtuse angles of such a rhombohedron are all equal and are so turned that two opposite solid angles are contained by equal obtuse angles, while each of the remaining six is contained by one obtuse and two acute angles. A direction equally inclined to the three edges that meet in one of the obtuse solid angles is called the *axis* of the crystal, and a plane through the axis perpendicular to a face of the rhomb is called the *principal plane* of that face.

Now it is found that when a cylindrical stream of light, coming directly from a luminous source, falls normally upon a rhomb of Iceland spar, it is subdivided into two refracted streams : the one, called the ordinary stream, traverses the crystal without deviation ; the other passes obliquely through the rhomb with its axis in the principal plane of the face of entry and emerges parallel to the first, from which it will be entirely distinct provided

* Voigt, *Komp. der Theor. Phys.* II. p. 531.

the diameter of the incident stream does not exceed about one-tenth of the length of the rhomb. These two streams have practically the same intensity, and the phenomenon is unaltered by a rotation of the rhomb about an axis normal to its end faces.

The case is however different if either of these emergent streams be transmitted through a second rhomb with its end faces parallel to those of the first: for then the relative brightness of the two streams, into which it is in general divided, depends upon the orientation of the second rhomb, and in certain cases one of these streams entirely vanishes. Thus the ordinary stream emergent from the first rhomb gives rise to an ordinary stream alone, when the principal planes of the faces of entry of the two rhombs are parallel, and to an extraordinary stream alone when these planes are at right angles; while the reverse is the case with the extraordinary stream of the first rhomb.

Hence while a stream of light coming directly from a luminous source exhibits properties that are alike on all sides of its direction of propagation, in the streams emergent from a rhomb of spar different directions round their axes are no longer of equal value. The streams may in fact be said to have acquired sides or to be polarised. The sides of the stream must in some way be connected with fixed planes in the rhomb and considerations of symmetry lead to their being referred to the principal plane of the face of entry or to the plane perpendicular to it. Either of these planes might be selected, but it is assumed that the ordinary stream has its sides or is polarised in the principal plane, and that the extraordinary stream is polarised in the perpendicular plane.

13. Before leaving this fundamental experiment of polarisation, a further point may be mentioned, that will prove of use subsequently. The direction of the axis of the extraordinary stream in the first rhomb is clearly independent of the diameter of the incident beam, so that the axes of the emergent pencils will be at a distance apart dependent only upon the length of the rhomb. It is then possible by increasing the diameter of the streams to make their perimeters intersect, giving rise to a complex stream in which three parts may be distinguished. The two outer parts are due to the streams ordinarily and extraordinarily refracted respectively and have equal intensities: the central part is formed by the superposition of these two streams and has twice the intensity of the two outer parts.

Now if this central part be examined with a second rhomb, it is found to exhibit no traces of polarisation and to behave exactly like common light. Thus a stream of common light has the same properties as that which results from the superposition of the two equally intense streams polarised at right-angles, into which a rhomb of Iceland spar divides a beam of common light incident upon it. Further, since the two streams traverse the rhomb with

different speeds, they will on emergence have a relative retardation dependent upon the length of the rhomb. Hence common light may be regarded as equivalent to the stream resulting from the superposition of two streams of equal intensity polarised at right-angles, whatever may be the retardation of the one stream with respect to the other.

14. In order to express the phenomenon of polarisation analytically, it becomes necessary to assume that a train of plane waves of polarised light may be represented at any instant by a vector d, the rectangular components of which may be written

$$u = A \cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - a\right\}, \quad v = B \cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - b\right\},$$

$$w = C \cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - c\right\}.$$

From these equations it follows that this vector always lies in the plane

$$\frac{u}{A}\sin(b-c) + \frac{v}{B}\sin(c-a) + \frac{w}{C}\sin(a-b) = 0,$$

and that its extremity in general describes an ellipse, the projections of which on the coordinate planes are given by

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} - 2\frac{uv}{AB}\cos(a-b) = \sin^2(a-b)$$

and two similar equations.

If the plane of xy be parallel to the plane of the elliptic path of the extremity of the vector, $C = 0$, and the angle θ that the axes of the ellipse make with the coordinate axes is given by

$$\tan 2\theta = \frac{2AB}{A^2 - B^2}\cos(a-b) = \tan 2\sigma \,.\, \cos(a-b),$$

where $\tan\sigma = B/A$, and if $\tan\beta$ be the ratio of the axes of the ellipse,

$$\sin 2\beta = \sin 2\sigma \,.\, \sin(a-b).$$

Now

$$\frac{v}{u} = \frac{B}{A}\cos(a-b) - \frac{B}{A}\sin(a-b)\tan\left\{\frac{2\pi}{\lambda}(\omega t - r) - a\right\}$$

gives the tangent of the angle that the vector d makes with the axis of x at any time t. As the time increases, $\tan\left\{\frac{2\pi}{\lambda}(\omega t - r) - a\right\}$ increases, and hence the vector moves from left to right or from right to left on the upper part of its path, according as $(B/A)\sin(a-b)$ or $AB\sin(a-b)$ is positive or negative: in the first case the motion is said to be right-handed and in the second left-handed. Thus A and B having the same signs, the motion is right-handed when $a-b$ is between 0 and π or between $-\pi$ and -2π, and left-handed if this angle lie between π and 2π or between 0 and $-\pi$.

15. It has been mentioned that the phenomenon of interference is to be ascribed to a superposition of the effects of different trains of waves without any modification of the waves themselves. From this it follows that the differential equations of the polarisation-vector d are linear, and this leads to a symbolical representation of the vector, that is often useful.

Since $2\cos\theta = e^{\iota\theta} + e^{-\iota\theta}$, and each exponential repeats itself on differentiation, all the terms in any one equation can be arranged in two groups, one containing $e^{\iota\theta}$ as a factor and the other containing $e^{-\iota\theta}$ as a factor: these two groups will be independent and each will satisfy the differential equations. Hence we may introduce one exponential alone, and then writing the result of our calculations in the form $P + \iota Q$, we have only to throw away the imaginary part or else to reject the real part and omit the ι, since the system of quantities P and the system Q must separately satisfy the conditions of the problem.

Thus when convenient the components of the polarisation-vector may be represented by the symbolical expressions

$$u = \bar{A}e^{\iota\frac{2\pi}{\lambda}(\omega t - r)}, \quad v = \bar{B}e^{\iota\frac{2\pi}{\lambda}(\omega t - r)}, \quad w = \bar{C}e^{\iota\frac{2\pi}{\lambda}(\omega t - r)},$$

the bars (–) placed over the letters A, B, C denoting that they may be complex. Let

$$\bar{A} = A' - \iota A'' = Ae^{-\iota a}, \quad \bar{B} = B' - \iota B'' = Be^{-\iota b}, \quad \bar{C} = C' - \iota C'' = Ce^{-\iota c};$$

then the actual components of the vector are

$$u = A\cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - a\right\} = A'\cos\frac{2\pi}{\lambda}(\omega t - r) + A''\sin\frac{2\pi}{\lambda}(\omega t - r),$$

$$v = B\cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - b\right\} = B'\cos\frac{2\pi}{\lambda}(\omega t - r) + B''\sin\frac{2\pi}{\lambda}(\omega t - r),$$

$$w = C\cos\left\{\frac{2\pi}{\lambda}(\omega t - r) - c\right\} = C'\cos\frac{2\pi}{\lambda}(\omega t - r) + C''\sin\frac{2\pi}{\lambda}(\omega t - r),$$

and we may remark, what will be of use later, that if $\bar{A}'$, $\bar{B}'$, $\bar{C}'$ be the expressions conjugate to $\bar{A}$, $\bar{B}$, $\bar{C}$,

$$A = \sqrt{\bar{A}\,.\,\bar{A}'}, \quad B = \sqrt{\bar{B}\,.\,\bar{B}'}, \quad C = \sqrt{\bar{C}\,.\,\bar{C}'},$$

$$\tan a = -\frac{1}{\iota}\frac{\bar{A} - \bar{A}'}{\bar{A} + \bar{A}'}, \quad \tan b = -\frac{1}{\iota}\frac{\bar{B} - \bar{B}'}{\bar{B} + \bar{B}'}, \quad \tan c = -\frac{1}{\iota}\frac{\bar{C} - \bar{C}'}{\bar{C} + \bar{C}'}.$$

Clearly $\pm A'$, $\pm B'$, $\pm C'$ and $\pm A''$, $\pm B''$, $\pm C''$ are the components of the polarisation-vector d, at the times for which

$$2\pi(\omega t - r)/\lambda = h\pi \quad \text{and} \quad (2h + 1)\pi/2$$

respectively, where h is an integer. Determining the condition that d has then a maximum or a minimum value, we obtain

$$A'\,.\,A'' + B'\,.\,B'' + C'\,.\,C'' = 0,$$

whence $\bar{A}^2+\bar{B}^2+\bar{C}^2$ is real, and this condition can always be satisfied by a proper choice of the origin of time. When this is so chosen,

$$d'^2 = A'^2 + B'^2 + C'^2,$$
$$\alpha' = A'/d', \quad \beta' = B'/d', \quad \gamma' = C'/d',$$

and similar expressions with doubly-accented letters give the semi-axes d', d'' of the elliptic path of the extremity of the polarisation-vector and their direction-cosines.

16. Before proceeding to a precise localisation of the polarisation-vector, it is necessary to obtain a measure of the intensity of a train of plane waves of light; and though this is scarcely possible, until some theory is formulated respecting the nature of the polarisation-vector, the following considerations lead to an estimate of the intensity, that is sufficient as a working hypothesis*.

Since the phenomena that are associated with a stream of light indicate that it is energy that is propagated by the waves, and since moreover the intensity of light from a given source varies inversely as the square of the distance from the source—the same law as obtains in the case of the rate at which energy is propagated across a given area perpendicular to the direction of flow—it is natural to measure the intensity of the stream by this quantity. It is thus necessary to express the energy in terms of the polarisation-vector, and this can only be done when the nature of the vector is itself determined. Since however energy is a scalar quantity, it must be expressed by an even power of the vector and this for present purposes may be taken as the second, for the variation of energy must be the same in sign as that of d^2, and if d^2 vanishes, so must the energy.

But light to be perceived must act for a finite period on the retina, and it is impossible to follow the rapid variations of the polarisation-vector during its vibrations. The intensity may thus be taken as measured by the mean value of the square of the vector for the time T required for light to affect the eye, and on account of the rapidity of the vibrations, T may be taken as an integral multiple of the period. Hence with monochromatic light

$$I = \frac{1}{n\tau}\int_0^{n\tau} (u^2+v^2+w^2)\,dt = A^2 + B^2 + C^2.$$

If the light be not monochromatic,

$$u = \Sigma A_n \cos\left\{\frac{2\pi}{\lambda_n}(\omega t - r) - a_n\right\}, \quad v = \Sigma B_n \cos\left\{\frac{2\pi}{\lambda_n}(\omega t - r) - b_n\right\},$$
$$w = \Sigma C_n \cos\left\{\frac{2\pi}{\lambda_n}(\omega t - r) - c_n\right\},$$

and since T is very great compared with τ_n, the terms in the expression for I that arise from the product of different cosines are vanishingly small and may be neglected: whence

$$I = \Sigma\,(A_n^2 + B_n^2 + C_n^2),$$

* Voigt, *loc. cit.* pp. 524, 529, 537.

and the intensity is the sum of the intensities of the different monochromatic constituents—a result that depends upon the mean value of the *square* of the polarisation-vector being taken as the measure of the intensity.

17. In 1816 Fresnel and Arago, in consequence of a discovery made by the latter, were led to investigate the conditions of the interference of polarised light. Postponing for the present any consideration of their experiments, it suffices for the completion of the specification of the polarisation-vector for a train of waves of monochromatic light to mention, that among other results they found that two polarised streams coming from the same stream, whether polarised or natural, are capable of interfering perfectly, when the polarisations are the same; that they do not interfere at all, if polarised in perpendicular planes; and that in intermediate cases, they interfere in intermediate degrees.

In order to determine the analytical significance of this result, we must investigate the conditions of the interference of two polarised streams, and for this purpose there is no occasion to consider the manner, in which they are related to the original stream, but it is sufficient to start with the component streams themselves*.

Taking the direction of propagation as the axis of z, let θ_1 and $90° + \theta_1$ be the azimuths of the axes of the ellipse that is the projection on the plane of the waves of the path of the extremity of the polarisation-vector, azimuths being measured round z from x to y, and let $\tan \beta_1$ be the ratio of the axes of this ellipse, β_1 lying numerically between 0° and 90°. Then if u_1 and v_1 be measured along the axes of this ellipse, the components of the polarisation-vector of the first stream may be represented by

$$u_1 = c_1 \cos \beta_1 e^{(T+a_1)\iota} = A_1 e^{(T+a_1)\iota},$$

$$v_1 = -\iota c_1 \sin \beta_1 e^{(T+a_1)\iota} = -\iota B_1 e^{(T+a_1)\iota},$$

$$w_1 = k_1 c_1 e^{(T+a_1+\epsilon_1)\iota},$$

where T is written for shortness in place of $2\pi(\omega t - z)/\lambda$ and β_1 is positive or negative according as the projection of the path on the plane of xy is described in a left- or a right-handed direction.

Let u_2, c_2, ... be for the second stream what u_1, c_1, ... are for the first, and let ρ_1, ρ_2 be the retardations of phase that occur before the recomposition of the streams: then resolving all the components along the axes of x, y and z, and writing $\delta = a_2 - \rho_2 - a_1 + \rho_1$,

$$u = (A_1 \cos \theta_1 + \iota B_1 \sin \theta_1) e^{(T+a_1-\rho_1)\iota} + (A_2 \cos \theta_2 + \iota B_2 \sin \theta_2) e^{(T+a_2-\rho_2)\iota}$$

$$= \{(A_1 \cos \theta_1 + A_2 \cos \theta_2 e^{\delta\iota}) + \iota (B_1 \sin \theta_1 + B_2 \sin \theta_2 e^{\delta\iota})\} e^{(T+a_1-\rho_1)\iota},$$

$$v = \{(A_1 \sin \theta_1 + A_2 \sin \theta_2 e^{\delta\iota}) - \iota (B_1 \cos \theta_1 + B_2 \cos \theta_2 e^{\delta\iota})\} e^{(T+a_1-\rho_1)\iota},$$

$$w = \{k_1 c_1 + k_2 c_2 e^{(\delta+\epsilon_2-\epsilon_1)\iota}\} e^{(T+a_1+\epsilon_1-\rho_1)\iota},$$

* Stokes, *Camb. Phil. Trans.* IX. Part 3, 399 (1852).

and the intensity is given by

$$\begin{aligned}I = &\{(A_1\cos\theta_1 + A_2\cos\theta_2 e^{\delta\iota}) + \iota(B_1\sin\theta_1 + B_2\sin\theta_2 e^{\delta\iota})\}\\
&\qquad\times\{(A_1\cos\theta_1 + A_2\cos\theta_2 e^{-\delta\iota}) - \iota(B_1\sin\theta_1 + B_2\sin\theta_2 e^{-\delta\iota})\}\\
&+\{(A_1\sin\theta_1 + A_2\sin\theta_2 e^{\delta\iota}) - \iota(B_1\cos\theta_1 + B_2\cos\theta_2 e^{\delta\iota})\}\\
&\qquad\times\{(A_1\sin\theta_1 + A_2\sin\theta_2 e^{-\delta\iota}) + \iota(B_1\cos\theta_1 + B_2\cos\theta_2 e^{-\delta\iota})\}\\
&+(k_1c_1 + k_2c_2e^{(\delta+\epsilon_2-\epsilon_1)\iota})(k_1c_1 + k_2c_2e^{-(\delta+\epsilon_2-\epsilon_1)\iota})\\
= &A_1^2 + A_2^2 + B_1^2 + B_2^2 + k_1^2c_1^2 + k_2^2c_2^2 + 2(A_1A_2 + B_1B_2)\cos(\theta_2-\theta_1)\cos\delta\\
&-2(A_1B_2 + A_2B_1)\sin(\theta_2-\theta_1)\sin\delta + 2k_1k_2c_1c_2\cos(\delta+\epsilon_2-\epsilon_1)\\
= &c_1^2(1+k_1^2) + c_2^2(1+k_2^2)\\
&+2c_1c_2\{\cos(\beta_2-\beta_1)\cos(\theta_2-\theta_1)\cos\delta - \sin(\beta_2+\beta_1)\sin(\theta_2-\theta_1)\sin\delta\\
&\qquad + k_1k_2\cos(\delta+\epsilon_2-\epsilon_1)\}.\end{aligned}$$

Now if there be no interference, the intensity must be independent of the relative retardation of phase $\rho_2-\rho_1$, and we must have

$$\cos(\beta_2-\beta_1)\cos(\theta_2-\theta_1) + k_1k_2\cos(\epsilon_2-\epsilon_1) = 0,$$

and

$$\sin(\beta_2+\beta_1)\sin(\theta_2-\theta_1) + k_1k_2\sin(\epsilon_2-\epsilon_1) = 0,$$

which conditions may be satisfied in an infinite number of ways, all of which appear equally admissible, unless recourse be had to other considerations.

There is however a case that leads to a definite conclusion; for it is found that there is no interference, when the two streams are both, say, the ordinary streams emergent from two rhombs of Iceland spar so placed that the planes of polarisation are at right-angles. In this case the one component stream is, so far as relates to its polarisation, what the other stream becomes on being turned about its axis through a right-angle. Writing then

$$\theta_2-\theta_1 = 90^\circ,\quad \beta_2=\beta_1,\quad k_2=k_1,\quad \epsilon_2=\epsilon_1,$$

the above conditions become

$$k_1^2 = 0,\quad \sin 2\beta_1 = 0;$$

that is, the polarisation-vector has no component in the direction of propagation of the stream and its vibrations are rectilinear.

Now the streams emergent from a rhomb of Iceland spar are said to be plane polarised, and thus in a stream of plane polarised light the polarisation-vector is transverse to the direction of propagation and its vibrations are rectilinear*. By symmetry these vibrations must be either in or perpendicular to the plane of polarisation: in what follows we shall assume that the latter is the case.

* Fresnel, *Mém. de l'Acad. des Sc.* VII. 56 (1821); *Œuvres complètes*, II. 490. Verdet, *Ann. de Ch. et de Phys.* (3) XXXI. 377 (1851); *Œuvres*, I. 73.

If a stream of plane polarised light be resolved into two streams, polarised at right-angles to one another, and these be recompounded after one has been retarded relatively to the other, the polarisation-vector of the resultant stream will have for its components

$$u = A\cos\theta\cos\left\{\frac{2\pi}{\lambda}(\omega t - z) + a\right\},\quad v = A\sin\theta\cos\left\{\frac{2\pi}{\lambda}(\omega t - z) + a - \delta\right\},$$

where θ and $90° - \theta$ are the angles that the plane of polarisation of the original stream makes with the planes of polarisation of the components. The vibrations of the polarisation-vector of the resultant stream are thus represented by an ellipse lying in the plane of the waves, and the light is said to be elliptically polarised.

In the particular case in which $\theta = 45°$ and the relative retardation of phase is $\pm(2n+1)\pi/2$, the vibrations are circular and the light is said to be circularly polarised.

18. Returning to the general conditions that express that there is no interference between polarised streams, and writing $k_1 = k_2 = 0$, we obtain

$$\cos(\beta_2 - \beta_1)\cos(\theta_2 - \theta_1) = 0,\quad \sin(\beta_2 + \beta_1)\sin(\theta_2 - \theta_1) = 0,$$

which are satisfied if

$$\cos(\theta_2 - \theta_1) = 0 \text{ and } \sin(\beta_2 + \beta_1) = 0,$$

or

$$\sin(\theta_2 - \theta_1) = 0 \text{ and } \cos(\beta_2 - \beta_1) = 0,$$

or

$$\cos(\beta_2 - \beta_1) = 0 \text{ and } \sin(\beta_2 + \beta_1) = 0.$$

The first case gives $\theta_2 - \theta_1 = 90°$, $\beta_2 = -\beta_1$, and these results express that the ellipses described are similar, their major axes at right-angles and the directions in which they are described are contrary.

The second pair of equations gives $\theta_2 - \theta_1 = 0°$ or $180°$, $\beta_2 = 90° + \beta_1$, which is merely a different manner of expressing the same result.

From the last pair of equations we have $\beta_2 = -\beta_1 = \pm 45°$, or the streams are circularly polarised in opposite directions—a special case of the former result.

Thus the intensity of the stream made up of the two components is only independent of any retardation, that the one has undergone relatively to the other before recomposition, when the one component stream is, so far as relates to its polarisation, what the other becomes when it is turned through an azimuth of 90° and has its nature reversed as regards right- and left-hand. Streams thus related are said to be oppositely polarised.

19. On the other hand the interference will be perfect, that is, the variations of intensity will be the greatest that the difference of intensity of the components admits of, so that if these be equal, the minima are absolutely zero, when the coefficient of $2c_1c_2$ has unity as its maximum value.

The maximum of $P\cos\delta - Q\sin\delta$ is $\sqrt{P^2+Q^2}$, so that the condition for perfect interference is

$$\cos^2(\beta_2-\beta_1)\cos^2(\theta_2-\theta_1)+\sin^2(\beta_2+\beta_1)\sin^2(\theta_2-\theta_1)=1$$
$$=\cos^2(\theta_2-\theta_1)+\sin^2(\theta_2-\theta_1)$$

or $$\cos^2(\theta_2-\theta_1)\sin^2(\beta_2-\beta_1)+\sin^2(\theta_2-\theta_1)\cos^2(\beta_2+\beta_1)=0.$$

This is only satisfied if

$$\sin^2(\theta_2-\theta_1)=0 \text{ and } \sin^2(\beta_2-\beta_1)=0,$$

or $$\cos^2(\theta_2-\theta_1)=0 \text{ and } \cos^2(\beta_2+\beta_1)=0,$$

or $$\sin^2(\beta_2-\beta_1)=0 \text{ and } \cos^2(\beta_2+\beta_1)=0$$

From the first pair of equations we have $\beta_2=\beta_1$, $\theta_2=\theta_1$, that is, the streams are identical as regards their polarisation.

The second case gives $\beta_2=90°-\beta_1$, $\theta_2-\theta_1=90°$, expressing the same result.

The third case gives $\beta_2=\beta_1=45°$, so that the streams are circularly polarised and of the same kind—a particular case of the former result.

Thus for perfect interference the polarisations of the two streams must be identical.

20. It now becomes necessary to determine the analytical representation of a stream of natural or unpolarised light*.

Experiment gives as the distinguishing characteristic of a stream of common light, that it can be resolved into two streams plane polarised in perpendicular planes; that the intensities of these streams are independent of the orientation of their planes of polarisation; and that the stream resulting from the recomposition of these components has the same property, whatever may be their relative retardation.

Now since this stream compounded of two streams that are plane polarised in rectangular planes, behaves in all respects as common light, and since each constituent is represented by a vector that is perpendicular to the direction of propagation, it follows that it must be possible to obtain an analytical representation of a stream of common light, in which no vector with a longitudinal component occurs. On the other hand a stream of monochromatic light with a polarisation-vector that is entirely transversal, must be polarised, whether elliptically, circularly or plane; whence it results that a stream of common light cannot be absolutely monochromatic.

Representing then the stream as the superposition of trains of waves of monochromatic light, let

$$u_n = c_n\cos\beta_n e^{\iota(T_n+a_n)} = A_n e^{\iota(T_n+a_n)},$$
$$v_n = -\iota c_n\sin\beta_n e^{\iota(T_n+a_n)} = -\iota B_n e^{\iota(T_n+a_n)}$$

* Stokes, *loc. cit.* Verdet, *Œuvres*, I. p. 281; *Ann. de l'école norm. supér.* II. 291 (1865).

be the components of the polarisation-vector for the nth constituent in directions making angles θ_n and $90° + \theta_n$ with the axis of x, where the axis of z being in the direction of propagation of the stream $T_n = 2\pi(\omega t - z)/\lambda_n$, and β_n is less than $90°$ and positive or negative according as this constituent is left- or right-handed.

Then the stream of common light may be represented by the two plane polarised components

$$u = \Sigma\,(A_n \cos\theta_n + \iota B_n \sin\theta_n)\, e^{\iota(T_n + \alpha_n)},$$
$$v = \Sigma\,(A_n \sin\theta_n - \iota B_n \cos\theta_n)\, e^{\iota(T_n + \alpha_n)}.$$

Now let the second component stream receive a retardation of phase δ relatively to the first, and let the stream of common light thus modified be resolved into two plane polarised components with their vectors in azimuths ϕ and $90° + \phi$ with respect to the axis of x: then for the first of these components the polarisation-vector is

$$U = \Sigma\,\{A_n(\cos\theta_n \cos\phi + \sin\theta_n \sin\phi e^{-\iota\delta}) + \iota B_n(\sin\theta_n \cos\phi - \cos\theta_n \sin\phi e^{-\iota\delta})\}\, e^{\iota(T_n + \alpha_n)},$$

and since the intensity is the sum of the intensities of the monochromatic constituents

$$\begin{aligned} I_\phi &= \Sigma\,(A_n^2 \cos^2\theta_n + B_n^2 \sin^2\theta_n)\cos^2\phi + \Sigma\,(A_n^2 \sin^2\theta_n + B_n^2\cos^2\theta_n)\sin^2\phi \\ &\quad + \sin\phi\cos\phi\,\{2\Sigma\,(A_n^2 - B_n^2)\sin\theta_n\cos\theta_n\cos\delta - 2\Sigma A_n B_n \sin\delta\} \\ &= \tfrac{1}{2}(P+Q)\cos^2\phi + \tfrac{1}{2}(P-Q)\sin^2\phi + \sin\phi\cos\phi\,(R\cos\delta - S\sin\delta), \end{aligned}$$

where

$$\begin{aligned} P &= \Sigma\,(A_n^2 + B_n^2) &&= \Sigma c_n^2, \\ Q &= \Sigma\,(A_n^2 - B_n^2)\cos 2\theta_n &&= \Sigma c_n^2 \cos 2\beta_n \cos 2\theta_n, \\ R &= \Sigma\,(A_n^2 - B_n^2)\sin 2\theta_n &&= \Sigma c_n^2 \cos 2\beta_n \sin 2\theta_n, \\ S &= 2\Sigma A_n B_n &&= \Sigma c_n^2 \sin 2\beta_n. \end{aligned}$$

But if the group of monochromatic constituents be equivalent to common light, I_ϕ must be independent of ϕ whatever δ may be, and for this to be the case, Q, R, S must separately vanish.

The effect of a change of coordinate axes is to write $\theta_n - \chi$ for θ_n $(n = 1, 2, 3, \ldots)$: this will leave P and S unaltered, while

$$Q \text{ becomes } \Sigma c_n^2 \cos 2\beta_n \cos 2(\theta_n - \chi) = Q\cos 2\chi + R\sin 2\chi,$$
$$R \text{ becomes } \Sigma c_n^2 \cos 2\beta_n \sin 2(\theta_n - \chi) = R\cos 2\chi - Q\sin 2\chi.$$

Hence the conditions that the group may be equivalent to common light are satisfied for any set of axes, if they be so for one set, and it is a matter of indifference with respect to what plane of polarisation the retardation δ is supposed to be introduced.

The conditions given above are then sufficient, as well as necessary, to characterise a stream of common light.

21. We may now find the condition that two polarised streams of a definite character may be together equivalent to a stream of common light.

Representing the first stream by the components

$$u' = \cos\beta' \Sigma c_n \cos(T_n + a_n), \quad v' = \sin\beta' \Sigma c_n \sin(T_n + a_n)$$

in directions making an angle θ' with the axes of x and y respectively, and employing doubly accented letters to denote quantities that refer to the second stream, the stream that results from the superposition of these trains of waves is characterised by

$$P = \Sigma c_n'^2 + \Sigma c_n''^2,$$

$$Q = \cos 2\beta' \cos 2\theta' \Sigma c_n'^2 + \cos 2\beta'' \cos 2\theta'' \Sigma c_n''^2,$$

$$R = \cos 2\beta' \sin 2\theta' \Sigma c_n'^2 + \cos 2\beta'' \sin 2\theta'' \Sigma c_n''^2,$$

$$S = \sin 2\beta' \Sigma c_n'^2 + \sin 2\beta'' \Sigma c_n''^2.$$

Writing that the intensities of the two components are as $k^2 : 1$, the condition that their mixture is equivalent to common light gives

$$\cos 2\beta' \cos 2\theta' + k^2 \cos 2\beta'' \cos 2\theta'' = 0, \quad \cos 2\beta' \sin 2\theta' + k^2 \cos 2\beta'' \sin 2\theta'' = 0,$$

$$\sin 2\beta' + k^2 \sin 2\beta'' = 0.$$

Transferring, squaring and adding, these equations give $k^4 = 1$, and since k^2 must be positive, $k^2 = 1$. Thus the streams must have equal intensities.

Since β' and β'' are supposed not to lie beyond the limits of $\pm 90°$, the last equation gives

$$\beta'' = -\beta' \quad \text{or} \quad \beta'' = \beta' \mp 90°,$$

the upper or lower sign being taken according as β' is positive or negative. Now clearly any solution may be expressed analytically in two ways, in which the values of β are complementary and the values of θ differ by $90°$, since either principal axis of the ellipse characterising the stream may be that for which the azimuth is θ. Accordingly the second solution may be rejected as being merely a different method of expressing the first, then substituting $\beta'' = -\beta'$ in the first two equations, they give

$$\cos 2\theta'' = -\cos 2\theta', \quad \sin 2\theta'' = -\sin 2\theta',$$

and hence θ' and θ'' differ by $90°$. The equations are also satisfied by

$$\beta'' = -\beta' = \pm 45°,$$

which is only a special case of the foregoing.

Thus common light is equivalent to any two oppositely polarised streams of half the intensity, and no two polarised streams can be together equivalent to common light, unless they are oppositely polarised and have their intensities equal.

22. Returning now to the case of a stream of light of the most general character, it is clear in the first place that the quantities P, Q, R, S are

restricted in value and so related that P^2 can never be less than $Q^2+R^2+S^2$; for

$$P^2-(Q^2+R^2+S^2)$$
$$=2\Sigma c_m{}^2c_n{}^2\{\sin^2(\beta_m-\beta_n)\cos^2(\theta_m-\theta_n)+\cos^2(\beta_m+\beta_n)\sin^2(\theta_m-\theta_n)\},$$

the summation extending to all values of m and n, and this expression is always positive, and only vanishes if

$$\sin(\beta_m-\beta_n)\cos(\theta_m-\theta_n)=0 \quad \text{and} \quad \cos(\beta_m+\beta_n)\sin(\theta_m-\theta_n)=0.$$

These conditions, as we have seen (§ 19), express that the polarisations of all the monochromatic constituents are identical, so that the stream is elliptically polarised with elliptic constants given by

$$\tan 2\theta=R/Q, \quad \sin 2\beta=S/P,$$

the polarisation being left- or right-handed according as S is positive or negative.

In general then P^2 exceeds $Q^2+R^2+S^2$, but it is always possible to find a positive quantity H, such that

$$(P-H)^2=Q^2+R^2+S^2,$$

and consequently the stream may be regarded as compounded of two groups, for one of which the constants are H, 0, 0, 0, representing a beam of common light, while for the other the constants are $P-H$, Q, R, S giving a stream of elliptically polarised light with elliptic constants determined from

$$\tan 2\theta=R/Q, \quad \sin 2\beta=S/(P-H).$$

If $S=0$, the second group is plane polarised, and if $Q=0$, $R=0$, its polarisation is circular.

A stream of the character just described is said to be partially polarised.

23. As examples of the above investigation, let us take the following cases:

(1) A polarising prism and a crystalline plate, set so as to give a stream of elliptically polarised light, are made to revolve together uniformly and rapidly with regard to the duration of impressions on the retina.

Let $$u'=c\cos\beta e^{\iota\{nt-\kappa z+a\}}, \quad v'=-\iota c\sin\beta e^{\iota\{nt-\kappa z+a\}}$$

represent one of the monochromatic constituents of the stream and θ the azimuth of its first axis at a given time, so that $\theta=\mu+\nu t$.

The components of this constituent along the axes are

$$u=\{c\cos\beta\cos(\mu+\nu t)+\iota c\sin\beta\sin(\mu+\nu t)\}\,e^{\iota\{nt-\kappa z+a\}}$$
$$=\tfrac{1}{2}c(\cos\beta+\sin\beta)\,e^{\iota\{(n+\nu)t-\kappa z+a+\mu\}}$$
$$+\tfrac{1}{2}c(\cos\beta-\sin\beta)\,e^{\iota\{(n-\nu)t-\kappa z+a-\mu\}},$$
$$v=-\iota\,\tfrac{1}{2}c(\cos\beta+\sin\beta)\,e^{\iota\{(n+\nu)t-\kappa z+a+\mu\}}$$
$$+\iota\,\tfrac{1}{2}c(\cos\beta-\sin\beta)\,e^{\iota\{(n-\nu)t-\kappa z+a-\mu\}}.$$

If the light be approximately monochromatic, β will be practically the same for all the constituents, and the stream will be composed of two oppositely circularly polarised streams represented by

$$u_1 = \tfrac{1}{2}(\cos\beta + \sin\beta)\,\Sigma c e^{\iota\{(n+\nu)t-\kappa z+a+\mu\}},$$

$$v_1 = -\iota\tfrac{1}{2}(\cos\beta + \sin\beta)\,\Sigma c e^{\iota\{(n+\nu)t-\kappa z+a+\mu\}},$$

and

$$u_2 = \tfrac{1}{2}(\cos\beta - \sin\beta)\,\Sigma c e^{\iota\{(n-\nu)t-\kappa z+a-\mu\}},$$

$$v_2 = \iota\tfrac{1}{2}(\cos\beta - \sin\beta)\,\Sigma c e^{\iota\{(n-\nu)t-\kappa z+a-\mu\}}.$$

We thus find $P=\Sigma c^2$, $Q=0$, $R=0$, $S=\sin 2\beta\Sigma c^2$ and the group is equivalent to a stream of common light of intensity $(1 \mp \sin 2\beta)\,\Sigma c^2$, together with a stream of circularly polarised light of intensity $\pm\sin 2\beta\Sigma c^2$ and of the same character as regards right- or left-hand, as the original stream would be, if the polariser and plate were stationary. The upper or lower sign must be taken according as β is positive or negative.

If the plate were set so as to give plane polarised light, we should have $\beta=0$ and the light would be completely depolarised.

(2) A plane polarised stream is transmitted through a thin crystalline plate, that is made to rotate uniformly and rapidly.

Let 2δ be the relative retardation of phase introduced by the plate, and θ the azimuth of the plane of polarisation of the least retarded stream within it at any time t, measured from the primitive plane of polarisation: then the emergent stream may be represented by the components

$$\begin{aligned} u &= \Sigma c\,(\cos^2\theta e^{\iota\delta} + \sin^2\theta e^{-\iota\delta})\,e^{\iota(nt-\kappa z+a)} \\ &= \Sigma c\cos\delta e^{\iota(nt-\kappa z+a)} + \iota\tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n+2\nu)t-\kappa z+a+2\mu\}} \\ &\qquad + \iota\tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n-2\nu)t-\kappa z+a-2\mu\}}, \\ v &= \Sigma c\sin\theta\cos\theta\,(e^{\iota\delta} - e^{-\iota\delta})\,e^{\iota(nt-\kappa z+a)} \\ &= \tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n+2\nu)t-\kappa z+a+2\mu\}} - \tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n-2\nu)t-\kappa z+a-2\mu\}}, \end{aligned}$$

polarised respectively in planes parallel and perpendicular to the original plane of polarisation.

Thus the stream is composed of three groups: one polarised in the primitive plane and represented by

$$u_1 = \Sigma c\cos\delta e^{\iota(nt-\kappa z+a)},$$

and two circularly polarised streams of the same intensity

$$u_2 = \iota\tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n+2\nu)t-\kappa z+a+2\mu\}},\quad v_2 = \tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n+2\nu)t-\kappa z+a+2\mu\}},$$

$$u_3 = \iota\tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n-2\nu)t-\kappa z+a-2\mu\}},\quad v_3 = -\tfrac{1}{2}\Sigma c\sin\delta e^{\iota\{(n-2\nu)t-\kappa z+a-2\mu\}}.$$

It is hence equivalent to a stream of common light of intensity $\Sigma c^2\sin^2\delta$, combined with a stream plane polarised in the same azimuth as the initial stream and of intensity $\Sigma c^2\cos^2\delta$.

CHAPTER III.

INTERFERENCE.

24. WE have seen that a train of waves may always be replaced by two trains polarised in perpendicular planes, and that the stream is equivalent to common light, provided the two component streams have the same intensity and no fixed relation exists between their corresponding monochromatic constituents.

In the case of common light, the modifications of the constituents of the one component must during the passage of the stream be identical in character with those of the corresponding constituent of the other, so long as no phenomena of polarisation supervene; for the characteristic property of common light is that all directions transverse to that of propagation are of equal value. Hence in considering the phenomena of common light, it is sufficient to take into account only one of the polarised trains of waves.

The phenomenon of interference lies in the forefront of physical optics and has already been appealed to for the purposes of illustrating the periodic character of a stream of light and of obtaining the form of the functions that characterise a train of luminous waves. We must now take up the subject in greater detail, in order to explain the appearances that result from the interference of streams of light, and to determine the conditions under which interference is possible and the limitations to which it is subjected.

25. When a number of trains of waves of the same simple harmonic type are propagated in one direction, the resultant train is of the same type; for

$$\Sigma A_n \cos\left\{\frac{2\pi}{\lambda}(\omega t - z) + a_n\right\}$$

$$= \cos\frac{2\pi}{\lambda}(\omega t - z)\,\Sigma A_n \cos a_n - \sin\frac{2\pi}{\lambda}(\omega t - z)\,\Sigma A_n \sin a_n$$

$$= A\cos\left\{\frac{2\pi}{\lambda}(\omega t - z) + \delta\right\},$$

where
$$A^2 = (\Sigma A_n \cos a_n)^2 + (\Sigma A_n \sin a_n)^2,$$
$$\tan\delta = (\Sigma A_n \sin a_n)/(\Sigma A_n \cos a_n).$$

In the special case of two component trains, the amplitude of the resultant is given by

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(a_1 - a_2)$$

and is equal to $A_1 \pm A_2$ according as the phases of the two components are the same or differ by half a period. In the latter case when the intensities of the components are equal, the amplitude of the resultant is zero and the component trains neutralise each other.

When the component trains travel in the same direction, the intensity is necessarily the same over the whole wave-front, but this is no longer the case, if they be inclined to one another at a small angle. The phenomenon then observed on a screen placed in the region common to the two streams is that known as interference fringes.

Suppose that we have two small sources of light, that are placed near one another and are of such a character that the corresponding monochromatic constituents of the streams emanating from them agree in amplitude and phase; and let us determine the effect produced on a screen parallel to the line joining the sources and at such a distance that the waves arriving at any point of it from the two sources may be regarded as sensibly plane and parallel.

Let S_1 and S_2 be the sources and X the point on the screen at which the effect is to be determined, then assuming for the present that the light is rigorously monochromatic, the phases at X will be accordant or completely discordant, according as

$$S_1X \sim S_2X = n\lambda \quad \text{or} \quad (2n+1)\lambda/2.$$

With centre X and radius XS_2 (S_2 being supposed nearer X than S_1) describe a circle in the plane XS_1S_2, cutting S_1S_2 produced in Q and XS_1 in P: then O being the middle point of S_1S_2, O' being the point on the line through X parallel to S_1S_2 that is equidistant from S_1 and S_2, we have

$$S_1P(S_1X + XS_2) = S_1S_2 . S_1Q = 2O'X . S_1S_2,$$

$$\therefore\ S_1X - S_2X = \frac{2O'X . S_1S_2}{S_1X + XS_2} \doteqdot \frac{O'X . S_1S_2}{OO'},$$

if X be near to O'.

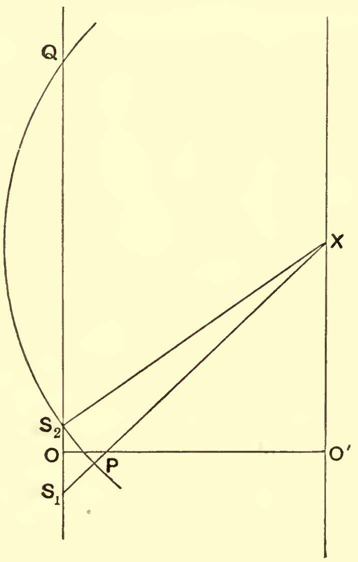


Fig. 9.

Taking the point O as the origin of a system of rectangular coordinates of which the line OS_2 is the axis of x and the normal to the screen is the axis

of z, then if the coordinates of X be x, y, z and $S_1S_2 = 2c$, the points at which the phases are accordant are given by

$$2cx/(y^2+z^2)^{\frac{1}{2}} = n\lambda,$$

and the points of complete discordance of phase are given by

$$2cx/(y^2+z^2)^{\frac{1}{2}} = (2n+1)\lambda/2.$$

Thus the points of accordance or of complete discordance lie on similar and similarly situated hyperbolas and the screen will be intersected by a series of bright and dark bands, that will appear nearly straight and perpendicular to the plane taken as that of xz, since the curvature of the hyperbolas is very small at their vertices.

The linear width of the bands in the plane of xz (from bright to bright, or from dark to dark) is

$$\Lambda = \lambda z/(2c).$$

26. Let us now consider the effect of interposing a plate of some medium between the screen and the sources of light.

Let S_2PQX be the ray from S_2 to X, meeting the plate in the points P and Q; and through Q draw QR parallel to the screen meeting S_2P produced in R and the line through P perpendicular to the screen in R': call d the thickness of the plate, μ its refractive index, and β its inclination to the screen.

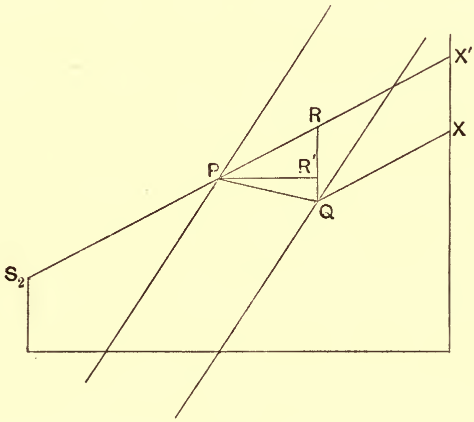


Fig. 10.

Then if i and r be the angles of incidence and refraction at the point P, and we suppose the angles i and β so small that the cubes and higher powers of their sines may be neglected, we have

$$PQ\ = d/\cos r \fallingdotseq d\,\{1+\sin^2 i/(2\mu^2)\},$$

$$PR' = d\cos(\beta - r)/\cos r \fallingdotseq d - d\sin^2\beta/2 + d\sin i\sin\beta/\mu,$$

$$QR\ = PQ\sin(i-r)/\cos(i-\beta) \fallingdotseq d\,(\mu-1)\sin i/\mu.$$

Now the optical length of path from S_2 to X is

$$\Delta_2 = S_2P + \mu PQ + QX = SX' - PR + \mu PQ = (z - PR')\sec(i - \beta) + \mu PQ$$

$$\doteqdot \left(z - d + \tfrac{1}{2}d\sin^2\beta - \frac{d}{\mu}\sin i \sin\beta\right)\{1 + \tfrac{1}{2}(\sin i - \sin\beta)^2\}$$

$$+ \mu d\left(1 + \frac{1}{2\mu^2}\sin^2 i\right)$$

$$\doteqdot z + (\mu - 1)d + \tfrac{1}{2}\left(z - \frac{\mu - 1}{\mu}d\right)(\sin i - \sin\beta)^2 + \frac{\mu - 1}{2\mu}d\sin^2\beta.$$

But
$$z\tan(i - \beta) = (x - c) + RQ,$$
or approximately
$$z(\sin i - \sin\beta) = (x - c) + \frac{\mu - 1}{\mu}d\sin i;$$
whence
$$\left(z - \frac{\mu - 1}{\mu}d\right)(\sin i - \sin\beta) = (x - c) + \frac{\mu - 1}{\mu}d\sin\beta.$$

Substituting in Δ_2 we have

$$\Delta_2 = z + (\mu - 1)d + \frac{\mu - 1}{2\mu}d\sin^2\beta + \tfrac{1}{2}\frac{\left(x - c + \frac{\mu - 1}{\mu}d\sin\beta\right)^2}{z - \frac{\mu - 1}{\mu}d}.$$

The optical length of path Δ_1 from S_1 to X is obtained from this expression by changing the sign of c, and hence the relative retardation of the two streams is

$$\Delta = \Delta_1 - \Delta_2 = \frac{2c}{z - \frac{\mu - 1}{\mu}d}\left\{x + \frac{\mu - 1}{\mu}d\sin\beta\right\}.$$

Thus the points of accordance of phase occur where

$$x = \frac{z - (\mu - 1)d/\mu}{2c}.n\lambda - \frac{\mu - 1}{\mu}d\sin\beta,$$

and the central band of the system, which corresponds to $n = 0$, is at the point

$$x = -\frac{\mu - 1}{\mu}d\sin\beta,$$

and this gives the shift due to the interposition of the plate.

If the plate be parallel to the screen and be traversed by the stream from S_2 alone, the shift is approximately $(\mu - 1)d.z/(2c)$ on the side of the stream that passes through the plate.

27. Since the light from the correlated sources is not strictly monochromatic, the only line of complete accordance of phase is that equidistant from the sources and there is no place of complete discordance of phase for

all pairs of constituents. Hence on receding from the line of complete accordance, the coincidence of the bands arising from the different monochromatic constituents becomes less and less complete, and finally all appearance of interference will be obliterated.

As an illustration of the result of the defect of the monochromatism of the light, the case may be considered in which this arises solely from the progressive motion of the molecules of the source as wholes*.

If ξ be the velocity of the molecules in the direction of propagation of the light, and δ_0 be the relative retardation of phase calculated on the assumption that the molecules are at rest, then the actual retardation of phase is

$$\delta = \delta_0 (1 + \xi/\omega).$$

Now the intensity corresponding to a retardation of phase δ is proportional to

$$2 (1 + \cos \delta),$$

and the number of molecules with velocities between ξ and $\xi + d\xi$ varies as

$$\exp(-\beta\xi^2)\, d\xi,$$

where $\beta = 4/(\pi u^2)$, u being the mean velocity; hence the intensity may be represented by

$$I = 2\int_{-\infty}^{\infty} \left(1 + \cos \delta_0 \cos \frac{\delta_0}{\omega}\xi - \sin \delta_0 \sin \frac{\delta_0}{\omega}\xi\right) e^{-\beta\xi^2}\, d\xi$$

$$= 2\sqrt{\frac{\pi}{\beta}}\left(1 + \cos \delta_0 e^{-\frac{\delta_0^2}{4\beta\omega^2}}\right) = \pi u \left\{1 + \cos \delta_0 e^{-\pi\left(\frac{u\delta_0}{4\omega}\right)^2}\right\}.$$

Hence the maximum intensity is

$$I_1 = \pi u \left\{1 + e^{-\pi\left(\frac{u\delta_0}{4\omega}\right)^2}\right\},$$

and the minimum intensity is

$$I_2 = \pi u \left\{1 - e^{-\pi\left(\frac{u\delta_0}{4\omega}\right)^2}\right\}.$$

Assuming with Michelson that the visibility of the fringes is given by

$$V = (I_1 - I_2)/(I_1 + I_2),$$

we have

$$V = e^{-\pi\left(\frac{u\delta_0}{4\omega}\right)^2},$$

and taking the limit of visibility as determined by $V = 1/40$,

$$\delta_0 = 4\frac{\omega}{u}\sqrt{\frac{\log_e 40}{\pi}},$$

or the limit of relative retardation Δ_0 is given by

$$\frac{\Delta_0}{\lambda} = \frac{\delta_0}{2\pi} = \frac{2}{\pi}\cdot\frac{\omega}{u}\sqrt{\frac{\log_e 40}{\pi}}.$$

* Lord Rayleigh, *Phil. Mag.* (5) xxvii. 298 (1889); Ebert, *Wied. Ann.* xxxvi. 466 (1889).

In the case of sodium vapour at 1000° C., $u = 1172$ metre/sec*, whence since $\omega = 3 \times 10^8$ metre/sec, we have

$$\Delta_0/\lambda = 180{,}000.$$

28. When the range of periods in the correlated streams extends over the visible spectrum, the limit of visibility depends upon the possibility of distinguishing chromatic variations: the central band is white and this is bordered by fringes, that on account of the limitation of the sensitiveness of the eye to periods extending over less than an octave appear to be sensibly black; to these succeed coloured bands, until a point is reached at which the annulments for waves of different periods are so numerous as not to affect the colour of the light.

The interference may however be rendered visible in this case by a spectroscopic analysis of the light. If the slit of the spectroscope be parallel to the direction of the fringes and be narrow in comparison with their breadth, a channelled spectrum is obtained, that is, a spectrum intersected by dark bands at right angles to its length, the centres of which occur at points corresponding to wave-lengths given by

$$\lambda = 2\Delta/(2n + 1),$$

Δ being the relative retardation of the streams and n an integer. As the slit is moved in a direction perpendicular to its length to places of continually higher relative retardation, the bands will travel along the spectrum from the blue to the red end, in the case of an ordinary refraction spectroscope closing up as they move.

The relative retardation Δ may be calculated from the number of bands between two parts of the spectrum corresponding to known wave-lengths; for if n and n' be the orders of the bands corresponding to wave-lengths λ and λ' respectively,

$$\Delta = (2n + 1)\,\lambda/2 = (2n' + 1)\,\lambda'/2,$$

whence if $n' - n = m$,

$$(2n + 1)/2 = m\lambda'/(\lambda - \lambda') \quad \text{and} \quad \Delta = m\lambda\lambda'/(\lambda - \lambda').$$

* The kinetic theory of gases gives that $p = \sigma u^2/3$, where p is the pressure and σ the density of a gas: the density of air at 0° C. and under normal pressure is 1/773·4, whence if s_0 be the density of the gas relatively to air at 0° C. and under normal pressure

$$u_0 = 48500/\sqrt{s_0}\,.\ \text{(cm/sec)},$$

and since by the gaseous laws $p/(s\theta) = p_0/(s_0\theta_0)$, where θ is the absolute temperature

$$u = u_0\sqrt{\frac{\theta}{\theta_0}} = 485\sqrt{\frac{\theta}{\theta_0 s_0}}\ \frac{\text{metre}}{\text{sec}}\,.$$

By Gay-Lussac's law the densities of two gases at the same temperature and under the same pressure are in the proportion of their molecular weights: thus the density of hydrogen relatively to air being ·0693, and sodium vapour being monatomic, we have in this case

$$s_0 = 23 \times {\cdot}0693 \div 2 \doteqdot {\cdot}8.$$

The number of bands that can in this way be rendered visible depends upon the resolving power of the spectroscope.

A second method of employing the spectroscope is to place the slit at right-angles to the direction of the fringes: the spectrum is then traversed by slightly curved bands running more or less along its length and approaching one another towards the violet end (fig. 11). The intercepts made by

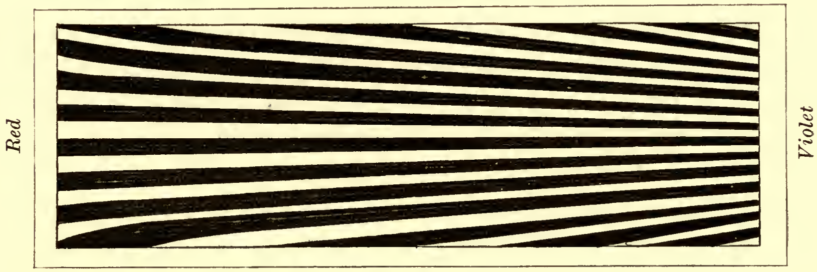


Fig. 11.

these bands on the lines of constant wave-length of the spectrum are equal and proportional to the length of the wave. If the slit intercept the central bright band, the fringes will be symmetrically placed with respect to a bright line.

29. In order to obtain visible interference, it is necessary that the streams emanating from the two sources be of such a character that all the corresponding monochromatic constituents have initially the same difference of phase: otherwise there would be no line of complete accordance of phase and the superposition of the systems of fringes due to the different constituents of the streams would tend to an obliteration of all appearance of interference.

Now though experiment shows that the streams from simple sources of the same nature, such as soda flames, are constant as regards their constituents, there is no reason to assume that the phases of these constituents are invariably related to one another. Were this the case, it would be possible to obtain interference fringes with streams of light from two distinct, though similar sources, which is found to be impossible. In order then to obtain interference fringes, it is essential that the streams should have come initially from a single source and should traverse paths that are optically nearly equivalent.

30. We will now consider four principal methods of obtaining interference fringes that may be classed together as being simple in theory and as having certain distinguishing characteristics.

The first method was devised by Fresnel* in 1816, in order to demon-

* Fresnel, *Œuvres Complètes*, I. 150, 327.

strate the fact of the interference of light by an experiment that was free from the objections brought against an earlier experiment of Young by the opponents of the wave-theory.

In Fresnel's experiment, light from a narrow slit falls upon two plane mirrors, inclined to one another at an angle of very nearly 180°: in this way two streams are obtained that partially overlap, and in their common part the phenomenon of interference is observed.

In order to calculate the position and width of the fringes, let us suppose that a plane through the line of intersection of the mirrors is the plane of yz, the axis of y being parallel to this line, and that the plane is so chosen as nearly to pass through the image of the luminous point in the plane bisecting the acute angle (2ω) between the mirrors.

Let the origin be so chosen that the coordinates of this image are ξ, η, 0 and let the screen on which the interference is observed be the plane $z = a + b$.

If a be the distance of the line of intersection of the mirrors from the origin, the equations of the mirrors may be written

$$\left.\begin{aligned} x \sin(\theta - \omega) + (z - a)\cos(\theta - \omega) = 0 \\ x \sin(\theta + \omega) + (z - a)\cos(\theta + \omega) = 0 \end{aligned}\right\}.$$

The coordinates of the luminous point are

$$x_0 = a \sin 2\theta + \xi \cos 2\theta, \quad y_0 = \eta, \quad z_0 = a + a\cos 2\theta - \xi \sin 2\theta,$$

and those of its image in the first mirror are

$$x_0 - 2\sin(\theta - \omega)\{x_0 \sin(\theta - \omega) + (z_0 - a)\cos(\theta - \omega)\} = a \sin 2\omega + \xi \cos 2\omega,$$

$$y_0,$$

$$z_0 - 2\cos(\theta - \omega)\{x_0 \sin(\theta - \omega) + (z_0 - a)\cos(\theta - \omega)\} = a - a\cos 2\omega + \xi \sin 2\omega.$$

Hence, the propagational speed of the light being taken as unity, the undulatory time of passage from the source to the point $(x, y, a+b)$ is for the stream reflected at the first mirror

$$\begin{aligned} V_1 &= \{(x - a\sin 2\omega - \xi\cos 2\omega)^2 + (y - \eta)^2 + (b + a\cos 2\omega - \xi \sin 2\omega)^2\}^{\frac{1}{2}} \\ &= \{(b + a\cos 2\omega)^2 - 2(x\cos 2\omega + b \sin 2\omega)\,\xi + (x - a\sin 2\omega)^2 + (y - \eta)^2 + \xi^2\}^{\frac{1}{2}} \\ &\doteqdot b + a\cos 2\omega - \frac{x\cos 2\omega + b\sin 2\omega}{b + a\cos 2\omega}\,\xi + \frac{1}{2}\,\frac{(x - a\sin 2\omega)^2 + (y-\eta)^2 + \xi^2}{b + a\cos 2\omega}. \end{aligned}$$

For the stream reflected at the second mirror, the undulatory time of passage V_2 between the same two points is obtained from V_1 by changing the sign of ω: hence the relative retardation of the streams, measured in length, is

$$\Delta = V_2 - V_1 = 2\,\frac{\sin 2\omega\,(b\xi + ax)}{b + a\cos 2\omega};$$

and the points of complete accordance of phase are given by

$$2 \sin 2\omega \frac{b\xi + ax}{b + a \cos 2\omega} = n\lambda,$$

and the linear width of the bands is

$$\Lambda = \lambda \frac{b + a \cos 2\omega}{2a \sin 2\omega}.$$

The phenomenon is in reality not so simple as here represented, as the streams being limited in extent, it is modified by variations of intensity near their edges or in other words by diffraction. When the incidence on the mirrors is nearly normal, the phenomenon is only affected by variations of intensity near the adjacent edges of the streams, the other limits being too remote to have any effect: but in the ordinary arrangement the light falls on the mirrors at nearly grazing incidence, and though the intensity is thereby increased, the streams are so narrow that the disturbance due to diffraction becomes very marked.

Let us now consider the result of a small motion of one of the mirrors parallel to itself. Suppose that the first mirror is moved towards the luminous point through a distance e in the direction of its normal: then assuming for simplicity that the luminous point is so placed that $\xi = 0$, the coordinates of its image in this mirror are

$$a \sin 2\omega + 2e \sin (\theta - \omega), \quad \eta, \quad a - a \cos 2\omega + 2e \cos (\theta - \omega),$$

and the value of V_1 becomes

$$[\{x - a \sin 2\omega - 2e \sin (\theta - \omega)\}^2 + (y - \eta)^2 + \{b + a \cos 2\omega - 2e \cos (\theta - \omega)\}^2]^{\frac{1}{2}}$$
$$\risingdotseq b + a \cos 2\omega - 2e \frac{b \cos (\theta - \omega) + a \cos (\theta + \omega)}{b + a \cos 2\omega} + \tfrac{1}{2} \frac{(x - a \sin 2\omega)^2 + (y - \eta)^2}{b + a \cos 2\omega};$$

to obtain V_2, we have merely to change the sign of ω and write $e = 0$, whence

$$\Delta = V_2 - V_1 = 2 \frac{a \sin 2\omega \,.\, x + e \{a \cos (\theta + \omega) + b \cos (\theta - \omega)\}}{b + a \cos 2\omega}.$$

Hence as the mirror is moved towards the source of light, the interference fringes move across the screen in a direction from the moving towards the fixed mirror and the displacement of the fringes is proportional to the shift of the mirror.

This method was employed by Fizeau and Foucault* for obtaining interference with a large relative retardation between the streams. Adopting the first of the spectroscopic methods of analysing the phenomena that have been described above, they obtained bands when the displacement of the

* *C. R.* XXI. 1155 (1845); *Ann. de Ch. et de Phys.* (3) XXVI. 138 (1849).

mirror was such as to give 141 bands between E and F of the spectrum, corresponding to a relative retardation of 1737 wave-lengths for the ray E.

31. In a second method of obtaining two correlated streams of light, Fresnel employed a biprism*. This instrument is a glass prism with a very large obtuse angle and, as far as regards light incident on its flat surface, may be regarded as made up of two prisms of very small refracting angles joined together by their bases. Hence a stream of light incident on the plane face of the prism is divided into two beams that are slightly bent towards one another, so that they overlap, and in the common part of the two streams the interference fringes are perceived.

Let the plane through the edge of the prism perpendicular to the opposite face be taken as the plane of yz, the edge being parallel to the axis of y; and suppose that the flat face is towards the luminous source which is in the plane of xy and very nearly in the axis of y.

If a be the distance of the edge of the prism from the origin and α_1, α_2 be the acute angles of the prism, the equations of its plane faces are

$$z = a - x \tan \alpha_1 \quad \text{and} \quad z = a + x \tan \alpha_2.$$

Let t be the distance of the flat face of the prism from its edge, $z = a + b$ the equation of the screen of observation, and suppose that the ray from the source $(\xi, \eta, 0)$ to the point (x, y) of the screen meets the faces of the half of the prism on the side of positive x in the points $(x_1, y_1, a - t)$ and (x_2, y_2, z_2) respectively. Then the undulatory time of passage through this half of the prism is

$$\begin{aligned} V_1 &= \sqrt{(x_1 - \xi)^2 + (y_1 - \eta)^2 + (a - t)^2} + \mu \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - a + t)^2} \\ &\quad + \sqrt{(x - x_2)^2 + (y - y_2)^2 + (a + b - z_2)^2} \\ &\fallingdotseq a - t + \mu (z_2 - a + t) + (a + b - z_2) \\ &\quad + \tfrac{1}{2} \left\{ \frac{(x_1 - \xi)^2 + (y_1 - \eta)^2}{a - t} + \mu \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{z_2 - a + t} + \frac{(x - x_2)^2 + (y - y_2)^2}{a + b - z_2} \right\} \\ &\fallingdotseq a + b + (\mu - 1) t - (\mu - 1) x_2 \tan \alpha_1 \\ &\quad + \tfrac{1}{2} \left\{ \frac{(x_1 - \xi)^2 + (y_1 - \eta)^2}{a - t} + \mu \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{t} + \frac{(x - x_2)^2 + (y - y_2)^2}{b} \right\}, \end{aligned}$$

with the conditions

$$\frac{\partial V_1}{\partial x_1} = \frac{x_1 - \xi}{a - t} + \mu \frac{x_1 - x_2}{t} = 0, \quad \frac{\partial V_1}{\partial y_1} = \frac{y_1 - \eta}{a - t} + \mu \frac{y_1 - y_2}{t} = 0,$$

$$\frac{\partial V_1}{\partial x_2} = -(\mu - 1) \tan \alpha_1 - \mu \frac{x_1 - x_2}{t} + \frac{x_2 - x}{b} = 0, \quad \frac{\partial V_1}{\partial y_2} = -\mu \frac{y_1 - y_2}{t} + \frac{y_2 - y}{b} = 0;$$

* Fresnel, *Œuvres*, I. 330.

whence

$$V_1 = a + b + (\mu - 1)t - (\mu - 1)\tan\alpha_1 . x_2$$
$$+ \tfrac{1}{2}\left[\left\{\frac{\mu^2(a-t)}{t^2} + \frac{\mu}{t}\right\}(x_1 - x_2)^2 + b\left\{(\mu - 1)\tan\alpha_1 + \frac{\mu}{t}(x_1 - x_2)\right\}^2\right.$$
$$\left. + \left\{\frac{\mu^2(a-t)}{t^2} + \frac{\mu}{t} + \frac{\mu^2 b}{t^2}\right\}(y_1 - y_2)^2\right]$$
$$= a + b + (\mu - 1)t + \tfrac{1}{2}b(\mu - 1)^2\tan^2\alpha_1 + \frac{\mu}{t}b(\mu - 1)\tan\alpha_1 . (x_1 - x_2)$$
$$- (\mu - 1)\tan\alpha_1 . x_2 + \tfrac{1}{2} . \frac{\mu}{t^2}\{\mu(a+b) - (\mu - 1)t\}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}.$$

But
$$x_2 = \xi - \frac{\xi - x - (\mu - 1)b\tan\alpha_1}{\mu(a+b) - (\mu - 1)t}\{\mu(a-t) + t\},$$
$$x_1 - x_2 = \frac{(\xi - x) - (\mu - 1)b\tan\alpha_1}{\mu(a+b) - (\mu - 1)t} . t,$$
$$y_1 - y_2 = \frac{\eta - y}{\mu(a+b) - (\mu - 1)t} . t,$$
$$\therefore\ V_1 = a + b + (\mu - 1)t + \tfrac{1}{2}b(\mu - 1)^2\tan^2\alpha_1$$
$$+ \mu b(\mu - 1)\tan\alpha_1\frac{\xi - x - (\mu - 1)b\tan\alpha_1}{\mu(a+b) - (\mu - 1)t}$$
$$- (\mu - 1)\tan\alpha_1\left[\xi - \frac{\xi - x - (\mu - 1)b\tan\alpha_1}{\mu(a+b) - (\mu - 1)t}\{\mu a - (\mu - 1)t\}\right]$$
$$+ \frac{\mu}{2}\frac{\{\xi - x - (\mu - 1)b\tan\alpha_1\}^2 + (\eta - y)^2}{\mu(a+b) - (\mu - 1)t},$$

which becomes on reduction

$$V_1 = a + b + (\mu - 1)t - \tfrac{1}{2}\frac{b\{\mu a - (\mu - 1)t\}(\mu - 1)^2\tan^2\alpha_1}{\mu(a+b) - (\mu - 1)t}$$
$$- \frac{\mu\eta y + \mu\xi x + (\mu - 1)\tan\alpha_1[\mu b\xi + \{\mu a - (\mu - 1)t\}x]}{\mu(a+b) - (\mu - 1)t}$$
$$+ \frac{\mu}{2} . \frac{\xi^2 + \eta^2 + x^2 + y^2}{\mu(a+b) - (\mu - 1)t}.$$

The undulatory time of passage V_2 to the same point of the stream that passes through the other half of the prism is obtained from V_1 merely by writing $-\tan\alpha_2$ for $\tan\alpha_1$: hence the relative retardation of the streams, measured in length in air, is

$$\Delta = V_2 - V_1 = \tfrac{1}{2}\frac{b\{\mu a - (\mu - 1)t\}(\mu - 1)^2(\tan^2\alpha_1 - \tan^2\alpha_2)}{\mu(a+b) - (\mu - 1)t}$$
$$+ \frac{(\mu - 1)(\tan\alpha_1 + \tan\alpha_2)[\mu b\xi + \{\mu a - (\mu - 1)t\}x]}{\mu(a+b) - (\mu - 1)t},$$

and the width of the bands is

$$\Lambda = \lambda \frac{\mu(a+b)-(\mu-1)t}{\{\mu a-(\mu-1)t\}(\mu-1)(\tan\alpha_1+\tan\alpha_2)}.$$

As an instrument for the production of interference fringes the biprism is more convenient than Fresnel's mirrors, as the latter are rather difficult to adjust, but as a measuring instrument it has the disadvantage that the phenomenon is complicated not only by diffraction, but also by an almost unavoidable imperfection in its construction. This arises from the fact that in polishing the faces of the prism it appears to be impossible to prevent a slight curvature near its edge, which is the very part through which the interfering portions of the streams pass. The result of this is that the deviation produced by the biprism, as calculated from the measured width of the bands, depends upon its distance from the source of light and is entirely different from that obtained from measurements with a spectrometer.

32. A third method of obtaining interference fringes is by means of Billet's* divided lens. This is a convergent lens of short focal length, divided by a plane through the principal axis into two halves, that can be separated from one another in a direction perpendicular to the plane of section by means of a screw. A second screw serves to adjust the sections to parallelism. The advantage of this instrument consists in the interfering streams being entirely separated during part of their course, so that either can be acted upon independently by the interposition of a retarding plate or otherwise: on the other hand the field is illuminated by the narrow stream that passes through the space between the halves of the lens and this tends to complicate the phenomenon.

In order to determine the relative retardation of the streams at a point of the screen, let us take the plane of xz through the principal axes of the two parts of the lens, the axis of z being parallel to and midway between them, and let the luminous point be in the plane $z=0$ and near the origin. Then an investigation exactly similar to that given in § 31, leads to the result that at the point (xy) of the screen (supposed perpendicular to the axis of z), the relative retardation of the streams that emanate from the point $(\xi, \eta, 0)$ and traverse each one half of the lens is

$$\Delta = 2\epsilon \frac{\left(b-t+\frac{t}{\mu}\frac{\mu-1}{r_1}F\right)\xi+\left(a+\frac{t}{\mu}\frac{\mu-1}{r_2}F\right)x}{a(b-t)-F(a+b-t)-F\frac{t}{\mu}\left\{1-\frac{\mu-1}{r_1}a-\frac{\mu-1}{r_2}(b-t)\right\}},$$

where 2ϵ is the separation of the halves of the lens,

t is the thickness, F the absolute value of the focal length of the lens,

a, b are the distances of the source and the screen from the side of the lens nearest the former,

* *Ann. de Ch. et de Phys.* (3) LXIV. 315 (1862); *Traité d'Optique*, I. 67 (1858).

and r_1, r_2 are the absolute values of the radii of the surfaces nearest the source and the screen respectively.

33. It has been assumed in what precedes that the dimensions of the source are so small that we may regard the light as coming from a luminous point. In practice the source is an elongated slit and it remains to determine under what conditions such an extension of the source is permissible and in what degree the phenomenon is thereby modified*.

Suppose that the slit, or in the case of Fresnel's mirrors its image in the plane bisecting the acute angle between them, is initially in the plane of xy with its centre at the origin and its central line coincident with the axis of y, and that it is then turned (1) about the line bisecting its length through an angle ϕ, and (2) round an axis through its centre normal to its new plane through an angle θ. When this has been done, it is necessary in the above formulæ for the relative retardation to write

$$a - \sin\phi\,(u \sin\theta + v\cos\theta) \quad \text{for } a$$

and

$$u\cos\theta - v\sin\theta \quad \text{for } \xi,$$

where u and v are the distances of a point of the slit from lines bisecting its width and its length respectively.

The intensity at the point (x, y) of the screen due to an element $du\,.\,dv$ at the point (u, v) of the slit will be proportional to

$$\left[1 + \cos\frac{2\pi}{\lambda}\{\alpha + \beta x + (\gamma\cos\theta - \beta'\sin\theta\sin\phi\,.\,x)\,u - (\gamma\sin\theta + \beta'\cos\theta\sin\phi\,.\,x)\,v\}\right] du\,dv\dagger,$$

where the values of α, β, γ, β' are given by the following schedule:

	Mirrors	Biprism	Divided Lens
α	0	$\frac{1}{2}\frac{(\mu-1)^2(\tan^2\alpha_1 - \tan^2\alpha_2)\,ab}{a+b}$	0
β	$\frac{2a\sin 2\omega}{a\cos 2\omega + b}$	$\frac{(\mu-1)(\tan\alpha_1+\tan\alpha_2)\,a}{a+b}$	$2\epsilon\frac{a}{ab - F(a+b)}$
γ	$\frac{2b\sin 2\omega}{a\cos 2\omega + b}$	$\frac{(\mu-1)(\tan\alpha_1+\tan\alpha_2)\,b}{a+b}$	$2\epsilon\frac{b}{ab - F(a+b)}$
β'	$\frac{2b\sin 2\omega}{(a\cos 2\omega + b)^2}$	$\frac{(\mu-1)(\tan\alpha_1+\tan\alpha_2)\,b}{(a+b)^2}$	$-2\epsilon\frac{Fb}{\{ab - F(a+b)\}^2}$

neglecting the thickness in the cases of the biprism and the divided lens.

* Fabry, *Thèse de Doctorat*, Marseille, 1892; *J. de Phys.* (3) I. 313 (1892).

† In the case of the biprism, $-\alpha'\sin\theta\sin\phi$ should be added to the coefficient of u and $\alpha'\cos\theta\sin\phi$ to that of v, where $\alpha' = \frac{1}{2}\frac{(\mu-1)^2(\tan^2\alpha_1 - \tan^2\alpha_2)\,b^2}{(a+b)^2}$: these terms are however very small.

Assuming that each element of the slit acts as an independent source of light—the condition most favourable for brightness*—the intensity due to the whole slit is proportional to

$$\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{k}{2}}^{\frac{k}{2}}\left[1+\cos\frac{2\pi}{\lambda}\{\alpha+\beta x+(\gamma\cos\theta-\beta'\sin\theta\sin\phi\,.\,x)\,u\right.$$
$$\left.-(\gamma\sin\theta+\beta'\cos\theta\sin\phi\,.\,x)\,v\}\right]du\,dv$$
$$=kl\left\{1+V\cos\frac{2\pi}{\lambda}(\alpha+\beta x)\right\},$$

where

$$V=\frac{\sin\frac{\pi}{\lambda}\{(\gamma\cos\theta-\beta'\sin\theta\sin\phi.x)\,k\}}{\frac{\pi}{\lambda}\{(\gamma\cos\theta-\beta'\sin\theta\sin\phi\,.\,x)\,k\}}\cdot\frac{\sin\frac{\pi}{\lambda}\{(\gamma\sin\theta+\beta'\cos\theta\sin\phi.x)\,l\}}{\frac{\pi}{\lambda}\{(\gamma\sin\theta+\beta'\cos\theta\sin\phi\,.\,x)\,l\}},$$

k being the width and l the length of the slit. Hence the intensity fluctuates between $kl\,(1\pm V)$ and according to Michelson's estimate the visibility of the fringes is measured by the absolute value of V.

When there is no tilt of the slit towards the interferential apparatus, $\phi=0$, and if besides $\theta=0$, the visibility is given by the absolute value of $\sin(\pi\gamma k/\lambda)/(\pi\gamma k/\lambda)$ and is independent of the length of the slit. The fringes will then vanish when k is of such a magnitude as to make γk a multiple of λ and the maxima of distinctness will occur when $\tan(\pi\gamma k/\lambda)=\pi\gamma k/\lambda$, the corresponding value of the visibility being the absolute value of $\cos(\pi\gamma k/\lambda)$.

The roots of the equation $\tan(\pi\gamma k/\lambda)=(\pi\gamma k/\lambda)$ may be calculated by the following method due to Lord Rayleigh†: assume

$$\pi\gamma k/\lambda=(m+1/2)\,\pi-y=U-y,$$

where y is a positive quantity that is small when $\pi\gamma k/\lambda$ is large; then substituting this value, we find $\cot y=U-y$, whence

$$y=\frac{1}{U}\left(1+\frac{y}{U}+\frac{y^2}{U^2}+\dots\right)-\frac{y^3}{3}-\frac{2y^5}{15}-\frac{17y^7}{315}-\dots,$$

and solving this equation by successive approximations, it will be found that

$$\frac{\pi\gamma k}{\lambda}=U-y=U-U^{-1}-\frac{2}{3}U^{-3}-\frac{13}{15}U^{-5}-\frac{146}{105}U^{-7}-\dots.$$

It is thus determined that the maxima of distinctness occur when

$$\gamma k/\lambda=0,\quad 1{\cdot}4303,\quad 2{\cdot}4590,\quad 3{\cdot}4709,\quad 4{\cdot}4747,\ \dots,$$

the corresponding values of the visibility being

$$1,\quad {\cdot}217,\quad {\cdot}128,\quad {\cdot}091,\quad {\cdot}079,\ \dots;$$

* Lord Rayleigh, *Phil. Mag.* (5) XXVIII. 81 (1889).

† *Theory of Sound*, Vol. I. § 207 (1894).

when $\gamma k/\lambda = 0$, the intensity is zero, but so long as $\gamma k/\lambda$ is small the distinctness of the fringes will be considerable. Now the linear breadth of the bands (from bright to bright or from dark to dark) being

$$\Lambda = \lambda/\beta,$$

the condition for maximum distinctness is that k must be a small fraction of $\beta\Lambda/\gamma$ or of $a\Lambda/b$ in the case of Fresnel's mirrors and in the cases of the biprism and the divided lens when the thickness is neglected: in other words the angle subtended by the breadth of the slit at the interferential apparatus must be a small fraction of that subtended by the width of the bands at the same point.

As the width of the slit is gradually increased, the distinctness of the fringes will gradually decrease: they then vanish and reappear again in the complementary position, since $\sin(\pi\gamma k/\lambda)/(\pi\gamma k/\lambda)$ changes sign on passing through the value zero; the distinctness then increases up to a maximum, that is about a fifth of the prime maximum of distinctness, and so on.

An interesting method of observing this phenomenon is to allow white light to pass and to subsequently analyse the mixture by a spectroscope with its slit placed at right-angles to the interference fringes. When the source of light is a narrow slit, the ordinary fan-like appearance already described is obtained, the bands being continuous along the whole length of the spectrum. As the source is gradually made wider, the bands become less distinct, the visibility decreasing most rapidly at the violet end, until a region without bands takes its rise at that end and passes along the spectrum to the red end, to be followed by a second such region and so on, the bands on the two sides of the bandless space being complementary.

In the general case in which the slit is tilted towards the interferential apparatus, the visibility depends upon the order of the bands, and when $\theta = 0$ is independent of the length of the slit at the point $x = 0$, its value then being the absolute value of $\sin(\pi\gamma k/\lambda)/(\pi\gamma k/\lambda)$. On moving away from this point the fringes become less and less distinct, vanish when $x = \lambda/(\beta' l \sin\phi)$ and then reappear as a set of fringes complementary to the former and so on. At a given point of the field, the visibility is only independent of the length of the slit if

$$\tan\theta = -\frac{\beta'}{\gamma}x\sin\phi = -\frac{x}{d}\sin\phi,$$

where

$$\begin{aligned} d &= a\cos 2\omega + b \quad \text{for the mirrors,} \\ &= a + b \quad \text{for the biprism,} \\ &= (a+b) - \frac{ab}{F} \quad \text{for the divided lens,} \end{aligned}$$

and the visibility at this point is then the absolute value of

$$\sin(\pi\gamma k\sec\theta/\lambda)/(\pi\gamma k\sec\theta/\lambda).$$

It thus follows that if the slit be inclined with its upper part towards the interferential apparatus, the effect of rotating it in its own plane from y towards x is to move the point of maximum distinctness in the direction of positive x.

34. In 1834 Lloyd* gave a method for obtaining interference fringes, that depends upon the interference of a direct stream of light with a stream from the same source reflected at nearly grazing incidence at a plane mirror. Though this method is not of great practical importance, it deserves mention on account of its theoretical interest.

If ξ be the distance of the source from the plane of the mirror, the relative retardation of the reflected and direct streams at the point of the screen distant x from its line of intersection with the plane of the mirror and on the same side as the source, is $2x\xi/d$, so far as it depends upon the distances traversed, where d is the distance of the source from the screen, which is supposed at right-angles to the plane of the mirror.

Assuming then that no change of phase is introduced at reflection, the position of the bands is given by

$$2x\xi/d = n\lambda/2,$$

where n is an integer, its even values giving the places of the bright bands and its odd values those of the dark bands.

In this case it is clear that at most only one-half of the system of fringes is visible and that only in a plane through the edge of the mirror, as otherwise the plane of symmetry, in which the central band lies, falls outside the region common to the two streams.

If however the phase of the reflected stream be accelerated at reflection by an amount $\mu\pi$, the position of the bands will be given by

$$2x\xi/d = (n + \mu)\,\lambda/2,$$

and while the linear breadth of the bands remains unaltered, the system is shifted away from the mirror by an amount

$$\mu\lambda d/(4\xi) = \mu\Lambda/2,$$

where Λ is the linear breadth of the bands. Lloyd deduced from his experiments that such a shift actually occurs and that it amounts to $\Lambda/2$, whence it follows that $\mu = 1$ or that the acceleration of phase is equal to π.

The effect of an extension of the source in Lloyd's experiment is in some respects essentially different from that determined in the former cases. Suppose that the source is a slit of light, with its plane initially parallel to the screen and its central line parallel to the mirror, and that it is then turned round the line bisecting its length through an angle ϕ and

* *Papers on Phys. Sc.* p. 149; *Trans. R. Ir. Acad.* XVII. 172 (1834).

next round the normal to its new plane through its centre through an angle θ. If u and v be the distances of an element of the slit from the lines bisecting its breadth and its length, we must write

$$d - \sin\phi\,(u \sin\theta + v\cos\theta) \quad \text{for } d,$$

and

$$c + u\cos\theta - v\sin\theta \qquad \text{for } \xi,$$

where c is the distance of the centre of the slit from the mirror, and proceeding as in the former case we find that the intensity at a given point of the screen is proportional to

$$kl\left\{1 + V\cos\frac{2\pi}{\lambda}\left(\frac{2cx}{d}\right)\right\},$$

where

$$V = \frac{\sin\frac{2\pi}{\lambda}\left\{\frac{x}{d}\left(\cos\theta + \frac{c}{d}\sin\theta\sin\phi\right)k\right\}}{\frac{2\pi}{\lambda}\left\{\frac{x}{d}\left(\cos\theta + \frac{c}{d}\sin\theta\sin\phi\right)k\right\}} \cdot \frac{\sin\frac{2\pi}{\lambda}\left\{\frac{x}{d}\left(\sin\theta - \frac{c}{d}\cos\theta\sin\phi\right)l\right\}}{\frac{2\pi}{\lambda}\left\{\frac{x}{d}\left(\sin\theta - \frac{c}{d}\cos\theta\sin\phi\right)l\right\}},$$

k and l being the breadth and the length of the slit.

In order that the visibility may be independent of the length of the slit, it is necessary that

$$\tan\theta = c\sin\phi/d,$$

which holds for any part of the field, and when this is the case

$$V = \sin\frac{2\pi}{\lambda}\left(\frac{x}{d}k\sec\theta\right)\Big/\left\{\frac{2\pi}{\lambda}\left(\frac{x}{d}k\sec\theta\right)\right\},$$

or if n be the order of the bands, so that $x/(d\lambda) = n/(2c)$,

$$V = \sin(\pi nk\sec\theta/c)/(\pi nk\sec\theta/c).$$

The arrangement most favourable for distinctness is when $\phi = 0$, $\theta = 0$.

Thus the case of Lloyd's mirror is characterised by the fact that, even with the most favourable orientation of the slit, the distinctness is dependent upon the order of the bands, the prime maximum of visibility occurring when k is a small fraction of c/n. The effect of a progressive widening of the slit is the same as in the former case.

This dependence of the visibility upon the order of the bands and their periodic disappearance may be easily observed with monochromatic light by leaving the width of the slit unaltered and moving the eyepiece, with which the bands are observed, away from the source, keeping it all the time in the doubly illuminated field.

35. In the cases hitherto considered, when white light is allowed to pass, there is an achromatic band, that is situated at the centre of symmetry of the system, where the interfering streams have traversed equal paths: the

achromatism of this band is complete. There will, however, be an incomplete achromatism, the band being achromatic only in the same sense as a telescope is achromatic, in the case in which there is coincidence of the fringes due to waves corresponding to the most brilliant part of the spectrum; and if in addition the width of the elementary system be a maximum or a minimum for some wave very nearly at the centre of the spectrum or in other words has the same value for two waves of finitely different frequencies, this coincidence of the fringes will occur for several bands, giving rise to an achromatic system*.

The relative retardation of phase of the interfering streams for light of wave-length λ at a point whose coordinate is x may be regarded as a function of x and λ: whence writing $\delta = \phi(x, \lambda)$ and expanding by Taylor's theorem, we have

$$\delta - \delta_0 = \frac{\partial \phi}{\partial x_0} \delta x + \frac{\partial \phi}{\partial \lambda_0} \delta \lambda_0 + \tfrac{1}{2} \frac{\partial^2 \phi}{\partial x_0^2} (\delta x)^2 + \frac{\partial^2 \phi}{\partial x_0 \partial \lambda_0} \delta x \,.\, \delta \lambda + \tfrac{1}{2} \frac{\partial^2 \phi}{\partial \lambda_0^2} (\delta \lambda)^2 + \dots,$$

where $$\delta_0 = \phi(x_0, \lambda_0).$$

Hence the condition for an achromatic fringe at the point x_0 is

$$\partial \phi / \partial \lambda_0 = 0,$$

and further the condition for an achromatic system at this place is

$$\frac{\partial^2 \phi}{\partial x_0 \partial \lambda_0} = 0;$$

when both these conditions are satisfied, δ becomes very approximately a function of x only throughout the region in question.

The following are cases of some importance:—

(1) When the fringes are viewed through a prism with its refracting edge parallel to the bands, each of the separate systems may be regarded as shifted through a space dependent upon the wave-length: then if $2c$ be the distance between the sources, and d be their distance from the screen,

$$\delta = \frac{2\pi}{\lambda} \cdot \frac{2c}{d} \{x + F(\lambda)\},$$

and the condition $\partial \delta / \partial \lambda_0 = 0$ gives

$$\lambda_0 F'(\lambda_0) - \{x + F(\lambda_0)\} = 0,$$

or the position of the achromatic fringe is given by

$$x = -F(\lambda_0) + \lambda_0 F'(\lambda_0).$$

Thus there is an abnormal shift of the central band, which is in addition to

* Lord Rayleigh, *Phil. Mag.* (5) XXVIII. 77, 189 (1889). Cf. also Cornu, *J. de Phys.* (2) I. 293 (1882). Mascart, *ibid.* (2) VIII. 445 (1889), (3) I. 509 (1892); *Phil. Mag.* (5) XXVII. 519 (1889); *C. R.* CVIII. 591 (1889). Macé de Lepinay, *J. de Phys.* (3) III. 241 (1894).

the normal shift introduced by the prism, since $F'(\lambda_0)$ is negative. This was first discovered by Potter* and the explanation was given by Airy†.

(2) When one of the streams passes through a dispersive plate

$$\delta = 2\pi \left\{ F(\lambda) + \frac{2c}{d} \cdot \frac{x}{\lambda} \right\},$$

where $F(\lambda)$ is the retardation in wave-lengths introduced by the plate.

The achromatic fringe is here determined by

$$F'(\lambda_0) - \frac{2c}{d}\frac{x}{\lambda_0^2} = 0 \quad \text{or} \quad x = \frac{d}{2c}\lambda_0^2 F'(\lambda_0).$$

This case is important as illustrating the difficulty of obtaining the refractive index of a plate by a measurement of the shift of the fringes caused by its introduction into the path of one of the interfering streams. With monochromatic light no band has a distinguishing characteristic that can afford a means of determining the number of complete bands that have been displaced through a given point by the interposition of the plate: while with white light, the motion of the centre of symmetry depends upon the dispersion of the plate and cannot be calculated until that is known‡.

(3) When the distance between the sources of the interfering streams is a function of the wave-length

$$\delta = \frac{2\pi}{\lambda} \cdot \frac{F(\lambda)}{d} x:$$

the position of the achromatic band is $x = 0$, and there will be an achromatic system if

$$\lambda_0 F'(\lambda_0) = F(\lambda_0);$$

the achromatism of this system will be complete, if $F(\lambda) \propto \lambda$.

This condition can easily be realised with Lloyd's mirror by the following arrangement suggested by Lord Rayleigh§. A series of real diffraction spectra are formed by white light from a slit, that falls successively on a grating and an achromatic lens: the central white image and all the spectra with the exception of that which is to form the proximate source of light, are intercepted by a screen. Then since the deviation of any colour from the central white image is proportional to λ, the condition for an achromatic system of fringes will be realised by an arrangement of the mirror, such that its plane passes through the position that would be occupied by the central white image.

A less perfect fulfilment of the achromatic condition is obtained by replacing the diffraction spectrum by one formed by a prism, adjusted so that

$$\lambda_0 F'(\lambda_0) = F(\lambda_0)$$

* Potter, *Phil. Mag.* II. 83, 276 (1833).

† Airy, *ibid.* II. 161, 451 (1833). Hamilton, *ibid.* II. 191, 284, 371 (1833).

‡ Stokes, *B. A. Report*, 1850, part 2, 20; *Math. and Phys. Papers*, II. 361.

§ Lord Rayleigh, *Phil. Mag.* (5) XXVIII. 86 (1889).

for the brightest part of the spectrum. Assuming Cauchy's law of dispersion, we may write

$$F(\lambda) = A - B\lambda^{-2},$$

and the condition for an achromatic system gives $3B = A\lambda_0^2$, whence

$$F(\lambda)/\lambda = A(\lambda^2 - \lambda_0^2/3)/\lambda^3.$$

As an illustration of the effect produced by the employment of the prismatic spectrum, let us determine the increase in the number of bands that can be observed, when the light has wave-lengths λ_0 and $\lambda_0 + \delta\lambda$*. When complete discrepance first occurs for wave-lengths λ and λ_0,

$$\frac{F(\lambda)}{\lambda}\frac{x}{d} = n, \quad \frac{F(\lambda_0)}{\lambda_0}\frac{x}{d} = n + \frac{1}{2},$$

$$\therefore \ 1 + \frac{1}{2n} = \frac{F(\lambda_0)/\lambda_0}{F(\lambda)/\lambda} = \frac{2}{3\lambda_0}\frac{\lambda^3}{\lambda^2 - \lambda_0^2/3} = \frac{2}{3}\frac{(\lambda/\lambda_0)^3}{(\lambda/\lambda_0)^2 - 1/3},$$

whence if $\lambda = \lambda_0 + \delta\lambda$,

$$\frac{1}{2n} = \frac{\frac{2}{3}(1+\delta\lambda/\lambda_0)^3}{(1+\delta\lambda/\lambda_0)^2 - \frac{1}{3}} - 1 = \frac{\left(\frac{\delta\lambda}{\lambda_0}\right)^2\left(1 + \frac{2}{3}\frac{\delta\lambda}{\lambda_0}\right)}{\frac{2}{3} + 2\frac{\delta\lambda}{\lambda_0} + \left(\frac{\delta\lambda}{\lambda_0}\right)^2},$$

and

$$\begin{aligned} n &= \frac{1}{2}\left(\frac{\lambda_0}{\delta\lambda}\right)^2\left\{\frac{2}{3} + 2\frac{\delta\lambda}{\lambda_0} + \left(\frac{\delta\lambda}{\lambda_0}\right)^2\right\}\left\{1 - \frac{2}{3}\frac{\delta\lambda}{\lambda_0} + \frac{4}{9}\left(\frac{\delta\lambda}{\lambda_0}\right)^2 - \dots\right\} \\ &= \frac{1}{2}\left(\frac{\lambda_0}{\delta\lambda}\right)^2\left\{\frac{2}{3} + \frac{14}{9}\frac{\delta\lambda}{\lambda_0} - \frac{1}{27}\left(\frac{\delta\lambda}{\lambda_0}\right)^2 + \dots\right\} \\ &= \frac{1}{3}\left(\frac{\lambda_0}{\delta\lambda}\right)^2\left\{1 + \frac{7}{3}\frac{\delta\lambda}{\lambda_0} - \frac{1}{18}\left(\frac{\delta\lambda}{\lambda_0}\right)^2 + \dots\right\}. \end{aligned}$$

This gives the order of the band at which complete discrepance first occurs for waves of length λ_0 and $\lambda_0 + \delta\lambda$, the adjustment being made for λ_0. When no prism is used, so that $F(\lambda)$ is constant, the corresponding value of n is $\lambda_0/(2\delta\lambda)$, so that the effect of the prism is to increase the number of bands in the ratio $2\lambda_0 : 3\delta\lambda$.

(4) A fourth case is that in which not only the separation of the sources but also their distance from the screen of observation varies with the wave-length of the light: in this case

$$\delta = \frac{2\pi}{\lambda}\frac{F(\lambda)}{\phi(\lambda)}x,$$

and the condition for an achromatic system is that

$$\frac{\partial}{\partial\lambda_0}\left\{\frac{F(\lambda_0)}{\phi(\lambda_0)}\right\} = 0.$$

This case may be realised with Billet's divided lens†; for since the focal

* Lord Rayleigh, *loc. cit.*

† Macé de Lepinay and Perot, *J. de Phys.* (2) IX. 376 (1890).

length depends upon the wave-length, the various coloured images that form the proximate sources of light are at different distances from the screen, and they are also at different distances apart, as they are situated on lines through the source and the optical centres of the two parts of the lens.

To determine the position of the achromatic system, we have (neglecting the thickness of the lens)

$$\delta = \frac{2\pi}{\lambda} \frac{2\epsilon a x}{ab - F(a+b)},$$

a and b being the distances of the lens from the source and the screen. Hence the distance of the screen at which the achromatic system is formed is given by

$$ab = (a+b)\frac{\partial(\lambda_0 F)}{\partial \lambda_0},$$

whence

$$b = \frac{a\,\partial(\lambda_0 F)/\partial\lambda_0}{a - \partial(\lambda_0 F)/\partial\lambda_0}.$$

Since this distance is independent of the separation (2ϵ) of the halves of lens, it is always possible to adjust the separation so that the position just determined falls within the region common to the interfering streams.

CHAPTER IV.

INTERFERENCE PRODUCED BY ISOTROPIC PLATES.

36. In the cases of interference considered in the last chapter, it is necessary that the dimensions of the source be strictly limited, and the phenomena are characterised by the fact that the fringes are visible throughout the region common to the interfering streams, whatever may be the distance of the screen or of the observing instrument from the interferential apparatus.

There are however cases of interference in which the limitation of the source is unnecessary and the fringes are then localised, requiring a definite focal adjustment of the instrument with which they are observed, if they are to be seen distinctly.

This distinction between the two classes of interference phenomena must not be insisted on too strongly; for in the case of the former class it is possible *theoretically* to obtain localised fringes with an extended source, while in the cases now to be considered interference bands, visible at all distances within the region common to the streams, can be obtained, provided the stream of light be limited by a properly orientated slit placed either before or after the interferential apparatus.

37. Suppose that light from a luminous point $S(x', y', z')$ is divided into two streams and that these, after traversing different routes, meet again at a

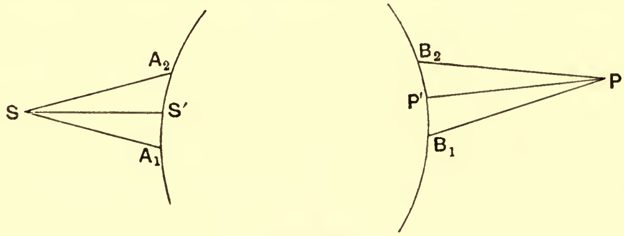


Fig. 12.

point $P(x, y, z)$: then if the suffixes (1) and (2) refer to the two streams, their relative retardation, measured in length in air, is at the point P

$$\Delta = V_2 - V_1 \qquad \text{.................................(1)},$$

where V denotes the undulatory time of passage between the two points, the propagational speed of light in air being taken as unity.

But if the initial and the final media be air and α, β, γ and α', β', γ' be the direction-cosines of the final and initial straight portion of a ray, the principle of least time gives for the variation of the undulatory time of passage

$$\delta V = \alpha\delta x + \beta\delta y + \gamma\delta z - \alpha'\delta x' - \beta'\delta y' - \gamma'\delta z' \quad \text{..............(2)},$$

whence the variation of the relative retardation is

$$\begin{aligned}\delta\Delta = {} & (\alpha_2 - \alpha_1)\,\delta x + (\beta_2 - \beta_1)\,\delta y + (\gamma_2 - \gamma_1)\,\delta z \\ & - (\alpha_2' - \alpha_1')\,\delta x' - (\beta_2' - \beta_1')\,\delta y' - (\gamma_2' - \gamma_1')\,\delta z' \quad \text{...........(3)},\end{aligned}$$

which is zero if

$$\begin{aligned}\delta x : \delta y : \delta z &:: \alpha_2 + \alpha_1 : \beta_2 + \beta_1 : \gamma_2 + \gamma_1, \\ \delta x' : \delta y' : \delta z' &:: \alpha_2' + \alpha_1' : \beta_2' + \beta_1' : \gamma_2' + \gamma_1'.\end{aligned}$$

If then, as is generally the case, the two waves issuing from S have very nearly the same form and position at P and their radii of curvature are large compared with their relative retardation, it follows that this relative retardation will remain unaltered when the initial and final points are displaced along the bisectors SS' and PP' of the angles between the initial and the final directions of the two rays that start from the one point and cross at the other point.

Let a_1', b_1', p_1', q_1' be the parameters of the initial straight part SA_1 and a_1, b_1, p_1, q_1 those of the final part B_1P of one of the rays between S and P and let similar quantities with the suffix (2) denote the parameters of the initial and the final parts SA_2, B_2P of the second ray; and suppose S displaced to S' on the bisector SS'.

The two rays from S' that meet at P are for the first part of their course very near to SA_1 and SA_2 respectively and have for their bisector SS'. If then $a_1' + \delta a_1'$, $b_1' + \delta b_1'$, $p_1' + \delta p_1'$, $q_1' + \delta q_1'$ be the parameters for the one and $a_2' + \delta a_2'$, $b_2' + \delta b_2'$, $p_2' + \delta p_2'$, $q_2' + \delta q_2'$ be those of the other,

$$\delta a_1' + \delta a_2' = \delta b_1' + \delta b_2' = \delta p_1' + \delta p_2' = \delta q_1' + \delta q_2' = 0.$$

The new parameters for the final portions of the rays are

$$\begin{aligned}& a_1 + \delta a_1, \quad b_1 + \delta b_1, \quad p_1 + \delta p_1, \quad q_1 + \delta q_1, \\ & a_2 + \delta a_2, \quad b_2 + \delta b_2, \quad p_2 + \delta p_2, \quad q_2 + \delta q_2,\end{aligned}$$

where δa_1, δa_2, ... are of the form

$$\begin{aligned}\delta a_1 &= A_1\delta a_1' + B_1\delta b_1' + P_1\delta p_1' + Q_1\delta q_1', \\ \delta a_2 &= A_2\delta a_2' + B_2\delta b_2' + P_2\delta p_2' + Q_2\delta q_2',\end{aligned}$$

A_1, A_2, ... depending upon the interferential apparatus and being given when that is known. But the relations connecting the parameters of B_2P to

those of SA_2 are only slightly different from those connecting the parameters of B_1P and SA_1; we can therefore write

$$\begin{aligned}\delta a_2 &= \quad A_1\delta a_2' + B_1\delta b_2' + P_1\delta p_2' + Q_1\delta q_2' \\ &= -A_1\delta a_1' - B_1\delta b_1' - P_1\delta p_1' - Q_1\delta q_1' = -\delta a_1;\end{aligned}$$

similarly $$\delta b_1 + \delta b_2 = \delta p_1 + \delta p_2 = \delta q_1 + \delta q_2 = 0,$$

which express that the rays at P that emanate from S' have the same bisector as those at the same point that start from S.

It follows that the relative retardation is completely determined if the position of PP' be given, without its being necessary to define the position of P on PP' or of S on SS'.

If $x = az + p$, $y = bz + q$ be the equations of PP', then

$$\Delta = f(a, b, p, q) \quad\quad (4).$$

Now to different points of the source correspond different directions of the line PP', and the condition for the distinctness of the fringes at P is that Δ must be stationary for all points of the source that contribute to the illumination of this point: if this condition be satisfied for the point P, it will be sensibly so for the neighbouring points.

Suppose that the fringes are observed with an optical instrument, the focal adjustment of which can be altered while its optic axis remains fixed in space, and let us take the axis of z along the optic axis, the origin being some point in the final medium distant D from that on which the instrument is focussed. Then the values of the parameters corresponding to the optic axis are

$$a = b = p = q = 0,$$

and these parameters will be small for all neighbouring directions.

Let $$\frac{x-\xi}{x_1-\xi} = \frac{y-\eta}{y_1-\eta} = \frac{z-\zeta}{D-\zeta} \quad\quad (5)$$

be the equations of a line near the optic axis, then the values of the parameters are

$$a = \frac{x_1-\xi}{D-\zeta}, \quad b = \frac{y_1-\eta}{D-\zeta}, \quad p = \frac{D\xi - x_1\zeta}{D-\zeta}, \quad q = \frac{D\eta - y_1\zeta}{D-\zeta} \quad\quad (6),$$

and the equation

$$\delta\Delta = \left(\frac{\partial f}{\partial a}\right)_0 a + \left(\frac{\partial f}{\partial b}\right)_0 b + \left(\frac{\partial f}{\partial p}\right)_0 p + \left(\frac{\partial f}{\partial q}\right)_0 q$$

gives

$$\delta\Delta = \left(\frac{\partial f}{\partial a}\right)_0 \frac{x_1-\xi}{D-\zeta} + \left(\frac{\partial f}{\partial b}\right)_0 \frac{y_1-\eta}{D-\zeta} + \left(\frac{\partial f}{\partial p}\right)_0 \frac{D\xi - x_1\zeta}{D-\zeta} + \left(\frac{\partial f}{\partial q}\right)_0 \frac{D\eta - y_1\zeta}{D-\zeta} \quad \ldots (7),$$

the suffix (0) denoting that in the partial differential coefficients the parameters are replaced by their common value zero. These coefficients are therefore constant, and $\delta\Delta$ may be written in the form

$$\delta\Delta = \frac{1}{D-\zeta}\{(A-P\zeta)x_1 + (B-Q\zeta)y_1 - (A-PD)\xi - (B-QD)\eta\} \quad \ldots(8).$$

At the point on which the optical instrument is focussed, $x_1 = y_1 = 0$, and hence the condition of visibility is

$$(A-PD)\xi + (B-QD)\eta = 0 \quad \ldots\ldots(9),$$

and if this condition be satisfied, the orientation of the fringe at this point is given by

$$y_1/x_1 = -(A-P\zeta)/(B-Q\zeta) \quad \ldots\ldots(10).$$

Now ξ and η being independent variables, the condition of visibility cannot in general be satisfied, unless a linear relation is established between them by limiting to one plane the final directions of the rays through P, as may be done by the introduction of a slit either before or after the interferential apparatus.

Suppose the slit introduced between the apparatus and the observing instrument in the plane $z = \zeta$ and let ϕ be the angle that the final plane of the rays through P makes with the plane of xz, then $\eta/\xi = \tan\phi$, where

$$\tan\phi = -(A-PD)/(B-QD) \quad \ldots\ldots(11):$$

thus the orientation of the slit depends upon the focal adjustment of the observing instrument, but is independent of the plane of the slit; on the other hand the orientation of the fringes given by (10) is independent of the focal adjustment of the instrument but depends upon the distance of the slit from the point observed.

If however the interferential apparatus be such that

$$A/B = P/Q = m \text{ say} \quad \ldots\ldots(12),$$

the condition for visibility becomes

$$(A-PD)(m\xi + \eta) = 0 \quad \ldots\ldots(13),$$

and without any limitation of the stream, the interference is visible, localised at the point given by

$$D = A/P \quad \ldots\ldots(14),$$

while by limiting the stream in such a way, that the final directions of the rays intersecting on the optic axis of the observing instrument lie in a plane making an angle $\tan^{-1}(-m)$ with that of xz, the localisation is destroyed and the interference becomes visible at all distances.

In this case $y_1/x_1 = -m$ and the bands are parallel to the plane, to which the final direction of the rays must be limited, in order that the localisation

of the fringes may disappear: however, if the slit be in the plane of localisation, y_1/x_1 is indeterminate and no bands are visible*.

38. Having obtained these general propositions respecting the visibility of interference fringes, we may now proceed to the consideration of the phenomena of interference produced by isotropic plates. In the first place it is necessary to calculate the intensities of the reflected and the transmitted light, when a train of plane waves of monochromatic light (λ) falls upon a parallel plate of index μ. The resultant reflected train is then made up of an infinite number of components, of which the first is reflected at the outer surface of the plate, while each of the remainder has been reflected an odd number of times within it. Similarly the first component of the resultant transmitted train passes through the plate without reflection and each of the remaining components passes out after an even number of internal reflections.

So far as it depends upon the distances travelled in the plate and in the surrounding medium, which we shall suppose to be air, the relative retardation of two successive components, measured in actual length in air, is

$$\Delta = 2\mu d \sec r - 2d \tan r \,.\, \sin i = 2\mu d \cos r \quad \text{...........(15)},$$

where d denotes the thickness of the plate, i is the angle of incidence, and r is the corresponding angle of refraction. Representing the polarisation-vectors by complex quantities, this retardation is expressed by the introduction of a factor $\exp(-\iota\delta)$ where $\delta = 2\pi\Delta/\lambda = \kappa\Delta$ is the relative retardation of phase.

At each reflection and refraction, the polarisation-vector is altered by a certain factor: this shall be supposed to be b for reflection and c for refraction in the case of progress to the plate from the surrounding medium, and to be e for reflection and f for refraction when the light proceeds to the surrounding medium from the plate. Further we may suppose that at these reflections and refractions there occur corresponding accelerations of phase, represented by β, γ, η, ϕ respectively: these will be expressed by the factors $\exp(\iota\beta)$, $\exp(\iota\gamma)$ ... .

Now between the factors of reflection and refraction and between the corresponding changes of phase there exist certain relations, that Stokes has determined from an application of the principle of reversion†.

Let O be a point on the interface of two transparent, homogeneous and isotropic substances and let IO be the direction of propagation of a wave in the first medium incident on the surface, OF, OR the directions of the normals of the reflected and refracted waves and OR' the normal of a

* Macé de Lepinay and Fabry, *J. de Phys.* (2) x. 5 (1891). Fabry, *Thèse de Doctorat*, Marseille, 1892; *J. de Phys.* (3) I. 313 (1892).

† *Camb. and Dub. Math. J.* IV. 1 (1849); *Math. and Phys. Papers*, II. 89.

reflected wave corresponding to an incident wave propagated in the direction RO and hence also that of a refracted wave due to a wave propagated in the direction FO. As we are dealing with one of the plane polarised components of a stream of common light, we may assume that the polarisation-vector of the incident waves is perpendicular to the plane of incidence and then by symmetry the vectors of the reflected and the refracted waves will be in the same direction.

Let z be measured from O negatively backwards along OI and positively forwards along OF, OR and OR', and let it denote the equivalent length of path in vacuum: then writing for shortness

$$2\pi(\omega t - z)/\lambda = T,$$

the polarisation-vectors for the incident, reflected and refracted waves may be represented by

$$\epsilon^{T\iota}, \quad b\epsilon^{(T+\beta)\iota}, \quad c\epsilon^{(T+\gamma)\iota},$$

and it follows, from the principle of reversion, that the reflected and the refracted waves reversed must produce simply the incident wave reversed.

Now in order to represent this reversion, it is sufficient to change the signs of t and of z, or which is the same, those of β and γ. The reversed reflected wave then gives rise to waves with polarisation-vectors

$$b^2\epsilon^{(T-\beta+\beta)\iota} \text{ and } bc\epsilon^{(T-\beta+\gamma)\iota}$$

propagated respectively along OI and OR', and the reversed refracted wave gives rise to waves propagated in the same two directions, for which the polarisation-vectors are respectively

$$cf\epsilon^{(T-\gamma+\phi)\iota} \text{ and } ce\epsilon^{(T-\gamma+\eta)\iota}.$$

Hence we must have

$$b^2 + cf\epsilon^{(\phi-\gamma)\iota} = 1, \text{ and } bc\epsilon^{(\gamma-\beta)\iota} + ce\epsilon^{(\eta-\gamma)\iota} = 0,$$

or

$$b\epsilon^{(2\gamma-\beta-\eta)\iota} + e = 0.$$

Whence equating real and imaginary parts, we obtain

$$\phi = \gamma, \quad cf = 1 - b^2, \quad \beta + \eta = 2\gamma, \quad e = -b \quad \text{..............(16).}$$

Returning now to the light reflected from or transmitted by the parallel plate, let us suppose for the sake of obtaining a result that will be of use to us later, that the plate is slightly opaque, and let the polarisation-vector be reduced in the proportion of 1 to $1 - qdx$ in traversing a distance dx within the plate: then writing for shortness $\exp(-qd\sec r) = g$, 1 to g will be the proportion in which the vector is reduced by the defect of transparency in a single transit.

Measuring now z positively forward along the directions of propagation of the reflected and the transmitted streams, and denoting the maximum

value of the polarisation-vector of the incident train by unity, the symbolical representation of the reflected train is

$$\left\{b\epsilon^{\beta\iota} + cefg^2\epsilon^{(\gamma+\eta+\phi-\delta)\iota}\sum_0^\infty e^{2n}g^{2n}\epsilon^{n(2\eta-\delta)\iota}\right\}\epsilon^{\kappa(\omega t-z)\iota}$$

$$=\left\{b\epsilon^{\beta\iota} + \frac{cefg^2\epsilon^{(\gamma+\eta+\phi-\delta)\iota}}{1-e^2g^2\epsilon^{(2\eta-\delta)\iota}}\right\}\epsilon^{\kappa(\omega t-z)\iota}$$

$$= b\,\frac{1-g^2\epsilon^{(2\eta-\delta)\iota}}{1-b^2g^2\epsilon^{(2\eta-\delta)\iota}}\,.\,\epsilon^{\{\kappa(\omega t-z)+\beta\}\iota} \qquad \text{.......................(17)}$$

if we may suppose that the slight opacity of the plate does not invalidate the relations (16).

In like manner the expression for the transmitted stream is

$$\frac{(1-b^2)g}{1-b^2g^2\epsilon^{(2\eta-\delta)\iota}}\,.\,\epsilon^{\{\kappa(\omega t-z)+\gamma+\phi\}\iota} \qquad \text{.....................(18).}$$

Hence the intensity of the reflected light is

$$b^2\,\frac{(1-g^2)^2+4g^2\sin^2(\delta/2-\eta)}{(1-b^2g^2)^2+4b^2g^2\sin^2(\delta/2-\eta)} \qquad \text{.....................(19),}$$

and that of the transmitted light is

$$\frac{(1-b^2)^2g^2}{(1-b^2g^2)^2+4b^2g^2\sin^2(\delta/2-\eta)} \qquad \text{.....................(20).}$$

The corresponding intensities in the case of a perfectly transparent plate, obtained from the above expressions by writing $g=1$, are

$$\frac{4b^2\sin^2(\delta/2-\eta)}{(1-b^2)^2+4b^2\sin^2(\delta/2-\eta)} \quad \text{and} \quad \frac{(1-b^2)^2}{(1-b^2)^2+4b^2\sin^2(\delta/2-\eta)} \text{...(21).}$$

39. It has been assumed in the above investigation that the reflection and refraction takes place at a definite surface, up to which the media on the two sides retain their homogeneity without any change. That such a state of things really exists is in itself extremely improbable, and indeed the observed phenomena of the reflection and refraction of polarised light appear to indicate that the passage from one homogeneous medium to another is through a very thin transition-layer, within which a rapid variation of properties occurs: if the thickness of this layer be comparable with the wave-length of light, we shall see that a change of phase at reflection and refraction will result. So long as the distances with which we are concerned exceed a few wave-lengths, no great error will probably be introduced by ignoring the transition-layer, but that our results cannot be applied to the case of extremely thin plates is shown at once by the fact that the expression for the intensity of the reflected light does not vanish with the thickness, as it should of course do.

It has also been supposed that the disturbance within the plate is fully represented by waves with transversal polarisation-vectors. If the existence

of a transition-layer be denied, the changes of phase must be attributed to undulations with a longitudinal vector, that are called into existence at reflection and refraction and would be themselves capable of producing transversal waves on encountering a reflecting surface. These longitudinal waves must be of the nature of superficial undulations becoming insensible at a very short distance from the surface, and they may therefore be left out of our calculations, so long as the plate that we are considering is not very thin.

Now the case of thin plates is the very one to which we want to apply our calculations, and for such plates, as we have just seen, our formulæ no longer hold. We may however obtain a result that agrees very well with observed facts, if we neglect any changes of phase at the reflections and refractions, as well as the transition-layers, or the superficial undulations, to which they appear to be due. Writing then $\eta = 0$, we have as the intensities of the reflected and the transmitted light

$$b^2 \frac{(1-g^2)^2 + 4g^2 \sin^2(\delta/2)}{(1-b^2g^2)^2 + 4b^2g^2 \sin^2(\delta/2)} \quad \text{and} \quad \frac{(1-b^2)^2 g^2}{(1-b^2g^2)^2 + 4b^2g^2 \sin^2(\delta/2)} \quad \ldots(22)$$

in the case of a semi-transparent plate, and

$$\frac{4b^2 \sin^2(\delta/2)}{(1-b^2)^2 + 4b^2 \sin^2(\delta/2)} \quad \text{and} \quad \frac{(1-b^2)^2}{(1-b^2)^2 + 4b^2 \sin^2(\delta/2)} \quad \ldots\ldots(23),$$

when the transparency is perfect.

It follows then that the reflected light becomes a minimum and in the case of perfect transparency vanishes, when $\delta = 2n\pi$, or when

$$2\mu d \cos r = n\lambda \quad \ldots\ldots(24),$$

n being an integer.

40. Suppose now that the light is not strictly monochromatic, but is made up of a number of constituents with periods only slightly different from one another. If the thickness of the plate be very great compared with the wave-length, then δ will vary enormously for a very small change in λ, and $\sin(\delta/2)$ will assume all values from -1 to $+1$. This being the case, the intensity of the reflected light, that of the incident light being taken as unity, may be represented by

$$\frac{1}{\pi}\int_0^\pi b^2 \frac{(1-g^2)^2 + 4g^2 \sin^2\zeta}{(1-b^2g^2)^2 + 4b^2g^2 \sin^2\zeta}\, d\zeta$$
$$= 1 - \frac{(1-b^2g^2)^2 - b^2(1-g^2)^2}{\pi} \int_0^\pi \frac{d\zeta}{(1-b^2g^2)^2 \cos^2\zeta + (1+b^2g^2)^2 \sin^2\zeta},$$

provided we may assume that the intensity of the constituent streams varies but slightly with the wave-length. Similarly the intensity of the transmitted light is

$$\frac{(1-b^2)^2 g^2}{\pi} \int_0^\pi \frac{d\zeta}{(1-b^2g^2)^2 \cos^2\zeta + (1+b^2g^2)^2 \sin^2\zeta}.$$

Now writing $\tan \xi = (b/a) \tan \zeta$, we see at once that

$$\int \frac{d\zeta}{a^2 \cos^2 \zeta + b^2 \sin^2 \zeta} = \frac{\xi}{ab}.$$

Hence the intensities in the two cases are

$$1 - \frac{(1-b^2g^2)^2 - b^2(1-g^2)^2}{1-b^4g^4} = b^2 + \frac{(1-b^2)^2 b^2 g^4}{1-b^4g^4}$$

and $$\frac{(1-b^2)^2 g^2}{1-b^4g^4},$$

the result that would be obtained by summing the *intensities* of the different components into which the incident stream is divided by the reflections and the refractions*.

41. We have now to justify the application of our formulæ to the cases that actually occur, in which the faces of the film are not necessarily parallel and in which the light incident upon it consists not of a train of plane waves but of a number of distinct streams coming from the various points of an extended source placed at a finite distance from the plate.

Suppose that the film of index μ is included between two media of index μ', of which the upper one is a thick parallel plate, while the lower boundary of the film is either a spherical surface of very large radius or a plane not necessarily parallel to the faces of the plate.

Let S be a point of the source and let us determine the relative retardation at some point P of the streams, that emanate from S and have been reflected at the outer and the inner surfaces of the film respectively.

In the case of the stream externally reflected, the ray through P lies entirely in one plane and is projected on the upper surface of the film in the straight line SA_1AA_2P, the points A_1, A, A_2 denoting the places at which a change of direction occurs: on the other hand the ray internally reflected lies

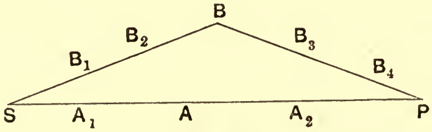


Fig. 13.

in general in two planes and its projection on the upper surface of the film is the broken line $SB_1B_2BB_3B_4P$, refraction or reflection taking place at the points indicated by B_1, B_2, ...

Let i and r' be the angles of incidence and refraction at A_1, then r' is the angle of incidence at A and r' and i are the angles of incidence and emergence at A_2.

* Kirchhoff, *Vorl. über Math. Optik.* p. 164.

Let $\left.\begin{array}{l} i_1,\ r_1' \\ r_1',\ r_2 \\ r_3,\ r_3' \\ r_3',\ i_4 \end{array}\right\}$ be the angles of incidence and refraction at $\left\{\begin{array}{l} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array}\right.$

and θ, θ_1, θ_2 the angles that SAP, SB, BP respectively make with some fixed straight line on the upper surface of the film.

Denote the heights of S and P above the surface of the thick plate by c and C, the thickness of the plate by h and that of the film measured normally to the faces of the plate by t. Then

$$\Delta = c(\sec i_1 - \sec i) + C(\sec i_4 - \sec i) + \mu' h(\sec r_1' + \sec r_3' - 2\sec r') \\ + \mu t(\sec r_2 + \sec r_3) \quad \text{..................(25)},$$

with the conditions

$$0 = c(\tan i \cos\theta - \tan i_1 \cos\theta_1) + C(\tan i \cos\theta - \tan i_4 \cos\theta_2) \\ + h(2\tan r' \cos\theta - \tan r_1' \cos\theta_1 - \tan r_3' \cos\theta_2) \\ - t(\tan r_2 \cos\theta_1 + \tan r_3 \cos\theta_2) \quad \text{..............(26)},$$

$$0 = c(\tan i \sin\theta - \tan i_1 \sin\theta_1) + C(\tan i \sin\theta - \tan i_4 \sin\theta_2) \\ + h(2\tan r' \sin\theta - \tan r_1' \sin\theta_1 - \tan r_3' \sin\theta_2) \\ - t(\tan r_2 \sin\theta_1 + \tan r_3 \sin\theta_2) \quad \text{..............(27)}.$$

Multiplying the last two equations by $\sin i \cos\theta$ and $\sin i \sin\theta$ respectively and adding them to the former, we obtain

$$\Delta = c\left(\frac{1 - \sin i \sin i_1 \cos\theta\cos\theta_1 - \sin i \sin i_1 \sin\theta \sin\theta_1}{\cos i_1} - \cos i\right) \\ + C\left(\frac{1 - \sin i \sin i_4 \cos\theta\cos\theta_2 - \sin i \sin i_4 \sin\theta\sin\theta_2}{\cos i_4} - \cos i\right) \\ + \mu' h\left(\frac{1 - \sin r' \sin r_1' \cos\theta\cos\theta_1 - \sin r' \sin r_1' \sin\theta\sin\theta_1}{\cos r_1'}\right. \\ \left. + \frac{1 - \sin r' \sin r_3' \cos\theta\cos\theta_2 - \sin r' \sin r_3' \sin\theta\sin\theta_2}{\cos r_3'} - 2\cos r'\right) \\ + \mu t\left(\frac{1 - \sin r \sin r_2 \cos\theta\cos\theta_1 - \sin r \sin r_2 \sin\theta\sin\theta_1}{\cos r_2}\right. \\ \left. + \frac{1 - \sin r \sin r_3 \cos\theta\cos\theta_2 - \sin r \sin r_3 \sin\theta\sin\theta_2}{\cos r_3}\right) \quad \text{........(28)},$$

where r is the angle of entry into the film corresponding to an angle of incidence i on the first surface of the plate, so that $\sin r = \sin i/\mu$.

Hence

$$\Delta = c\,\frac{1 - \cos\epsilon_1}{\cos i_1} + C\,\frac{1 - \cos\epsilon_4}{\cos i_4} + \mu' h\left(\frac{1 - \cos\epsilon_1'}{\cos r_1'} + \frac{1 - \cos\epsilon_3'}{\cos r_3'}\right) \\ + \mu t\left(2\cos r + \frac{1 - \cos\epsilon_2}{\cos r_2} + \frac{1 - \cos\epsilon_3}{\cos r_3}\right) \quad \text{..................(29)},$$

where ϵ denotes the angle between corresponding parts of the rays, so that for instance

$$\cos \epsilon_2 = \cos r \cos r_2 + \sin r \sin r_2 \cos (\theta_1 - \theta).$$

When the angles ϵ are very small, which will be the case when t is very small, we have neglecting ϵ^2

$$\Delta = 2\mu t \cos r.$$

To the same degree of approximation, the stream that is reflected $(2p-1)$ times within the film is retarded relatively to that reflected at its upper surface by an amount

$$\Delta_p = 2\mu (t_1 + t_2 + \ldots + t_p) \cos r,$$

where t_1, t_2 ... denote the distances below the upper surface of the points at which the reflections at the lower side of the film occur. Since however the importance of the successive components decreases very rapidly as their order becomes greater, we may, provided the thickness varies only very slowly, write the above expression for the retardation as

$$\Delta_p = p \,.\, 2\mu t \cos r,$$

where t is the thickness at the point of reflection of the externally reflected stream, and in that case the intensity of the reflected light is given by the expression already obtained for the case of a parallel plate. If however the incidence be very oblique and the variation of the thickness be not very small, there may be a considerable departure from the theoretical simplicity assumed in the above investigation*.

42. If now we pass to another point of the source, we obtain for the intensity at P an expression of the same form, in which r and t have new values, and since there is no regular interference between streams that start from different points of a source, the resulting intensity is the sum of all such expressions for those points of the source that contribute to the illumination of P.

In general then there will be no visible interference at P, unless at this point Δ has the same value for all the points that send light to P, or which is the same, for all points of the upper surface of the film, that are included in the area traced upon it by the rays through P, that meet the object glass of the optical instrument, with which the interference is observed.

The condition of visibility then is

$$\begin{aligned} d\Delta &= 2\mu \cos r dt - 2\mu t \sin r dr \\ &= 2\mu \cos r dt - 2t \tan r \cos i di = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (30), \end{aligned}$$

for all points of the film utilised.

43. As a first application of these considerations, let us take the case in which the film is a parallel plate. Then $dt = 0$ and the condition of visibility

* Macé de Lepinay, *J. de Phys.* (2) IX. 121 (1890).

is that $di = 0$, which expresses the fact that the interference is localised at infinity.

The bands are arcs of circles and have this peculiarity that the order of the band decreases as the angle of incidence increases; for at normal incidence

$$\Delta_0 = 2\mu t = n_0\lambda,$$

where n_0 is not necessarily integral, and at an angle of entry r

$$\Delta = 2\mu t \cos r = n\lambda,$$

whence $$\Delta_0 - \Delta = 2\mu t (1 - \cos r) = 4\mu t \sin^2 \frac{r}{2} = (n_0 - n)\lambda.$$

To determine the angular width of a band, corresponding to a change of n into $n - 1$, we have

$$2\mu t \sin r dr = \lambda,$$

whence $$di = \frac{\mu\lambda}{t} \cdot \frac{\cos r}{\sin 2i}.$$

Thus the bands are very broad at normal and at grazing incidence, and their minimum separation, corresponding to the minimum value of $\cos r \,.\, \operatorname{cosec} 2i$, occurs when

$$\tan^4 i = \frac{\mu^2}{\mu^2 - 1}.$$

These bands were first observed by Haidinger with plates of mica*.

44. An interesting case of these bands occurs when the plate is less dense than the surrounding medium and the angle of incidence is very nearly that corresponding to total reflection: they are then known as Herschel's bands†. This case may be realised by employing a parallel plate of air bounded on one side by the face of a prism.

Denoting by r and i the angles of incidence and refraction at the plate of

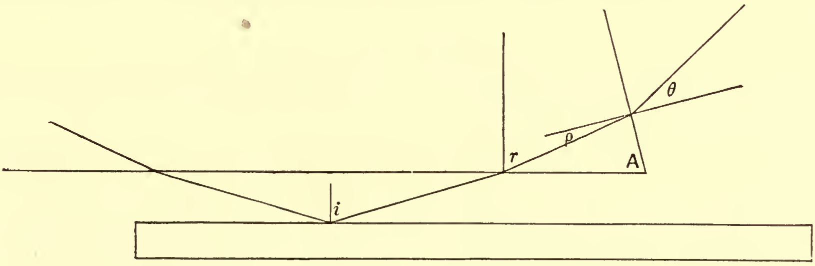


Fig. 14.

* Haidinger, *Pogg. Ann.* LXXVII. 219 (1849); *Wien. Ber.* XIV. 295 (1854). Mascart, *Ann. de Ch. et de Phys.* (4) XXIII. 126 (1871). Lummer, *Wied. Ann.* XXIII. 49 (1884).

† Herschel, *Phil. Trans.* XCIX. 274 (1809). Mascart, *C. R.* CVIII. 596 (1889); *Phil. Mag.* (5) XXVII. 524 (1889); *J. de Phys.* (2) VIII. 445 (1889): (3) I. 509 (1892). Lord Rayleigh, *Phil. Mag.* (5) XXVIII. 197 (1889).

air, and by ρ and θ the angles of incidence and emergence at the surface of the prism, the angle of which is A, let r_0, $\pi/2$, ρ_0, θ_0 be the values of these angles, when the light meets the plate at the critical angle. Then

$$\Delta = 2t \cos i = n\lambda \quad \text{.........................(31)},$$

$$\left.\begin{aligned} \sin i &= \mu \sin r & 1 &= \mu \sin r_0 \\ \sin \theta &= \mu \sin \rho & \sin \theta_0 &= \mu \sin \rho_0 \\ A &= \rho + r = \rho_0 + r_0 \end{aligned}\right\} \quad \text{..............(32)}.$$

Since the differences $\theta - \theta_0$, $\rho - \rho_0$, $r - r_0$ are very small, these equations give

$$\left.\begin{aligned} \theta - \theta_0 &= \mu(\rho - \rho_0) \cos \rho_0 / \cos \theta_0 = -\mu(r - r_0) \cos \rho_0 / \cos \theta_0 \\ \sin i &= 1 + \mu(r - r_0) \cos r_0 = 1 - (\theta - \theta_0) \cos r_0 \cos \theta_0 / \cos \rho_0 \\ \cos^2 i &= 2(\theta - \theta_0) \cos r_0 \cos \theta_0 / \cos \rho_0 \end{aligned}\right\} \quad \text{......(33)},$$

whence
$$\Delta^2 = n^2\lambda^2 = 8t^2 \frac{\cos r_0 \cos \theta_0}{\cos \rho_0} (\theta - \theta_0) \quad \text{.................(34)},$$

and the angular width of the n^{th} band in monochromatic light λ, corresponding to a change of n into $n + 1$, is given by

$$\delta\theta = \frac{n\lambda^2}{4t^2} \cdot \frac{\cos \rho_0}{\cos r_0 \cos \theta_0} \quad \text{..........................(35)}.$$

Hence the width of the n^{th} band is approximately proportional to the order, to the square of the wave-length and to the inverse square of the thickness.

Let us now consider the phenomenon in white light*. Since $\cos \theta_0$, $\cos \rho_0$ and $\cos r_0$ vary but slowly with the wave-length, we may write

$$\Delta = n\lambda = h(\theta - \theta_0)^{\frac{1}{2}} \quad \text{..........................(36)},$$

where h may be regarded as constant, and θ_0 is a function of the wave-length.

Now
$$d\theta_0 = \frac{\sin A}{\cos \theta_0 \cos r_0} d\mu,$$

and since the coefficient of $d\mu$ may be considered as constant, θ_0 is a linear function of μ, and we may write

$$\theta_0 = a + b\lambda^{-2} \text{...............................(37)}.$$

Now the bands will be superposed for all colours for which the wave-length is near a certain value chosen arbitrarily, if the differential coefficient of the deviation with respect to the wave-length be zero, the order n being supposed to be constant. The deviation of the fringes of the same order relatively to more remote colours, whether of greater or less wave-length, is then in the same direction, and the band may be said to be achromatised for the colour of concordance.

* Macé de Lepinay, *J. de Phys.* (3) III. 163 (1894).

The condition for achromatism for light of wave-length λ is

$$\theta - \theta_0 = -\frac{\lambda}{2} \cdot \frac{\partial \theta_0}{\partial \lambda} = b\lambda^{-2} \quad\text{.......................(38),}$$

and the deviation of the band achromatised for this radiation is

$$\theta = a + 2b\lambda^{-2} \quad\text{................................(39),}$$

the order of the band being given by

$$n\lambda = h(\theta - \theta_0)^{\frac{1}{2}} = h\sqrt{b}/\lambda \quad\text{...........................(40).}$$

Eliminating λ between these last two equations, we obtain

$$\theta = a + 2n\sqrt{b}/h \quad\text{................................(41),}$$

and the width of the bands is $2\sqrt{b}/h$, which is practically constant.

Thus in white light the coloured bands are nearly equidistant, though in monochromatic light their width varies as the square of the wave-length.

45. Let us next consider the case in which the film is contained between a thick parallel plate of thickness h and a plane surface inclined to the faces of the plate at a small angle α*.

Let us take as origin the point in which the edge of the wedge-shaped film is met by a plane through the optic axis of the observing microscope normal to the thick plate, and let ψ be the angle between this plane and one perpendicular to the edge of the film.

Then denoting by R the radius-vector to the point in which the upper surface of the film is met by the ray that emerges from the thick plate in the direction of the optic axis and by a the distance of the same point from the edge of the wedge,

$$t = a \tan \alpha = R \cos \psi \, . \tan \alpha,$$

$$\therefore \; dt = \tan \alpha (\cos \psi dR - R \sin \psi d\psi) = t(\cos \psi dR - R \sin \psi d\psi)/a.$$

If C be the height above the top of the thick plate of the point P on which the microscope is focussed

$$R = \text{const} + C \tan i + h \tan r',$$

$$\begin{aligned} \therefore \; dR &= C \sec^2 i di + h \sec^2 r' dr' = (C \sec^2 i + h \cos i \sec^3 r'/\mu')\, di \\ &= (D \sec i + h \cos i \sec^3 r'/\mu')\, di, \end{aligned}$$

where D is the distance of P from the top of the plate measured along the optic axis of the microscope.

Hence the condition of visibility (30) becomes

$$\frac{\mu \cos r}{a} (\cos \psi dR - R \sin \psi d\psi) = \frac{\tan r \cos i}{D \sec i + h \cos i \sec^3 r'/\mu'} dR,$$

or writing

$$dR = \xi, \quad Rd\psi = \eta,$$

$$\xi \cos \psi - \eta \sin \psi = \frac{a}{\mu} \cdot \frac{\sin r \sec^2 r \cos^2 i}{D + h \cos^2 i \sec^3 r'/\mu'} \xi \quad\text{...........(42).}$$

* Macé de Lepinay, *J. de Phys.* (2) IX. 121 (1890).

Now ξ and η being independent variables, this relation cannot in general be satisfied, unless the final directions of the rays that pass through P be limited to one plane by the introduction of a slit: the points at which the rays are reflected at the upper surface of the film will then form an element of a line, and if this make an angle ϕ with the trace of the plane of incidence, $\eta/\xi = \tan\phi$ and

$$D = -\frac{h\cos^2 i}{\mu'\cos^3 r'} + \frac{a}{\mu}\cdot\frac{\cos^2 i\sin r}{\cos^2 r}\cdot\frac{\cos\phi}{\cos(\phi+\psi)} \quad\text{..............(43)},$$

whence it follows that the plane of localisation depends upon the orientation of the slit.

If the fringes be visible with an extended source, the value of D must be independent of ϕ: this occurs

(1) when $\psi = 0°$ or $180°$, the plane of localisation being then given by

$$D = -\frac{h\cos^2 i}{\mu'\cos^3 r'} \pm \frac{a}{\mu}\frac{\cos^2 i\sin r}{\cos^2 r},$$

the upper or lower sign being taken, according as $\psi = 0°$ or $180°$;

(2) when $r = 0°$ or the incidence is normal; the fringes are then localised at the point

$$D = -h/\mu',$$

that is at the apparent upper surface of the film;

(3) when $r = 90°$ or the light meets the film at the critical angle. The plane of localisation is then at infinity.

We may notice that the expression $-h\cos^2 i\sec^3 r'/\mu'$ gives the position of the first focal line of the pencil, that emanates from a point on the lower surface of the thick plate and has its axis on emergence along the optic axis of the microscope.

46. As a final application of the formulæ, let us take the case of Newton's rings formed by reflection from a thin film included between a thick parallel plate and a convex surface of the same substance of very small curvature*.

Take the point of contact of the surfaces as the origin of a rectangular system of coordinates, the upper surface of the film being the plane of xy, that of xz being parallel to the plane normal to the plate through the axis of the microscope, with which the rings are observed, and the axis of x being directed towards the luminous source.

Let $(x, y, 0)$ be the cartesian and (R, ψ) the polar coordinates of the point, in which the top surface of the film is met by the ray that emerges

* Macé de Lepinay, *loc. cit.* Cf. also, Feussner, *Marburg. Ber.* (1880) 1; (1881) 1; (1882) 1; *Wied. Ann.* XIV. 545 (1881). Wangerin, *Pogg. Ann.* CXXXI. 497 (1867); *Wied. Ann.* XL. 738 (1890). Sohncke and Wangerin, *Wied. Ann.* XII. 1, 201 (1881); XX. 177, 391 (1883). Gumlich, *ibid.* XXVI. 337 (1885). Flux, *Phil. Mag.* (5) XXIX. 217 (1890).

from the plate in the direction of the optic axis of the microscope: then if $(x+\xi,\ y+\eta,\ 0)$ be the coordinates of a point near to $x, y, 0$, we have

$$2t = R^2/\rho,$$

ρ being the radius of the spherical surface, and

$$2dt = \frac{2RdR}{\rho} = \frac{2R}{\rho}(\xi\cos\psi + \eta\sin\psi) = \frac{4t}{R}(\xi\cos\psi + \eta\sin\psi),$$

and $$\xi = (D\sec i + h\cos i\sec^3 r'/\mu')\,di,$$

D having the same meaning as in the last case.

Hence we have for the condition of visibility

$$\frac{2\mu}{R}(\xi\cos\psi + \eta\sin\psi) = \frac{\sin r\sec^2 r\cos^2 i}{D + h\cos^2 i\sec^3 r'/\mu'}\,\xi,$$

and introducing the relation $\eta/\xi = \tan\phi$, the plane of localisation of the fringes is given by

$$D = -\frac{h}{\mu'}\cdot\frac{\cos^2 i}{\cos^3 r'} + \frac{R}{2\mu}\,\frac{\sin r\cos^2 i}{\cos^2 r}\cdot\frac{\cos\phi}{\cos(\phi-\psi)}\ldots\ldots\ldots\ldots\ldots(44),$$

an equation that gives the same results as were obtained for the case of a wedge-shaped film.

47. In the case of curved interference fringes, the retardation of phase δ is to be regarded as a function of x, y and λ, and the equation

$$\delta = \phi(x, y, \lambda)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

in which λ is regarded as a constant, determines the form of the fringes as seen in homogeneous light.

If the light be white, the bands will be in general coloured, but those points will be achromatic for which

$$\frac{d\delta}{d\lambda_0} = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

This condition gives a relation between x and y, and determines a curve that may be called the achromatic curve, but inasmuch as the value of δ is not constant along it, this curve is not an achromatic *band*. The achromatic bands are a system of infinitely short lines, that exist only at the points of intersection of the achromatic curve with the lines $\delta = \text{const}$.

In the case of Newton's rings, the thickness of the film at the point (x, y) measured from its thinnest point is

$$t = a + b(x^2 + y^2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

whence $$\delta = \frac{4\pi}{\lambda}\{a + b(x^2+y^2)\}\cos i\ \ldots\ldots\ldots\ldots\ldots\ldots(48),$$

and the achromatic curve is

$$a + b(x^2 + y^2) = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49).$$

It is thus wholly imaginary, if a and b be both positive and finite: but if $a = 0$ there is an achromatic point $x = 0$, $y = 0$.

The result is however different when the rings are viewed through a prism. We may then suppose that each monochromatic system is shifted as a whole parallel to the axis of x by an amount dependent upon the wave-length of the light. The apparent coordinates being ξ and η, so that

$$\xi = x - f(\lambda), \quad \eta = y \qquad (50),$$

the equation of the rings as seen through the prism is

$$\delta = \frac{4\pi \cos i}{\lambda} [a + b\{\xi + f(\lambda)\}^2 + b\eta^2] \qquad (51),$$

and the equation of the achromatic curve is

$$\{\xi + f(\lambda_0) - \lambda_0 f'(\lambda_0)\}^2 + \eta^2 = \lambda_0^2 \{f'(\lambda_0)\}^2 - a/b \qquad (52),$$

which represents a circle with its centre on the axis of ξ.

If $a = 0$, the curve is real and passes through the point

$$\xi + f(\lambda_0) = 0, \quad \eta = 0,$$

that is, the image of the point of contact ($x = 0$, $y = 0$) in light of wave-length λ_0. At the point

$$\xi = -f(\lambda_0) + 2\lambda_0 f'(\lambda_0), \quad \eta = 0,$$

in which the circle again meets the axis, the bands are parallel to the achromatic curve and are specially developed.

As a increases from zero, the radius of the achromatic circle decreases, the centre remaining fixed, so that the two points in which the circle cuts the axis are on the same side of the image of $x = 0$, $y = 0$. When a is such that

$$a/b = \lambda^2 \{f'(\lambda_0)\}^2,$$

the circle reduces to a point, given by

$$x = \xi + f(\lambda_0) = \lambda_0 f'(\lambda_0), \quad y = \eta = 0,$$

and since there are two coincident achromatic points on the axis, the condition is satisfied for an achromatic system. We then have

$$a/b = x^2,$$

so that
$$t = a + bx^2 = 2a,$$

and hence for an achromatic system, the thickness at the point must be due half to curvature and half to imperfect contact at the place of nearest approach of the surfaces*.

48. It has already been stated that a defect in the monochromatism of the light leads to the final obliteration of the interference fringes as the

* Lord Rayleigh, *Phil. Mag.* (5) XXVIII. 203 (1889).

relative retardation of the interfering streams increases, and the visibility of the phenomenon has been determined in the simple case of a source emitting radiations that are grouped about some principal period with intensities given by Maxwell's law.

Another instance of a somewhat different character is afforded by Fizeau's* celebrated experiment with Newton's rings viewed at normal incidence in the light from a soda-flame. Roughly speaking, this light may be said to consist of two systems of radiations, the wave-lengths of which differ from one another by about one-thousandth of that of the one with the lower frequency. When the film is very thin, the difference of phase is sensibly the same for both systems, so that the maxima due to each coincide and the rings will have their greatest possible distinctness. As the thickness of the film is increased, the rings will move in towards their centre, becoming less and less distinct, and when the distance between the surfaces of the film is of such a magnitude that the relative retardation of phase for one radiation exceeds that for the other by half a period, the maxima of the one will be superposed on the minima due to the other and the rings will be no longer visible. A further increase in the thickness of the film will cause a reappearance of the fringes, the distinctness of which will increase up to a maximum, corresponding to the case of the relative retardation of phase for the one radiation being a complete period in excess of that for the other.

If the light from the soda-flame had the simple character stated above, these phenomena would be repeated indefinitely and the visibility of the rings would be the same at the successive maxima of distinctness: this however is not the case, and it becomes important to determine the manner in which the visibility of interference phenomena depends upon the radiations from a complex source and to investigate whether the variation in the visibility as the relative retardation increases affords a means of discovering these radiations.

49. Suppose then that $f(\lambda)\,d\lambda$ is the intensity of illumination due to streams, the wave-lengths of which are comprised between λ and $\lambda + d\lambda$, and that Δ is the relative retardation in actual length in air introduced by the interferential apparatus: then the intensity due to these streams is

$$2\left(1 + \cos\frac{2\pi}{\lambda}\Delta\right) f(\lambda)\,d\lambda \quad\text{.......................(53)},$$

and if the radiations from the source are grouped about some principal radiation, the total intensity is obtained by integrating this expression between the limits λ_1 and λ_2—the wave-lengths of the extreme constituents of the complex stream.

* Fizeau, *Ann. de Ch. et de Phys.* (3) LXVI. 429 (1862).

Let $\lambda^{-1} = (1+x)\lambda_0^{-1}$, the values of x corresponding to λ_1 and λ_2 being $-x_1$ and x_2 respectively, then writing $f(\lambda)\,d\lambda = \phi(x)\,dx$ and $\Delta = p\lambda_0$, we have

$$I = 2\int_{-x_1}^{x_2} \phi(x)\,dx + 2\int_{-x_1}^{x_2} \cos 2\pi p(1+x)\,.\,\phi(x)\,dx$$
$$= 2P + 2C\cos 2\pi p - 2S\,.\,\sin 2\pi p \qquad\qquad (54),$$

where

$$P = \int_{-x_1}^{x_2} \phi(x)\,dx, \quad C = \int_{-x_1}^{x_2} \cos 2\pi px\,.\,\phi(x)\,dx, \quad S = \int_{-x_1}^{x_2} \sin 2\pi px\,.\,\phi(x)\,dx.$$

If the interval $x_1 + x_2$ be small, the variations of C and S with p may be neglected, and the maxima and minima of the intensity occur when

$$C\sin 2\pi p + S\cos 2\pi p = 0,$$

the value then being

$$I = 2\{P \pm \sqrt{C^2+S^2}\} \qquad\qquad (55),$$

whence the visibility of the fringes is given by

$$V_1^2 = (C^2+S^2)/P^2 \qquad\qquad (56).$$

If now the radiations from the source form several groups such as that just considered and the values of x for their principal radiations be $\alpha_1, \alpha_2, \ldots$, then replacing x by $\alpha_n + z$ and $\phi_n(\alpha_n+z)$ by $\psi_n(z)$, we have

$$C = \Sigma\int \cos 2\pi p(\alpha_n+z)\,\psi_n(z)\,dz = \Sigma C_n\cos 2\pi p\alpha_n - \Sigma S_n \sin 2\pi p\alpha_n,$$

$$S = \Sigma\int \sin 2\pi p(\alpha_n+z)\,\psi_n(z)\,dz = \Sigma S_n\cos 2\pi p\alpha_n + \Sigma C_n \sin 2\pi p\alpha_n,$$

and the visibility is given by

$$(\Sigma P)^2 V^2 = \{\Sigma(C_mC_n + S_mS_n)\cos 2\pi p(\alpha_n - \alpha_m)$$
$$+ \Sigma(C_mS_n - C_nS_m)\sin 2\pi p(\alpha_n - \alpha_m)\} \qquad\qquad (57).$$

When each group is symmetrical,

$$(\Sigma P)^2 V^2 = \Sigma P_m P_n V_m V_n \cos 2\pi p(\alpha_n - \alpha_m) \qquad\qquad (58),$$

and if the groups be alike, except for a constant factor h that may represent intensity,

$$V^2 = \frac{\Sigma h_m h_n \cos 2\pi p(\alpha_n - \alpha_m)}{(\Sigma h)^2} V_1^2 \qquad\qquad (59),$$

where V_1 denotes the visibility for a single group.

The most interesting case is that in which the intensities in the groups are distributed in accordance with Maxwell's law, or $\phi(x) = \exp(-k^2x^2)$. When the coefficient k is very large, the exponential diminishes very rapidly and the important terms are those near the principal radiation, for which $x = 0$: taking the limits of integration as $\pm\infty$,

$$V_1 = \exp(-\pi^2p^2/k^2) \qquad\qquad (60),$$

and in the case of two groups, for which the intensities are as $1 : r$, the visibility is

$$V = e^{-\frac{\pi^2 p^2}{k^2}} \sqrt{\frac{1 + r^2 + 2r \cos 2\pi p (\alpha_1 - \alpha_2)}{1 + 2r + r^2}} \quad \text{..............(61).}$$

50. Conversely suppose that the visibility is found to be represented by

$$V = 2^{-X^2/C^2} \sqrt{\frac{1 + r^2 + 2r \cos 2\pi X/D}{1 + 2r + r^2}} \quad \text{..................(62),}$$

X being the retardation in length, and let us determine the radiations present in the streams.

The form of the expression shows that the source is double; that its components have the intensity-ratio $1 : r$, and that in each the light is distributed according to the exponential law expressed by its first term.

From a comparison of (62) with the expression for the visibility in the case of a double source of which the constituents are known, we have, if λ be the mean wave-length,

$$\alpha_2 - \alpha_1 = \lambda/D:$$

but λ_1 and λ_2 being the wave-lengths of the principal radiations

$$\alpha_2 - \alpha_1 = \lambda(\lambda_2^{-1} - \lambda_1^{-1}) \fallingdotseq (\lambda_1 - \lambda_2)\lambda^{-1},$$

and hence on a scale of wave-lengths the distance between the principal radiations is

$$\lambda_1 - \lambda_2 = \lambda^2/D \quad \text{..................................(63).}$$

Again comparing the exponentials, we have

$$\pi^2 p^2/k^2 = (X/C)^2 \log_e 2 = (p\lambda/C)^2 \log_e 2,$$

$$\therefore \quad k = \frac{\pi C}{\lambda} \frac{1}{\sqrt{\log_e 2}}.$$

But if ϵ be the "half-width" of the spectral line—the value of x that makes $\phi(x) = \cdot 5$—

$$k\epsilon = \sqrt{\log_e 2}, \quad \text{or} \quad \epsilon = \frac{\log_e 2}{\pi} \frac{\lambda}{C},$$

and on a scale of wave-lengths the "half-width" of the spectral line is

$$\lambda\epsilon = \frac{\log_e 2}{\pi} \frac{\lambda^2}{C} \fallingdotseq 0 \cdot 22 \frac{\lambda^2}{C} \quad \text{........................(64).}$$

Hence the expression for the visibility gives the ratio of the intensities of the components, their width and the distance between them; but the order in which they are arranged in the spectrum remains indeterminate*.

* Michelson, *Phil. Mag.* (5) XXXI. 256, 338 (1891); XXXIV. 280 (1892); *Travaux et Mémoires du Bureau Intern. des Poids et Mesures*, XI. 129 (1895).

Fabry and Perot* have however succeeded in determining this by Fizeau's method (§ 48); but instead of using Newton's rings, they employed Haidinger's fringes formed by the light transmitted through a parallel plate of air contained between two thick plates of glass, the adjacent surfaces of which were lightly silvered. By thus increasing the reflecting power of the faces of plate, the dark bands are made much blacker, while the bright rings are rendered very fine, and in consequence the rings produced by radiations extremely near to one another can be easily separated by a progressive increase of the thickness of the plate.

51. In order to determine the expression for the visibility, Michelson† employed the refractometer described in Chapter II, deducing the form of the function from his measures by a graphic method. A reference to the description of the apparatus already given will show that, so far as interference is concerned, the streams comport themselves as if they were reflected at the first and second surfaces of a film of air contained between the image of the mirror M_1 in the silvered surface of the glass plate G_1 (which image is called the plane of reference) and the surface of the mirror M_2, since when the silver coating of G_1 is very thin, the change of phase on reflection at it amounts to π whether the reflection takes place in air or in the glass‡. There is here clearly no question of multiple reflections within the film, and the dark bands will occur when

$$2t \cos i = (2n+1)\lambda/2,$$

and the bright bands, when

$$2t \cos i = n\lambda.$$

When M_2 is parallel to the plane of reference, the fringes are concentric circles localised at infinity, while if M_2 be inclined to this plane and the plane of incidence be perpendicular to their line of intersection, the fringes are straight lines, parallel to this line and localised on the surface of the film.

For the determination of the visibility Michelson adopted the first of these two cases of interference. The mirror M_2 was first adjusted to coincidence with the plane of reference, in which case the two streams have traversed equal distances, and it was then displaced through 1 mm., giving a difference of path of 2 mm. and the visibility was estimated, and so on.

These eye estimates of the visibility having been checked and corrected by previous observations of fringes having a visibility that could be calculated, a curve was drawn by taking the differences of path as abscissæ and the visibilities as ordinates, and the equation of the curve was then found by trial.

* Fabry and Perot, *Ann. de Ch. et de Phys.* (7) XII. 459 (1897); XVI. 115, 289 (1899); XXII. 564 (1901); *Astrophys. J.* XIII. 265 (1901).

† *loc. cit.* p. 115.

‡ Edser and Stansfield, *Nature*, LVI. 504 (1897).

The following examples will serve as specimens of the results obtained by Michelson*. In the figures the curves drawn in full on the right represent the visibilities given by the observations, the dotted curves represent the equations adopted as the expressions for the visibility: the figures on the left give the character of the spectral lines deduced from the curves of visibility.

(1) The visibility-curve of the red hydrogen line ($\lambda = 6{\cdot}56 \times 10^{-4}$) practically agrees with

$$V = 2^{-(X/19)^2} \cos \cdot 7/30,$$

the form $\cos r/D$ being written for $\sqrt{1 + r^2 + 2r \cos 2\pi X/D}/(1 + r)$, so that it is practically the same as that due to a double source, the components of which have the intensity-ratio 7 : 10.

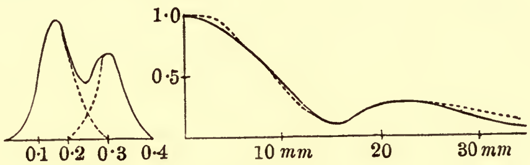


Fig. 15.

The distance between the components of the line is

$$\frac{1}{30} \times (6{\cdot}56 \times 10^{-4})^2 = 1{\cdot}4 \times 10^{-8} \text{ mm.} = 0{\cdot}14 \text{ divisions of Rowland's scale.}$$

The width of each component on the same scale is 0·099.

(2) For the red cadmium line ($\lambda = 6{\cdot}44 \times 10^{-4}$) the visibility-curve agrees with

$$V = 2^{-(X/138)^2}:$$

this then is a remarkably simple line of breadth 0·013 on Rowland's scale, and the red cadmium line thus affords a specially homogeneous source of light.

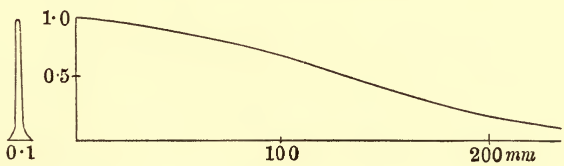


Fig. 16.

(3) The orange-red oxygen line ($\lambda = 6{\cdot}16 \times 10^{-4}$) gives for the curve of visibility

$$V = 2^{-(X/34)^2} \left\{ 0{\cdot}36 + 0{\cdot}32 \cos 2\pi \frac{X}{2{\cdot}69} + 0{\cdot}16 \cos 2\pi \frac{X}{4{\cdot}85} + 0{\cdot}16 \cos 2\pi \frac{X}{1{\cdot}73} \right\}^{\frac{1}{2}}.$$

* *loc. cit.* p. 138.

This expression indicates that the source consists of three simple and similar lines, the intensity-ratios of which are 2 : 2 : 1, the last being at the end of the series: the width of the lines is 0·05 and their distances apart are 1·41 and 0·78.

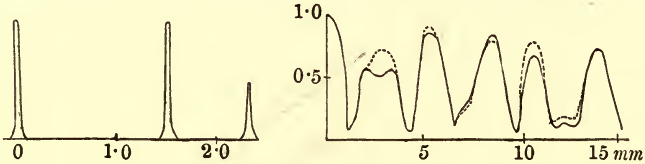


Fig. 17.

(4) The green thallium line ($\lambda = 5{\cdot}35 \times 10^{-4}$) gives

$$V = \frac{1}{3} \cos {\cdot}2/160 \left\{ 4V_1^2 + V_2^2 + 4V_1V_2 \cos 2\pi \frac{X}{25{\cdot}3} \right\}^{\frac{1}{2}},$$

$$V_1 = 2^{-(X/246)^2}, \quad V_2 = 2^{-(X/188)^2}.$$

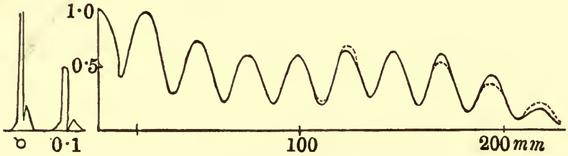


Fig. 18.

The light thus consists of two sources, for which the intensity-ratio is 2 : 1, and each of these is a doublet, the elements of which are determined from

$$V_1 = 2^{-(X/246)^2} \cos {\cdot}2/160 \quad \text{and} \quad V_2 = 2^{-(X/188)^2} \cos {\cdot}2/160.$$

Thus the components of each doublet have the intensity-ratio 5 : 1 and for each the distance between the components is 0·018 on Rowland's scale; the width of each component is for the one source 0·005 and for the other is 0·007 on the same scale. The distance between the doublets is 0·113.

CHAPTER V.

DIFFERENTIAL EQUATIONS OF THE POLARISATION-VECTOR.

52. Before proceeding further, it is necessary to determine the differential equations that the polarisation-vector must satisfy in the case of an isotropic, dispersionless, transparent medium, that is, one in which waves travel with the same speed, whatever their period and their direction.

It has been shown in Chapter II that the phenomenon of interference indicates that the result of a superposition of trains of waves of light is represented by a summation of their separate effects without any modification of the waves themselves. Since then in a train of waves the vibrations of the polarisation-vector are in the plane of the waves, provided they are identical over the whole extent of the wave-front, the components of the polarisation-vector d must satisfy the equations*

$$u = \Sigma\phi_n(\omega t - r), \quad v = \Sigma\chi_n(\omega t - r), \quad w = \Sigma\psi_n(\omega t - r) \quad \text{.........(1)},$$

$$ul + vm + wn = 0 \quad \text{..............................(2)},$$

where ϕ_n, χ_n, ψ_n are periodic functions, ω is the propagational speed of light, l, m, n denote the direction-cosines of the normal to the wave-front and

$$r = lx + my + nz.$$

Eliminating the arbitrary functions and the direction-cosines, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \omega^2\nabla^2 u, \quad \frac{\partial^2 v}{\partial t^2} = \omega^2\nabla^2 v, \quad \frac{\partial^2 w}{\partial t^2} = \omega^2\nabla^2 w \quad \text{...............(3)}.$$

Writing now
$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

we have
$$\delta = -\frac{1}{\omega}\left(\frac{\partial u}{\partial t} l + \frac{\partial v}{\partial t} m + \frac{\partial w}{\partial t} n\right) = 0$$

by (2). Hence the condition

$$\delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{............................(4)},$$

* Voigt, *Kompendium der Theoretischen Physik*, Bd. II. 554.

which expresses that the polarisation-vector has no convergence anywhere, may be regarded as equivalent to (2), and the differential equations required are given by (3) and (4). These equations may clearly be written in the symmetrical form

$$\dot{d} = -\operatorname{curl} \varpi, \quad \dot{\varpi} = \operatorname{curl} e \quad \text{..........(5)},$$

where the components of the vector e are given by

$$(e_1, e_2, e_3) = \tfrac{1}{2}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)\{\omega^2(u^2+v^2+w^2)\} \quad \text{..........(6)}.$$

53. As a first application of these equations, let us determine the nature of the vibrations when the waves are unhomogeneous, that is when they are no longer identical over the whole front. Using bars ($-$) over the letters to denote complex quantities, let us assume as a solution of the equations

$$u = \bar{\alpha} d, \quad v = \bar{\beta} d, \quad w = \bar{\gamma} d, \quad d = A \exp\{\iota(\bar{l}x + \bar{m}y + \bar{n}z + st)\} \quad \text{......(7)},$$

where

$$\bar{\alpha}^2 + \bar{\beta}^2 + \bar{\gamma}^2 = 1 \quad \text{..........(8)},$$

$$\bar{l}x + \bar{m}y + \bar{n}z = \frac{2\pi}{\lambda}\{x \cos i + y \cos j + z \cos k + \iota\nu(x \cos I + y \cos J + z \cos K)\} \quad \text{.........(9)};$$

$\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are then the complex direction-cosines of the vector d,

$\cos i$, $\cos j$, $\cos k$ are the direction-cosines of the normal to the planes of like phase,

$\cos I$, $\cos J$, $\cos K$ are the direction-cosines of the normal to the planes of like amplitude, and

ν is the coefficient of extinction of the waves along this normal.

Then equation (4) gives

$$\bar{\alpha}\bar{l} + \bar{\beta}\bar{m} + \bar{\gamma}\bar{n} = 0 \quad \text{..........(10)},$$

and from each of equations (3)

$$s^2 = \omega^2(\bar{l}^2 + \bar{m}^2 + \bar{n}^2) \quad \text{..........(11)}.$$

Separating the real and imaginary parts of this equation, we have

$$\Omega^2 = \omega^2(1-\nu^2), \quad 0 = \nu(\cos i \cos I + \cos j \cos J + \cos k \cos K) \quad \text{...(12)},$$

Ω being the propagational speed of unhomogeneous waves of period τ: since Ω is real if the waves be propagated without change of type, $\nu < 1$.

Also since we are assuming that $\nu \neq 0$, it follows that

$$\cos i \cos I + \cos j \cos J + \cos k \cos K = 0 \quad \text{..........(13)},$$

which expresses that the planes of like phase are at right angles to those of like amplitude, and this being so

$$\bar{l}^2 + \bar{m}^2 + \bar{n}^2 = \frac{4\pi^2}{\lambda^2}(1-\nu^2) \quad \text{..........(14)}.$$

Taking the axis of z in the direction of the wave-normal and the axis of x along the normal to the planes of like amplitude, we have

$$\bar{l} = \iota \frac{2\pi}{\lambda} \nu, \quad \bar{m} = 0, \quad \bar{n} = \frac{2\pi}{\lambda},$$

and $$\bar{\alpha} : \bar{\gamma} :: 1 : -\iota\nu.$$

It follows then that the vibrations are elliptical and no longer in the plane of the wave; but though this is so, they are still of the nature of transversal vibrations, for the equation $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ is still satisfied, and this is the distinguishing characteristic of such vibrations.

54. An important question in the problem of wave-propagation is that of the direction in which any peculiarity of phase or amplitude is propagated in a stream of light and the speed with which it travels. In a simple train of waves there is no distinguishing mark by which any portion is identified, and consequently the determination of the velocity of light is generally effected by measuring the propagational speed of some peculiarity impressed upon the train: this will only give the wave-velocity if the singularity travel at the same rate as the waves.

We will now consider this point, taking the case of a medium that is characterised by equations (3)*.

Let U, V, W be three functions of the rectangular coordinates of any point in the medium, such that

$$U = \alpha_1, \quad V = \alpha_2, \quad W = \alpha_3 \quad\quad (15),$$

in which the parameters α_1, α_2, α_3 are given all possible values, form a system of conjugate or orthogonal surfaces: and let us take U, V, W as new coordinates.

In order to transform the equations (3) to this new system of coordinates, we note that $\nabla^2 u$ is the divergence of a vector, the components of which are $\partial u/\partial x, \partial v/\partial y, \partial w/\partial z$, and therefore the volume integral of $\nabla^2 u$ taken throughout any region is equal to the surface integral of the vector over the boundary of the region, that is, to $\int (\partial u/\partial n)\, dS$, where n is the normal to dS drawn outwards. Let us apply this theorem to the small rectangular parallelopiped, the faces of which are parts of the six surfaces U, $U + dU$, V, $V + dV$, W, $W + dW$.

If $a\,dU$, $b\,dV$, $c\,dW$ be the lengths of the edges of the parallelopiped, the pair of faces, forming part of the surfaces U and $U + dU$, contribute to the surface integral the amount

$$\frac{\partial}{\partial U}\left\{\frac{bc}{a}\frac{\partial u}{\partial U}\right\} dU\,dV\,dW,$$

* Poincaré, *Théorie Mathématique de la Lumière*, II. p. 114.

and similarly for the other two pairs of faces. Hence since the volume of the parallelopiped is $abc\,dU.dV.dW$, we have at once

$$\nabla^2 u = \frac{1}{abc}\left\{\frac{\partial}{\partial U}\left(\frac{bc}{a}\frac{\partial u}{\partial U}\right) + \frac{\partial}{\partial V}\left(\frac{ca}{b}\frac{\partial u}{\partial V}\right) + \frac{\partial}{\partial W}\left(\frac{ab}{c}\frac{\partial u}{\partial W}\right)\right\} \quad \text{......(16)},$$

and the equations (3) become in the new system of coordinates

$$\frac{abc}{\omega^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial U}\left(\frac{bc}{a}\frac{\partial u}{\partial U}\right) + \frac{\partial}{\partial V}\left(\frac{ca}{b}\frac{\partial u}{\partial V}\right) + \frac{\partial}{\partial W}\left(\frac{ab}{c}\frac{\partial u}{\partial W}\right) \quad \text{.........(17)},$$

and two similar equations.

Now let us suppose that the surfaces $W = \alpha_3$ are a system of parallel surfaces: then $c = 1$ and ab is proportional to the section at any point of a small tube of normals to the surfaces; and if these surfaces be the wave-fronts we may write

$$u = A \exp\{\iota(nt - \kappa W)\},$$

where $n = 2\pi/\tau$, $\kappa = 2\pi/\lambda$ and A is a function of (U, V, W, t).

Substituting this value in (17) and writing $ab = \sigma$, we obtain

$$\frac{\sigma}{\omega^2}\left\{\frac{\partial^2 A}{\partial t^2} + 2\iota n\frac{\partial A}{\partial t} - n^2 A\right\} = \frac{\partial}{\partial U}\left(\frac{b}{a}\frac{\partial A}{\partial U}\right) + \frac{\partial}{\partial V}\left(\frac{a}{b}\frac{\partial A}{\partial V}\right) + \frac{\partial}{\partial W}\left(\sigma\frac{\partial A}{\partial W}\right)$$

$$- \iota\kappa A\frac{\partial\sigma}{\partial W} - 2\iota\kappa\sigma\frac{\partial A}{\partial W} - \kappa^2 A\sigma \quad \text{.........(18)}.$$

Since $n = \kappa\omega$, the terms involving A cancel, and if the differential coefficients of A be all finite, we may neglect the terms that do not involve the large quantity κ and we obtain

$$\frac{\sigma}{\omega}\frac{\partial A}{\partial t} = -\sigma\frac{\partial A}{\partial W} - \tfrac{1}{2}A\frac{\partial\sigma}{\partial W}$$

or

$$\frac{\sqrt{\sigma}}{\omega}\frac{\partial A}{\partial t} = -\sqrt{\sigma}\frac{\partial A}{\partial W} - A\frac{\partial\sqrt{\sigma}}{\partial W},$$

and as σ is independent of t,

$$\frac{1}{\omega}\frac{\partial(A\sqrt{\sigma})}{\partial t} + \frac{\partial(A\sqrt{\sigma})}{\partial W} = 0 \quad \text{.......................(19)},$$

whence

$$A\sqrt{\sigma} = f(U, V, W - \omega t) \quad \text{.......................(20)}.$$

Thus any singularity of phase or amplitude is propagated along the normal to the wave-front with the speed ω, and since the amplitude varies as $1/\sqrt{\sigma}$, the intensity is inversely as the section of the beam of light.

55. This result that the peculiarities of phase and amplitude travel with the speed of the waves, depends upon the assumption that the wave-velocity is the same for all waves whatever their period may be and cannot be applied to the case of dispersive media. The effect of impressing any distinguishing mark on a train of waves is to destroy its simple harmonic character, and if

the constituents of the altered stream travel with different velocities, it by no means follows that the group thus formed is propagated at the same rate as the original train.

If the original train be characterised by the vector

$$u = a\cos(nt - \kappa x) \qquad (21),$$

we can represent the group by the vector

$$u' = a_1\cos\{(n+\delta n_1)t - (\kappa+\delta\kappa_1)x + \alpha_1\} + a_2\cos\{(n+\delta n_2)t - (\kappa+\delta\kappa_2)x + \alpha_2\} \qquad (22),$$

where δn_1, δn_2, ... and $\delta\kappa_1$, $\delta\kappa_2$, ... represent small variations of n and κ. This may be written in the form

$$\cos(nt-\kappa x)\,\Sigma a\cos(\delta n\,.\,t - \delta\kappa\,.\,x + \alpha)$$
$$- \sin(nt-\kappa x)\,\Sigma a\sin(\delta n\,.\,t - \delta\kappa\,.\,x + \alpha) \qquad (23).$$

Now n and κ are connected by some relation, such as $n = \phi(\kappa)$, where the form of ϕ depends upon the nature of the medium: hence

$$\frac{\delta n_1}{\delta\kappa_1} = \frac{\delta n_2}{\delta\kappa_2} = \ldots = \frac{dn}{d\kappa} = U \text{ (say)} \qquad (24),$$

and the resultant group is represented by

$$F(Ut - x)\cos\{nt - \kappa x + \chi(Ut - x)\} \qquad (25).$$

Thus the peculiarities of phase and intensity travel with the speed

$$U = d(\kappa\omega)/d\kappa,$$

and this differs from the wave-velocity ω unless waves of all periods are propagated with one and the same velocity*.

56. Taking now the methods employed for the determination of the velocity of light, we see that the measurements depending upon astronomical aberration give the true wave-velocity ω, but that it is the group-velocity U that is found by the methods of Römer and Fizeau, since they both depend upon the rate of progress of a peculiarity of intensity.

Foucault's method requires further consideration†; in this experiment the subject of measurement is the deflection of a stream of light produced by the rotation of a mirror during the time of passage of the waves from the revolving to a fixed mirror and back again.

Now the motion of the mirror impresses a variation of wave-length along the fronts of the waves as they leave the mirror, making it greater on the side of the stream that is reflected at its receding part. Consequently if the medium be dispersive, that side of the stream will travel faster than the

* Lord Rayleigh, *Nature*, XXIV. 382; XXV. 52 (1881).

† Lord Rayleigh, *loc. cit.* Schuster, *Nature*, XXXIII. 439 (1886). Gibbs, *ibid.* p. 582. Gouy, *C. R.* CI. 502 (1885).

other, and there will be an aërial rotation of the waves during their passage between the mirrors. Since the waves are inverted by reflection at the fixed mirror, the side of the stream that leaves the receding part returns to the preceding part of the revolving mirror, and the aërial rotation of the stream is in the opposite direction to that of the mirror.

Hence if D be the distance between the mirrors, θ the angular velocity of the rotating mirror, θ' that of the aërial rotation of the waves, the angular deflection of the stream is

$$\chi = 2D(2\theta + \theta')/\omega \qquad (26).$$

Denoting by z distances measured along any wave-front in a direction perpendicular to the axis of rotation of the mirror, we have

$$\theta' = d\omega/dz = d\omega/d\lambda \,.\, d\lambda/dz \qquad (27);$$

but $d\lambda/dz$ is the angle between corresponding elements on two wave-fronts in the same phase, and this angle is due in part to the rotation of the mirror and in part to the aërial rotation of the waves; hence if τ be the period

$$d\lambda/dz = (2\theta + \theta')\tau = (2\theta + \theta')\lambda/\omega \qquad (28),$$

and

$$\theta' = (2\theta + \theta')\frac{\lambda}{\omega}\frac{d\omega}{d\lambda} \qquad (29);$$

whence

$$2\theta + \theta' = \frac{2\theta}{1 - \dfrac{\lambda}{\omega}\dfrac{d\omega}{d\lambda}} = \frac{2\theta}{U}\omega \qquad (30),$$

and

$$\chi = \frac{4D\theta}{U} \qquad (31).$$

Thus by this method it is the group-velocity U and not the wave-velocity that is determined.

CHAPTER VI.

HUYGENS' PRINCIPLE*.

57. In the last chapter it was assumed, when considering the propagation of a stream of light, that infinite space is filled with an homogeneous medium, in which no foreign substance occurs, except such as exists at a centre of luminous disturbance, and we must consider the effect that is produced by the introduction into the ether of media that differ from it in their optical properties. In this way we shall determine, in what degree the wave-theory accounts for the rectilinear propagation of light and thence leads to the laws of geometrical optics.

Let us consider a portion T of the ether bounded by a surface S, on the outside of which luminous sources and different bodies may be distributed in any manner, and let us determine the disturbance at any point within this space T.

Let $U(x, y, z, r)$ be a function of the coordinates (x, y, z) of any variable point and its distance r from the given point, this function together with its first differential coefficients being single-valued, finite and continuous for all points of T and of its bounding surface S: then denoting by d total differentiation and by ∂ partial differentiation with respect to x, y, z, r we have the identity

$$\frac{d}{dx}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial r \partial x}\frac{\partial r}{\partial x} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial r \partial x}\frac{\partial x}{\partial r},$$

and

$$\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right) = \frac{1}{r}\frac{\partial^2 U}{\partial x^2} + \frac{1}{r}\frac{\partial^2 U}{\partial r \partial x}\frac{\partial x}{\partial r} - \frac{1}{r^2}\frac{\partial U}{\partial x}\frac{\partial x}{\partial r},$$

whence

$$\Sigma \frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right) = \frac{1}{r}\nabla^2 U + \frac{1}{r}\left\{\frac{d}{dr}\left(\frac{\partial U}{\partial r}\right) - \frac{\partial^2 U}{\partial r^2}\right\} - \frac{1}{r^2}\left\{\frac{dU}{dr} - \frac{\partial U}{\partial r}\right\},$$

* Kirchhoff, *Berl. Ber.* (1882) 641; *Wied. Ann.* XVIII. 663 (1883); *Ges. Abh. Nachtrag*, p. 22; *Vorles. über Math. Optik*, p. 22. Beltrami, *N. Cim.* (3) XXVI. 233 (1889); *Rend. Lincei* (5) I. [1] 99 (1892); IV. [2] 29, 51 (1895). Maggi, *Annali di Mat.* (2) XVI. 21 (1888). Potier, *C. R.* CXII. 220 (1891). Bruhnes, *Mém. des Fac. de Lille*, IV. No. 16 (1895); *J. de Phys.* (3) IV. 6 (1895). Carvallo, *C. R.* CXX. 88 (1895). Gutzmer, *Crelle's J.* CXIV. 333 (1895). Morera, *N. Cim.* (4) II. 17 (1895).

which may be transformed into

$$\frac{1}{r^2}\frac{d}{dr}\left(U - r\frac{\partial U}{\partial r}\right) + \Sigma\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right) + \frac{1}{r}\left(\frac{\partial^2 U}{\partial r^2} - \nabla^2 U\right) = 0 \quad \text{......(1).}$$

Now draw a cone from the pole to the element dS; its cross-section at a distance r is $r^2 \sin\theta d\theta d\phi$, but this is also equal to

$$\cos \widehat{rn} \,.\, dS \quad \text{or} \quad r^2 \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS,$$

where n is the normal to the element drawn inwards; then

$$\begin{aligned}\int\frac{d}{dr}\left(U - r\frac{\partial U}{\partial r}\right)\frac{dT}{r^2} &= \iiint\frac{d}{dr}\left(U - r\frac{\partial U}{\partial r}\right) dr \,.\, \sin\theta d\theta d\phi \\ &= \iint_S\left(U - r\frac{\partial U}{\partial r}\right)\sin\theta d\theta d\phi - \left(U - r\frac{\partial U}{\partial r}\right)_0\iint \sin\theta d\theta d\phi \\ &= \int\left(U - r\frac{\partial U}{\partial r}\right)\frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS - 4\pi U_0 \\ &= \int\frac{\partial}{\partial r}\left(\frac{U}{r}\right)\frac{\partial r}{\partial n} dS - 4\pi U_0 \quad \text{..............................(2);}\end{aligned}$$

also
$$\int\Sigma\frac{d}{dx}\left(\frac{1}{r}\frac{\partial U}{\partial x}\right) dT = \int\left\{\frac{1}{r}\frac{\partial U}{\partial x} dydz + \ldots\right\} = -\int\frac{U_n}{r} dS,$$

where
$$U_n = \Sigma\frac{\partial U}{\partial x}\frac{\partial x}{\partial n} \quad \text{..............................(3).}$$

Multiplying then (1) by dT and integrating over the whole space T, we obtain

$$4\pi U_0 = \int\left\{\frac{\partial}{\partial r}\left(\frac{U}{r}\right)\frac{\partial r}{\partial n} - \frac{U_n}{r}\right\} dS + \int\left\{\frac{\partial^2 U}{\partial r^2} - \nabla^2 U\right\}\frac{dT}{r} \quad \text{.........(4).}$$

If $\phi(x, y, z, t)$ be a function of x, y, z, t that satisfies the equation

$$\frac{\partial^2\phi}{\partial t^2} = \omega^2\nabla^2\phi \quad \text{..............................(5),}$$

and if U be what ϕ becomes when $t - r/\omega$ is written for t, then

$$\frac{\partial^2 U}{\partial t^2} = \omega^2\frac{\partial^2 U}{\partial r^2} = \omega^2\nabla^2 U, \quad U_0 = \phi(x_0, y_0, z_0, t) = \phi_0,$$

whence the volume-integral in (4) vanishes, and we obtain

$$4\pi\phi_0 = \int\left\{\frac{\partial}{\partial n}\frac{\phi(t - r/\omega)}{r} - \frac{\phi_n(t - r/\omega)}{r}\right\} dS \quad \text{..............(6),}$$

in the first term of the second member the differentiation with respect to the normal being operative only on the radius-vector r, while in the second term $t - r/\omega$ is written for t after the differentiation.

This equation still holds, if the luminous points be within the space T and the pole at which the effect is required be without it, provided we regard n as the normal drawn outwards, that is, into the portion of space in which the pole is situated, and ascribe to the function ϕ the ordinary properties of a potential function at infinity. For in this case the integrations have to be extended over the whole of space outside S, and hence the surface-integrals consist of two parts, of which the one is extended over S and has the value just determined, and the second is extended over the surface of a sphere of large radius and vanishes on account of the second of the above assumptions.

58. Let us next suppose that the closed surface S either includes the luminous points as well as the pole, or else includes neither the one nor the other.

Consider the first case and imagine a closed curve drawn on S dividing it into two parts, S_1 and S_2, and through this curve a surface S_3 described so as to include the pole between S_3 and S_1 and to exclude all the luminous points. Then, denoting by Ω the integrand on the right side of (6) and supposing the normals to S_1 and S_2 to be directed inwards, that to S_3 to be directed into the space containing the pole, we have

$$4\pi\phi_0 = \int_{S_1} \Omega dS + \int_{S_3} \Omega dS = -\int_{S_2} \Omega dS + \int_{S_3} \Omega dS,$$

whence

$$\int_{S_1} \Omega dS + \int_{S_2} \Omega dS = 0,$$

or the surface-integral over the closed surface S is zero.

Similarly for the second case: we imagine the surface S_3 drawn through the closed curve on S, so that the pole alone is contained between S_1 and S_3; then, as in the first case,

$$4\pi\phi_0 = \int_{S_3} \Omega dS - \int_{S_1} \Omega dS = \int_{S_3} \Omega dS + \int_{S_2} \Omega dS,$$

whence

$$\int_{S_1} \Omega dS + \int_{S_2} \Omega dS = 0.$$

From this result it follows that the surface-integral is the same for any two unclosed surfaces S and S' having the same bounding curve, provided neither the pole alone, nor the luminous points alone are included in the space between these surfaces. For

$$\int_{S} \Omega dS + \int_{S'} \Omega dS = 0,$$

if the normals to the elements have the same sign, when they are directed either within or without the included space. Hence if we regard the normals

to the two surfaces to be positive, when they are similarly directed with respect to corresponding elements

$$\int_S \Omega dS = \int_{S'} \Omega dS.$$

59. Let us assume that we have a single luminous point and let us call r the distance of the element dS from this point and r_0 the distance of dS from the pole at which the effect is required.

Then if we assume

$$\phi(t) = \frac{A}{r_1} e^{\iota\kappa(\omega t - r_1)}$$

where $\kappa = 2\pi/\lambda$, we have

$$\frac{1}{r_0}\phi(t - r_0/\omega) = \frac{A}{r_1 r_0} e^{\iota\kappa(\omega t - r_1 - r_0)},$$

and $$\frac{\partial}{\partial n}\frac{\phi(t - r_0/\omega)}{r_0} = \frac{A}{r_1}\frac{\partial\left(\dfrac{1}{r_0}\right)}{\partial n} e^{\iota\kappa(\omega t - r_1 - r_0)} - \iota\kappa\frac{A}{r_1 r_0}\frac{\partial r_0}{\partial n} e^{\iota\kappa(\omega t - r_1 - r_0)},$$

since in forming this expression r_0 alone is to be regarded as variable.

On the other hand in forming

$$\phi_n(t - r_0/\omega)$$

we have to differentiate $\phi(t)$ with respect to n, and after differentiation to write $t - r_0/\omega$ for t: hence

$$\frac{\phi_n(t - r_0/\omega)}{r_0} = \frac{A}{r_0}\frac{\partial\left(\dfrac{1}{r_1}\right)}{\partial n} e^{\iota\kappa(\omega t - r_1 - r_0)} - \iota\kappa\frac{A}{r_1 r_0}\frac{\partial r_1}{\partial n} e^{\iota\kappa(\omega t - r_1 - r_0)}.$$

Thus

$$\Omega = \left\{\frac{A}{r_1}\frac{\partial\left(\dfrac{1}{r_0}\right)}{\partial n} - \frac{A}{r_0}\frac{\partial\left(\dfrac{1}{r_1}\right)}{\partial n}\right\} e^{\iota\kappa(\omega t - r_1 - r_0)} + \iota\kappa\frac{A}{r_1 r_0}\left(\frac{\partial r_1}{\partial n} - \frac{\partial r_0}{\partial n}\right) e^{\iota\kappa(\omega t - r_1 - r_0)}.$$

Since λ is a very small quantity, the first term in the expression for Ω is of very slight importance in comparison with the second, and we may write

$$\Omega \doteqdot \iota\kappa\frac{A}{r_1 r_0}\left(\frac{\partial r_1}{\partial n} - \frac{\partial r_0}{\partial n}\right) e^{\iota\kappa(\omega t - r_1 - r_0)} \qquad \text{.....................(7).}$$

If now the surface of resolution be a wave-surface,

$$\partial r_1/\partial n = 1, \qquad \partial r_0/\partial n = -\cos\theta,$$

where θ is the angle between the normal to dS and the line joining the element to the pole, and if the primary disturbance at dS be represented by

$$\frac{A}{r_1}\cos\frac{2\pi}{\lambda}(\omega t - r_1) \qquad \text{.............................(8),}$$

the actual disturbance at the pole due to the element dS is given by

$$\Omega = -\frac{2\pi}{\lambda}\frac{A}{r_1 r_0}(1+\cos\theta)\sin\frac{2\pi}{\lambda}(\omega t - r_1 - r_0) \quad \text{...........(9)},$$

which is Stokes' law of the secondary disturbance*.

60. We now require the value of $\int \Omega dS$ over a surface that is not closed, and this may easily be determined in the case in which Ω has the special value (7) and λ is very small.

With the luminous point and the pole as foci draw a series of spheroids $r_1 + r_0 = \zeta = \text{const.}$, determining a series of curves on the surface S, and define $F(\zeta)$ by the equation

$$F(\zeta) = \pm \int \frac{A}{r_1 r_0}\frac{\partial}{\partial n}(r_1 - r_0)\, dS,$$

the integration being extended over the part of the surface between the lines $\zeta = Z$ and $\zeta = \zeta$ and the $\pm$ sign being taken according as $Z \lesseqgtr \zeta$, so that $F(\zeta)$ increases with ζ whether it be greater or less than Z, if for example $\frac{A}{r_1 r_0}\frac{\partial}{\partial n}(r_1 - r_0)$ be positive. Then if $d\zeta$ be taken positive,

$$\frac{dF}{d\zeta}d\zeta = \int \frac{A}{r_1 r_0}\frac{\partial}{\partial n}(r_1 - r_0)\, dS,$$

in which the integration is extended over the region of the surface between the curves corresponding to the values ζ and $\zeta + d\zeta$.

Let ζ_0 and ζ_1 be the least and the greatest values of ζ on the surface S, then $\int \Omega dS$ is of the form

$$\iota\kappa e^{\iota T}\int_{\zeta_0}^{\zeta_1}\frac{dF}{d\zeta}e^{-\iota\kappa\zeta}d\zeta;$$

whence, integrating by parts,

$$4\pi\phi_0 = -\left[\frac{dF}{d\zeta}e^{-\iota\kappa\zeta}\right]_{\zeta_0}^{\zeta_1}e^{\iota T} + e^{\iota T}\int_{\zeta_0}^{\zeta_1}\frac{d^2F}{d\zeta^2}e^{-\iota\kappa\zeta}d\zeta \quad \text{...........(10)}.$$

Consider first the term

$$\int_{\zeta_0}^{\zeta_1}\frac{d^2F}{d\zeta^2}e^{-\iota\kappa\zeta}d\zeta,$$

and let us divide the interval ζ_0 to ζ_1 into partial intervals, such that in some $\frac{d^2F}{d\zeta^2}$ remains finite, while in the remainder $\frac{d^2F}{d\zeta^2}$ becomes very great of the order κ. Now we may neglect the intervals in which $\frac{d^2F}{d\zeta^2}$ remains finite; for if $\zeta'' - \zeta'$ be one of these intervals, we may assume that within this interval $\frac{d^2F}{d\zeta^2}$ always either increases or decreases, so that the sign of $\frac{d^3F}{d\zeta^3}$

* Stokes, *Camb. Phil. Trans.* IX. 1 (1849); *Math. and Phys. Papers*, II. 243.

remains the same; otherwise we have only to subdivide the interval into smaller parts for which this is the case. Integrating then by parts

$$\int_{\zeta'}^{\zeta''} \frac{d^2F}{d\zeta^2} e^{-\iota\kappa\zeta}\, d\zeta = -\left[\frac{1}{\iota\kappa}\frac{d^2F}{d\zeta^2} e^{-\iota\kappa\zeta}\right]_{\zeta'}^{\zeta''} + \frac{1}{\iota\kappa}\int_{\zeta'}^{\zeta''} \frac{d^3F}{d\zeta^3} e^{-\iota\kappa\zeta}\, d\zeta,$$

and the integrated term may be neglected, since $e^{-\iota\kappa\zeta}$ is < 1 and $\dfrac{d^2F}{d\zeta^2}$ is finite so that the ratio $\dfrac{d^2F}{d\zeta^2}\Big/\kappa$ is very small; also since $\dfrac{d^3F}{d\zeta^3}$ has the same sign throughout the region of integration

$$\int_{\zeta'}^{\zeta''} \frac{d^3F}{d\zeta^3} e^{-\iota\kappa\zeta}\, d\zeta < \int_{\zeta'}^{\zeta''} \frac{d^3F}{d\zeta^3} d\zeta < \frac{d^2F''}{d\zeta^2} - \frac{d^2F'}{d\zeta^2},$$

and this quantity is finite, so that its quotient by κ may be neglected.

Hence the integral in question may be neglected except at parts of the surface at which $\dfrac{d^2F}{d\zeta^2}$ becomes very great of the same order of magnitude as κ.

But we have
$$\frac{dF}{d\zeta} d\zeta = \int \frac{A}{r_0 r_1} \frac{\partial (r_1 - r_0)}{\partial n} dS,$$

and $\dfrac{A}{r_1 r_0} \dfrac{\partial (r_1 - r_0)}{\partial n}$ is finite and continuous: hence the only portions of the surface that can contribute anything to the integral are those at which $d\zeta = 0$.

Consider next the term
$$\left[\frac{dF}{d\zeta} e^{-\iota\kappa\zeta}\right]_{\zeta_0}^{\zeta_1};$$

excluding the case in which $d\zeta = 0$ at any point of the surface, the maximum and minimum values of ζ lie on the bounding curve of S and if we further assume that for no finite portion of this curve ζ is constant, the lines ζ_0 and ζ_1 only touch this curve, in which case $\int dS$ and hence also $\int \dfrac{A}{r_0 r_1} \dfrac{\partial (r_1 - r_0)}{\partial n} dS$ is at these limits an infinitesimal of a higher order than $d\zeta$. Whence for each of these points $\dfrac{dF}{d\zeta} = 0$ and the integrated term of $\int \Omega dS$ vanishes.

Let us now consider in what cases $d\zeta = 0$ at a point of the surface S. Let $g(x, y, z) = 0$ be the equation to the surface, and x, y, z the coordinates of a point on it, at which $d\zeta = 0$ for a displacement of the same on the surface, then

$$\left.\begin{aligned} \frac{\partial \zeta}{\partial x} &\equiv \frac{\partial r_1}{\partial x} + \frac{\partial r_0}{\partial x} = L\frac{\partial g}{\partial x} \\ \frac{\partial \zeta}{\partial y} &\equiv \frac{\partial r_1}{\partial y} + \frac{\partial r_0}{\partial y} = L\frac{\partial g}{\partial y} \\ \frac{\partial \zeta}{\partial z} &\equiv \frac{\partial r_1}{\partial z} + \frac{\partial r_0}{\partial z} = L\frac{\partial g}{\partial z} \end{aligned}\right\} \text{ or } \left\{\begin{aligned} &\cos r_1x + \cos r_0x = M \cos nx \\ &\cos r_1y + \cos r_0y = M \cos ny \\ &\cos r_1z + \cos r_0z = M \cos nz, \end{aligned}\right.$$

where L is an undetermined multiplier and $M = L\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}$.
The conditions for the vanishing of $d\zeta$ can then be satisfied in two ways, either

$$M = 0 \text{ and } \cos r_1 x = -\cos r_0 x, \quad \cos r_1 y = -\cos r_0 y, \quad \cos r_1 z = -\cos r_0 z,$$

or
$$M \neq 0 \text{ and } \frac{\partial \zeta}{\partial x} : \frac{\partial \zeta}{\partial y} : \frac{\partial \zeta}{\partial z} :: \frac{\partial g}{\partial x} : \frac{\partial g}{\partial y} : \frac{\partial g}{\partial z}.$$

In the first case the line joining the luminous point to the pole cuts the surface at the point (x, y, z); and in the second, the spheroid $\zeta = \text{const.}$ touches the surface at this point.

Hence it follows that $\int \Omega dS$ extended over the surface S vanishes when κ is very great, except in the following cases:

(1) when for a finite portion of its bounding curve $r_1 + r_0 = \text{const.}$,

(2) when the line connecting the luminous point and the pole cuts the surface,

(3) when there is contact of any order between the surface and the spheroid $r_1 + r_0 = \text{const.}$

The last case however does not really form an exception: for, as we have seen, $\int \Omega dS$ depends only upon the bounding curve of S and hence in the cases in which the spheroid touches the surface, we can substitute for S another surface with the same contour, for which this is not the case.

The value of $\int \Omega dS$ extended over the surface, when the line from the luminous point to the pole cuts it, may be determined in the following manner. Complete the surface S by a surface including the source and not cut again by the line in question: then the normal to the complete surface being everywhere directed outwards, we have

$$\int \Omega dS = 4\pi\phi_0,$$

the integral being extended over the whole surface: but for the part completing S, we have

$$\int \Omega dS = 0;$$

hence
$$\int_S \Omega dS = 4\pi\phi_0,$$

the integral being extended over S and the normal being directed away from the luminous point. When the normal is directed towards this point, so as to make an obtuse angle with the line joining it to the pole, we have

$$\int_S \Omega dS = -4\pi\phi_0 \quad \text{.............................(11).}$$

The case, in which the line from the source to the pole passes through the bounding curve or infinitely near it, is at present excluded, the value of the integral being then indeterminate.

61. Let us apply these results to the case in which some foreign substance is present in the ether external to the luminous point and to the pole at which the effect is to be determined.

Exclude this body from the region of integration by a surface S drawn infinitely near its surface and let σ be a surface excluding the luminous point: then the normal being reckoned positive when it is directed into the region of integration, we have

$$4\pi\phi_0 = \int \Omega d\sigma + \int \Omega dS,$$

in which equation the value of Ω is changed from what we had before on account of the change in the values of ϕ and ϕ_n occasioned by the introduction of the body.

If, as we are free to assume, the surface σ be a very small sphere, the introduction of the body into the field will cause a comparatively small alteration in the values of ϕ and ϕ_n on the surface of σ, except in certain cases, such as that in which the body is a concave spherical reflector with the luminous point at its centre, and since the sphere is very small, the influence of this change on the integral over its surface is also very small. If then Φ_0 denote the value of ϕ_0 before the introduction of the body

$$\int \Omega d\sigma = 4\pi\Phi_0,$$

and

$$4\pi\phi_0 = 4\pi\Phi_0 + \int \Omega dS \quad \text{.........................(12)},$$

from which equation ϕ_0 can be in general determined, if Φ_0 and the values of ϕ and ϕ_n on S be known.

Let us now suppose that the body is opaque and has a black surface that reflects no light: then on the side turned towards the luminous point, the disturbance is the same as if the body were not there and on the side turned away from the luminous point there is no disturbance at all. Hence in (12) the integration is to be extended over the part of S alone that is turned towards the source of light and is bounded by the tangent cone to S from the luminous point, and in Ω the values of ϕ and ϕ_n are the same as they would be if the body were absent.

It follows then that for points outside the cone, $\int \Omega dS = 0$ and hence $\phi_0 = \Phi_0$, or the presence of the body is without effect: while for points within the cone on the side of the body opposite to the source,

$$\int \Omega dS = -4\pi\Phi_0$$

from (11), and $$4\pi\phi_0 = 4\pi\Phi_0 - 4\pi\Phi_0 = 0,$$
or there is no light at such points.

We are thus led to the laws of geometrical shadows. This result has however been obtained on the assumption that the wave-length is infinitesimal and we have excluded the cases in which the line from the source to the pole passes very nearly through the boundary of S and in which a finite portion of this curve is on one of the spheroids $r_1 + r_0 = \text{const.}$ We then have the phenomena of diffraction.

62. As a further application of the analytical expression of Huygens' principle, let us determine how the vibrations of the polarisation-vector change as we pass along a ray, that is, in what manner ϕ_0 depends upon z_0, where z_0 is the distance of the point considered from a wave-surface, this distance being measured along a normal to the surface*.

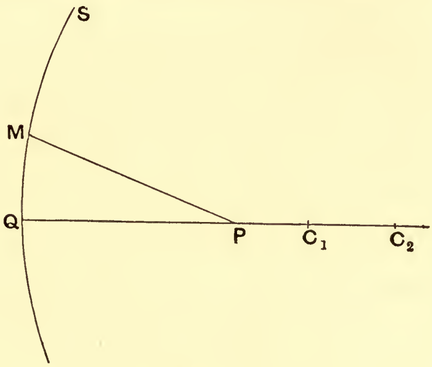


Fig. 19.

Let S represent a wave-surface, P the point at which the effect is to be determined, PQ the normal to S, and C_1, C_2 the centres of principal curvature of the surface at the point Q.

In the system of curvilinear coordinates U, V, W employed in § 54, let the surface $W = \text{const.}$ represent the wave-surface, then the primary disturbance on the surface S may be taken as

$$\phi(t) = Ae^{\iota\kappa(\omega t - W)} \quad\text{.............................(13)},$$

A being a function of U, V, W, all the differential coefficients of which remain finite: taking this surface as that of resolution, the effect at P is given by

$$\phi_0(t) = \frac{\iota\kappa}{4\pi} e^{\iota\kappa(\omega t - W)} \int \frac{A}{r}(1 + \cos\theta)\, e^{-\iota\kappa r}\, dS \quad\text{...........(14)},$$

r being the distance of P from the element dS and θ the angle that the radius-vector makes with the normal to the element.

* Poincaré, *Théorie Math. de la Lumière*, II. 174.

Let us now take Q as the origin of a system of rectangular coordinates, the tangent plane to the surface at Q being the plane of xy and the principal sections of the surface at this point being those of xz and yz: then if ϕ be the angle that the element dS at the point (x, y, z) makes with the plane of xy,

$$dS = dx\,dy/\cos\phi,$$

and

$$\phi_0(t) = \frac{\iota\kappa}{4\pi} e^{\iota\kappa(\omega t - W)} \iint \frac{A(1+\cos\theta)}{r\cos\phi} e^{-\iota\kappa r}\,dx\,dy \quad \text{.........(15)}.$$

Now by drawing a series of spheres round P as centre, so as to determine a number of curves on S, it is easy to show by reasoning analogous to that employed in § 60 that the above integral extended over any part of S is zero, unless this part has contact of any order with one of this system of spheres: it follows then that we may confine the integration to a small area including the point Q, the dimensions of which are actually very small, though large in comparison with the wave-length λ.

The factor $\frac{A}{r}\frac{1+\cos\theta}{\cos\phi}$ will not in general vary very rapidly and will have sensibly the same value over the whole of this area, and we may therefore assign to it the value that it has at the point Q, where

$$r = QP = z_0, \quad \cos\theta = \cos\phi = 1, \quad A = A_0,$$

so that

$$\frac{A}{r}\frac{1+\cos\theta}{\cos\phi} \text{ becomes } \frac{2A_0}{z_0}.$$

On the other hand the variations of $\exp(-\iota\kappa r)$ are rapid, for the differential coefficient of this expression contains κ as a factor and is consequently very great: it thus becomes necessary to determine its value for points on the surface near to Q.

Let f_1, f_2 be the principal radii of curvature at Q, then the equation to the surface is approximately

$$z = \frac{x^2}{2f_1} + \frac{y^2}{2f_2},$$

whence

$$r = \sqrt{x^2 + y^2 + (z_0 - z)^2} \fallingdotseq z_0 + \mu_1 x^2 + \mu_2 y^2,$$

where

$$\mu_1 = \frac{f_1 - z_0}{2f_1 z_0}, \quad \mu_2 = \frac{f_2 - z_0}{2f_2 z_0};$$

hence

$$\phi_0(t) = \frac{\iota\kappa}{2\pi z_0} A_0 e^{\iota\kappa(\omega t - W - z_0)} \iint e^{-\iota\kappa(\mu_1 x^2 + \mu_2 y^2)}\,dx\,dy.$$

Let us now write $\xi = x\sqrt{\kappa}$, $\eta = y\sqrt{\kappa}$, and take, as we are at liberty to do, the form of the area, over which the integration is to be extended, as a small rectangle with its edges parallel to the coordinate axes: then since the dimensions of this area are very large compared with the small quantity λ the limits of the integration for ξ and η are $\pm\infty$, and we have

$$\phi_0(t) = \frac{\iota}{2\pi z_0} A_0 e^{\iota\kappa(\omega t - W - z_0)} \int_{-\infty}^{\infty} e^{-\iota\mu_1\xi^2}\,d\xi \int_{-\infty}^{\infty} e^{-\iota\mu_2\eta^2}\,d\eta \quad \text{......(16)},$$

and since
$$\int_{-\infty}^{\infty} e^{\pm \iota u^2}\, du = (1 \pm \iota)\sqrt{\frac{\pi}{2}},$$

we have
$$\int_{-\infty}^{\infty} e^{-\iota\mu\xi^2}\, d\xi = (1 \mp \iota)\sqrt{\frac{\pi}{\pm 2\mu}},$$

according as μ is positive or negative.

Let us suppose in the first place that the points are in the order Q, P, C_1, C_2, then μ_1 and μ_2 are both positive, and

$$\phi_0(t) = \frac{\iota}{2\pi z_0} A_0 e^{\iota\kappa(\omega t - W - z_0)} \frac{\pi}{2} \frac{(1-\iota)^2}{\sqrt{\mu_1\mu_2}}$$
$$= \sqrt{\frac{f_1 f_2}{(f_1 - z_0)(f_2 - z_0)}} A_0 e^{\iota\kappa(\omega t - W - z_0)} \quad \text{..........(17).}$$

Secondly, if the points occur in the order Q, C_1, P, C_2, μ_1 is negative and μ_2 is positive, whence

$$\phi_0(t) = \frac{\iota}{2\pi z_0} A_0 e^{\iota(\omega t - W - z_0)} \frac{\pi}{2} \frac{(1+\iota)(1-\iota)}{\sqrt{-\mu_1\mu_2}}$$
$$= \iota \sqrt{\frac{f_1 f_2}{(z_0 - f_1)(f_2 - z_0)}} A_0 e^{\iota\kappa(\omega t - W - z_0)} \quad \text{..........(18).}$$

Finally, when P is further from the surface than both C_1 and C_2, both μ_1 and μ_2 are negative, and

$$\phi_0(t) = \frac{\iota}{2\pi z_0} A_0 e^{\iota(\omega t - W - z_0)} \frac{\pi}{2} \frac{(1+\iota)^2}{\sqrt{\mu_1\mu_2}}$$
$$= -\sqrt{\frac{f_1 f_2}{(z_0 - f_1)(z_0 - f_2)}} A_0 e^{\iota(\omega t - W - z_0)} \quad \text{..........(19).}$$

Now the actual effect being represented by the real part of the above expressions, we see that on traversing a ray from a wave-surface the phase changes suddenly by $\pi/2$ on passing through either of the centres of curvature, and in calculating the retardation it thus becomes necessary to subtract $\lambda/4$ from the actual length of path on crossing either of the points*.

The points C_1 and C_2 are called the focal points of the ray and near them the disturbance becomes very great. When the focal points coincide, the retardation is obtained from the actual distance by subtracting $\lambda/2$ on traversing this common point.

63. In considering the application of Huygens' principle to the determination of the effect of a black screen of any form placed in the vicinity of a luminous point, we excluded certain cases and to these we must now turn our attention.

* For an experimental verification of this result see Gouy, *Ann. de Ch. et de Phys.* (6) XXIV. 197 (1891).

Suppose that we have a luminous point C with a perfectly black screen near it: round the screen draw a closed surface at all points infinitely near that of the screen itself and divide this surface into two parts by the line of contact of the surface with a tangent cone having its vertex at C. Let us call the part of the surface turned towards the luminous point S' and that turned

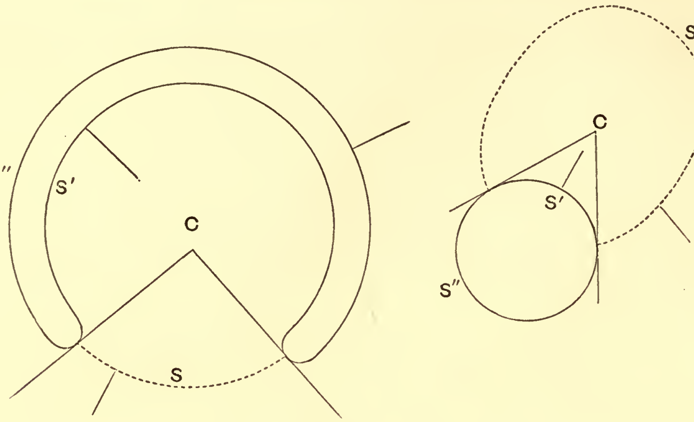


Fig. 20.

away from it S'', and let us complete these surfaces by another surface S, in such a way that C is entirely enclosed by the surfaces $S+S'$ and $S+S''$, while the point O at which the effect is to be determined is excluded.

Then if ϕ denote any one of the components of the polarisation-vector or an allied function satisfying the relation

$$\frac{\partial^2\phi}{\partial t^2} = \omega^2\nabla^2\phi \quad \text{.................................(20)},$$

and ϕ_0 be the value of ϕ at the point O, Φ the value that ϕ would have at any point, if the screen were removed, we have at all points of S and S'

$$\phi = \Phi, \quad \frac{\partial\phi}{\partial n} = \frac{\partial\Phi}{\partial n} \quad \text{...........................(21)},$$

and at all points of S''

$$\phi = 0, \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{..............................(22)},$$

and taking $S+S''$ as the surface of resolution, Huygens' principle gives

$$\phi_0 = \frac{1}{4\pi}\int_{S+S''}\Omega dS = \frac{1}{4\pi}\int_S \Omega dS \quad \text{....................(23)},$$

since the integral extended over S'' vanishes, where

$$\Omega = \frac{\partial}{\partial n}\frac{\phi(t-r_0/\omega)}{r_0} - \frac{\phi_n(t-r_0/\omega)}{r_0} \quad \text{.................(24)},$$

the normal being reckoned positive when it is directed on the side of S on which the point O is situated.

This expression for Ω may be simplified, if the only part of the surface of resolution, at which ϕ and $\partial\phi/\partial n$ do not vanish, be plane*.

Write

$$\psi = \frac{1}{2\pi}\int \frac{\partial}{\partial n}\frac{\phi(t-r_0/\omega)}{r_0}dS, \quad \chi = -\frac{1}{2\pi}\int\frac{\phi_n(t-r_0/\omega)}{r_0}dS,$$

so that $$2\phi_0 = \psi + \chi.$$

Both ψ and χ satisfy the differential equation for ϕ and it is easy to show that they also satisfy the surface conditions, if the surface be plane. For writing

$$\psi' = \frac{1}{2\pi}\int\frac{\partial}{\partial n}\frac{\phi(t)}{r_0}dS,$$

then ψ' is the potential of a double layer, the density being $\pm\,\phi(t)/2\pi$: such a potential shows a discontinuity at the surface equal to $2\phi(t)$. On the other hand $\psi - \psi'$ is continuous at the surface, as it remains finite when $r_0 = 0$, and hence ψ shows a discontinuity equal to $2\phi(t)$, whence if the surface be plane ψ takes at the surface the same value as ϕ and consequently the same holds for χ.

Again, let $$\chi' = -\frac{1}{2\pi}\int\frac{\phi_n(t)}{r_0}dS,$$

so that χ' is the potential due to a surface covered with an attracting mass, of which the density is $-\phi_n(t)/2\pi$. Then $\chi - \chi'$ and its differential coefficients are continuous at the surface, and since χ' is continuous, χ is so as well. Also

$$\frac{\partial\chi}{\partial n} = \frac{\partial(\chi-\chi')}{\partial n} + \frac{\partial\chi'}{\partial n},$$

and $\partial\chi'/\partial n$ is discontinuous and changes suddenly by $2\phi_n(t)$ on crossing the surface: hence since $\partial(\chi-\chi')/\partial n$ is continuous, $\partial\chi/\partial n$ shows a discontinuity at the surface equal to $2\phi_n(t)$, and therefore at the plane surface

$$\frac{\partial\chi}{\partial n} = \phi_n(t) = \frac{\partial\phi}{\partial n}.$$

Thus the differential coefficient of χ at the surface is the same as that of ϕ, and the same must hold for ψ.

Hence both ψ and χ satisfy all the conditions, and taking the axis of z in the direction of the normal, we have for the special case under consideration

$$2\pi\phi_0(t) = \int\frac{\partial}{\partial z}\frac{\phi(t-r_0/\omega)}{r_0}dS \quad \text{......................(25)}.$$

Let $$\phi(t) = \frac{A}{r_1}e^{\iota\kappa(\omega t - r_1 - \delta)} \quad \text{..............................(26)};$$

* Schuster, *Phil. Mag.* (5) XXXVII. 543 (1894).

then retaining those terms alone that contain κ as a factor, we have

$$\phi_0(t) = -\frac{A}{\lambda} e^{\iota\kappa\omega t} \int \iota \frac{1}{r_1 r_0} \frac{\partial r_0}{\partial z} e^{-\iota\kappa(r_1+r_0+\delta)} dS$$

$$\doteqdot -\frac{A}{\lambda} \frac{1}{r_1 r_0} \frac{\partial r_0}{\partial z} e^{\iota\kappa\omega t} \int \iota e^{-\iota\kappa(r_1+r_0+\delta)} dS \text{ ....(27)},$$

since $r_1, r_0, \partial r_0/\partial z$ vary but slowly over the part of the surface that contributes sensibly to the value of the integral.

A point in the aperture or very near to it being taken as origin, let x_1, y_1, z_1 be the coordinates of the luminous point and x_0, y_0, z_0 those of the point at which the effect is considered: then, if

$$\rho_1^2 = x_1^2 + y_1^2 + z_1^2, \quad \rho_0^2 = x_0^2 + y_0^2 + z_0^2,$$

we have

$$r_1 = \rho_1 - \frac{xx_1 + yy_1}{\rho_1} + \frac{1}{2} \frac{x^2 + y^2}{\rho_1},$$

$$r_0 = \rho_0 - \frac{xx_0 + yy_0}{\rho_0} + \frac{1}{2} \frac{x^2 + y^2}{\rho_0},$$

the dimensions of the aperture being supposed to be small, and writing $\delta + \rho_0 + \rho_1 = -\epsilon$ we obtain

$$\phi_0(t) = -\iota \frac{A}{\lambda} \frac{1}{\rho_1 \rho_0} \frac{\partial \rho_0}{\partial z} e^{\iota\kappa(\omega t+\epsilon)}$$

$$\times \iint e^{\iota\kappa\left\{x\left(\frac{x_1}{\rho_1}+\frac{x_0}{\rho_0}\right)+y\left(\frac{y_1}{\rho_1}+\frac{y_0}{\rho_0}\right)-\frac{x^2+y^2}{2}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)\right\}} dxdy \text{..........(28)}.$$

64. Let us first suppose that the terms involving the square of the coordinates of a point of the aperture vanish: this will occur when

$$\rho_0 = \rho_1 = \infty \quad \text{or} \quad \text{when } \rho_0 = -\rho_1.$$

When $\rho_0 = \rho_1 = \infty$, the source of light is at infinity and the waves incident upon the aperture are plane: the secondary waves from the aperture are parallel and interfere at an infinitely distant point. On account of the optical equivalence of paths between the conjugate foci of a lens system, this case may be realised with a spectrometer, in which both collimator and telescope are focussed on infinity and the diffraction screen is placed between them.

When $\rho_0 = -\rho_1$ and ρ_1 is negative, the wave incident on the aperture is spherical and concave to its direction of propagation, and the centre of the sphere is on the screen of observation. This case is obtained upon a screen by placing the aperture between the screen and a lens adjusted to give upon it an image of the radiant point.

When $\rho_0 = -\rho_1$ and ρ_1 is positive, the incident wave is spherical and convex to its direction of propagation: the diffraction phenomena are virtual

and apparently formed on a screen through the source of light. This is the case of an aperture held in front of the eye or of the object-glass of a telescope adjusted for distant vision of the source of light.

In these cases we have what are known as Fraunhofer's diffraction phenomena.

When the term $(x^2+y^2)(\rho_1^{-1}+\rho_0^{-1})/2$ is not zero, we have Fresnel's diffraction phenomena: these occur near the limits of the geometrical shadow of the screen, and in the main correspond to the case in which the line from the radiant point to that at which the effect is required passes very nearly through the edge of the screen. In the case of this kind of diffraction we have often to deal with apertures that are very large, but as the only effective part of the aperture is that near the point in which it is met by the line from the radiant point to the pole, we may still employ the values of r_1 and r_0 obtained above and may extend the limits of integration as far as we please from the limiting line of the aperture, provided we go far enough.

65. The above results have been deduced from the formula

$$4\pi\phi_0 = \int \Omega dS,$$

in which the integration is extended over the apertures in the diffraction screen, but we might employ a formula, in which we have to integrate over the opaque parts: for describing a small sphere σ round the luminous point, Huygens' principle gives

$$4\pi\phi_0 = \int_\sigma \Omega dS + \int_{S'} \Omega dS + \int_{S''} \Omega dS,$$

where the second and third integrals are extended over the parts of the surface surrounding the screen, that are turned towards and away from the radiant point respectively.

Now the introduction of the screen has only a very slight effect on the values of ϕ and $\partial\phi/\partial n$ on the sphere σ, and since this surface is very small we have

$$\int_\sigma \Omega dS = 4\pi\Phi_0,$$

where Φ_0 is the value of ϕ_0 before the introduction of the screen; also since ϕ and $\partial\phi/\partial n$ are zero at all points of S'',

$$\int_{S''} \Omega dS = 0,$$

whence

$$4\pi\phi_0 = 4\pi\Phi_0 + \int_{S'} \Omega dS.$$

Suppose now that we have two cases, that only differ from one another by

the interchange of the opaque and transparent portions of the screen: then in the one case we have

$$4\pi\phi_0 = \int \Omega dS,$$

the integration being extended over the apertures and the normal to dS being directed away from the radiant point: in the second case we have

$$4\pi\phi_0' = 4\pi\Phi_0 + \int \Omega' dS,$$

the integration being now taken over the opaque parts of the screen and the normal to dS being directed towards the luminous point. But the opaque parts in the second case being transparent parts of the first case, and the normals in the two cases being oppositely directed, we have

$$\int \Omega' dS = -\int \Omega dS,$$

whence

$$\phi_0' = \Phi_0 - \phi_0 \dots .$$

Now Φ_0 is the value of ϕ_0 at the point under consideration when there is no screen. In the case of Fraunhofer's diffraction phenomena this is zero, except at the image of the radiant point, and therefore at all other points

$$\phi_0' = -\phi_0.$$

Thus the intensity at all points, except at the image of the radiant point, is the same in the two cases and the pattern has the same form when the diffraction screen is generally transparent and studded over with opaque discs, as when it is generally opaque and perforated with exactly corresponding apertures*.

In the case of Fresnel's phenomena this is not so, for then $\Phi_0 \neq 0$ and the disturbances corresponding to Φ_0 and ϕ_0 have different phases and give rise to interference, that modifies the intensity and changes the character of the pattern.

* Babinet, *C. R.* IV. 638 (1837).

CHAPTER VII.

FRAUNHOFER'S DIFFRACTION PHENOMENA*.

66. THE formula relating to the case of Fraunhofer's diffraction phenomena may be written in the form

$$\phi_0(t) = -\frac{A}{\lambda}\frac{1}{\rho_1\rho_0}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\iint \iota e^{\iota(px+qy)}\,dx\,dy \qquad \text{.................(1)}$$

where $$p = \frac{2\pi}{\lambda}\frac{\xi}{\rho_0}, \quad q = \frac{2\pi}{\lambda}\frac{\eta}{\rho_0},$$

ξ and η being the coordinates of the point considered on the screen of observation relatively to the image of the radiant point as origin.

Before proceeding to apply this formula to the diffraction patterns produced by apertures of special form, let us first consider some general properties of the solution†:

(*a*) If the wave-length vary, the aperture being given, the composition of the integral is unaltered, provided ξ and η be taken inversely as λ. Thus a diminution of λ leads to a simple proportional contraction of the diffraction pattern, accompanied by an augmentation of brilliancy proportional to λ^{-2}.

(*b*) If we write mx for x and ny for y, the wave-length remaining unaltered, then writing ξ/m for ξ and η/n for η, $\phi_0(t)$ becomes $mn\phi_0(t)$ and the intensity becomes m^2n^2I: hence the linear dimensions of the diffraction pattern are inversely as those of the aperture and the brilliancy at corresponding points is as the square of the dimensions of the aperture.

Thus it is possible to deduce from the pattern due to any aperture, that given by an aperture formed from it by an alteration of the abscissæ and the ordinates of its boundary in any given ratios: thus the pattern due to an elliptic boundary may be obtained from that given by a circular hole.

The shrinkage of the diffraction pattern consequent on the increase in the dimensions of the aperture has an important bearing on the theory of optical instruments. According to geometrical optics, the images of two

* Schwerd, *Die Beugungserscheinungen*, Mannheim, 1835.

† Lord Rayleigh, *Encycl. Brit.*, Article "Wave Theory," Vol. xxiv. p. 430.

radiant points are regarded as distinct, however close they may be: in other words, the pattern due to each is supposed to be infinitely small, or which is the same, the wave-length is assumed to be infinitesimal. The fact that the wave-length is finite imposes a limit on the resolving or separating power of an optical instrument.

In order that the image of a radiant point may be sharp, the illumination must become insensible at points very near the geometrical focus, and this can only be effected by discrepancies of phase among the secondary waves from the elements of the aperture. Whatever may be the discrepancy of phase that is required to cause a marked reduction in the illumination, it is clear that the larger the aperture the less it is necessary to deviate from the principal direction in order to obtain the specified discrepancy and consequently the smaller will be the image*.

(*c*) If the wave-length and the scale of the aperture increase in the same proportion, the size and form of the pattern remain unchanged.

(*d*) Suppose that there are n equal, similar and similarly situated apertures in the diffraction screen, and let a_h, b_h ($h=1, 2, \ldots n$) be the coordinates of corresponding points of the n apertures, and suppose moreover that these apertures are covered with retarding plates, δ_h being the retardation of phase introduced by that covering the hth aperture. Then

$$\phi_0(t) = -\frac{A}{\lambda}\frac{1}{\rho_1\rho_0}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\sum_1^n\iint \iota e^{\iota\{p(a_h+x')+q(b_h+y')-\delta_h\}}dx'dy' \quad \ldots\ldots(2),$$

the integration being extended over a single aperture. Writing

$$K = \sum_1^n \cos(pa_h + qb_h - \delta_h), \qquad \Sigma = \sum_1^n \sin(pa_h + qb_h - \delta_h),$$

$$c = \iint \cos(px+qy)\,dxdy, \qquad s = \iint \sin(px+qy)\,dxdy,$$

we have
$$\phi_0(t) = -\frac{A}{\lambda}\frac{1}{\rho_1\rho_0}\frac{\partial\rho_0}{\partial z}\iota(c+\iota s)(K+\iota\Sigma)e^{\iota\kappa\omega t} \quad \ldots\ldots\ldots\ldots\ldots(3),$$

and the intensity is

$$I = \frac{A^2}{\lambda^2}\frac{1}{\rho_1^2\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2(c^2+s^2)(K^2+\Sigma^2)$$
$$= (K^2+\Sigma^2)I_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where I_0 is the intensity due to a single aperture.

If the apertures be arranged so that their corresponding points are in lines parallel to the axis of x and equidistant from one another, then

$$a_h = (h-1)\sigma, \quad b_h = 0,$$

and if in addition

$$\delta_h = (h-1)\delta,$$

* Lord Rayleigh, *loc. cit.*

we have $$K + \iota\Sigma = \sum_1^n e^{\iota(h-1)(p\sigma-\delta)} = (e^{\iota n(p\sigma-\delta)} - 1)/(e^{\iota(p\sigma-\delta)} - 1),$$

whence

$$K^2 + \Sigma^2 = \sin^2\left(\frac{np\sigma-\delta}{2}\right)\Big/\sin^2\left(\frac{p\sigma-\delta}{2}\right) = \sin^2 nu/\sin^2 u, \text{ say,}$$

and $$I = n^2 \frac{\sin^2 nu}{n^2 \sin^2 u} I_0 \quad \text{..........(5).}$$

The maxima and minima values of the second factor of the right-hand member of this equation occur when

$$\frac{\sin nu}{n \sin u} \frac{n \sin u \cos nu - \cos u \sin nu}{n \sin^2 u} = 0,$$

giving the two equations

$$\frac{\sin nu}{n \sin u} = 0, \qquad \frac{n \sin u \cos nu - \cos u \sin nu}{n \sin^2 u} = 0 \quad \text{..........(6).}$$

The roots of $\sin nu/(n \sin u) = 0$ are given by $nu = k\pi$, k being an integer that is not a multiple of n: these values of u annul the expression for the intensity and thus determine the position of the minima. When k is divisible by n, the expression takes the form $0/0$, the value of which is found to be unity.

The maxima of the second factor in the expression for the intensity are determined by the roots of the second of the equations (6), or of

$$\tan nu = n \tan u \quad \text{..........(7).}$$

These values of u may be classed in two groups, according as they annul both $\tan nu$ and $\tan u$, or neither of these quantities. The first group of values are given by $u = m\pi$, and to these correspond what may be called the principal maxima, the value of the intensity becoming

$$I = n^2 I_0.$$

In addition to these maxima, there is a series of secondary maxima corresponding to the second group of the roots of (6). Now between two consecutive principal maxima there are $n-1$ minima: hence since maxima and minima must occur alternately, there are $n-2$ secondary maxima between two consecutive principal maxima. Writing (7) in the form

$$\frac{\sin^2 nu}{1 - \sin^2 nu} = \frac{n^2 \sin^2 u}{1 - \sin^2 u},$$

we deduce $$\frac{\sin^2 nu}{n^2 \sin^2 u} = \frac{1}{1 + (n^2-1)\sin^2 u},$$

so that at the secondary maxima the intensity is

$$I = \frac{n^2}{1 + (n^2-1)\sin^2 u} I_0,$$

whence it follows that the intensity at the secondary maxima is in general less than that at the principal maxima and the more so the greater the number of apertures.

Thus on certain lines parallel to the axis of y the illumination will be increased, while on others it will be annulled, and the pattern due to a single aperture thus appears to be traversed by parallel dark lines, that are the nearer together the greater the distance of the apertures from one another and the greater their number.

When the apertures are very numerous and very close together, the pattern of the single aperture may be very considerably modified, and in this case the effect of the factor I_0 in the expression for the intensity is chiefly shown by a reduction in the intensity of the successive principal maxima, some of which may actually disappear owing to the vanishing of I_0.

67. Having established these general results respecting Fraunhofer's diffraction phenomena, we may now pass to the consideration of the patterns produced by some of the more important forms of apertures.

In the first place let us suppose that we have a rectangular hole of width $2a$ in the direction of x and length $2b$ parallel to y, and that this has properties such that a disturbance of unit amplitude incident at a distance x from its central line becomes a disturbance of amplitude $\cos \alpha x$, where α is a constant*.

Such an aperture may be called a simple grating, the length of a complete period of which is $\sigma = 2\pi/\alpha$, so that if there be N such periods in it, we have $N\pi = a\alpha$, N being necessarily even.

Then the origin being taken at the centre of the aperture

$$\phi_0(t) = -\frac{\iota}{2}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\int_{-a}^{a}\int_{-b}^{b}\{e^{\iota(p+\alpha)x} + e^{\iota(p-\alpha)x}\}e^{\iota qy}\,dx\,dy$$

$$= -\frac{\iota}{2}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\,.\,2\left\{\frac{\sin(p+\alpha)a}{p+\alpha} + \frac{\sin(p-\alpha)a}{p-\alpha}\right\}\frac{2\sin qb}{q},$$

whence, remembering that $a\alpha$ is a multiple of 2π

$$\phi_0(t) = -\iota\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{4p}{p^2-\alpha^2}\sin pa\frac{\sin qb}{q}e^{\iota\kappa\omega t}$$

$$= -\iota\frac{A}{\lambda}\frac{4ab}{\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{p}{p\pm\alpha}\frac{\sin\frac{N\pi}{\alpha}(p\mp\alpha)}{\frac{N\pi}{\alpha}(p\mp\alpha)}\frac{\sin qb}{qb}e^{\iota\kappa\omega t} \quad\text{............(8)},$$

and the intensity is

$$I = \left(\frac{A}{\lambda}\frac{4ab}{\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{p}{p\pm\alpha}\right)^2\left\{\frac{\sin\frac{N\pi}{\alpha}(p\mp\alpha)}{\frac{N\pi}{\alpha}(p\mp\alpha)}\right\}^2\left\{\frac{\sin qb}{qb}\right\}^2 \quad\text{...............(9)}.$$

* Schuster, *Phil. Mag.* (5), XXXVII. 509 (1894).

The last two factors of this expression have the form $\sin^2 u/u^2$, the minima values of which occur when $u = m\pi$ $(m = 1, 2, \ldots)$ and the maxima values are given by $u = 0$ and the roots of the equation

$$u = \tan u$$

which has already been discussed in § 33.

Thus there are two diffraction patterns grouped about the points given by

$$p \mp \alpha = 0, \quad q = 0,$$

or

$$\xi = \pm \frac{\rho_0}{\sigma} \lambda, \quad \eta = 0,$$

and these are traversed by dark lines, of which the equations are

$$\xi = \pm \left(1 \pm \frac{m}{N}\right) \frac{\rho_0}{\sigma} \lambda, \quad \eta = \pm m \frac{\rho_0}{b} \lambda.$$

Within the rectangles contained by pairs of consecutive lines and not far from their centres the brightness rises to a maximum, but the intensity at these points falls considerably below that at the centres of the patterns.

If N be very great, the successive maxima along the axis of ξ are very close together, so that the whole light is concentrated near the lines

$$\xi = \pm \rho_0 \lambda / \sigma.$$

68. In the case of a luminous line parallel to the sides of length $2b$ of the rectangle, the intensity may be represented by

$$\int I dy_1 = \frac{\rho_1}{\rho_0} \int I d\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

the integration being extended from a large negative to a large positive value of η, the largeness being estimated by comparison with $\lambda \rho_0 / b$. Since b is supposed moderately large, the whole diffraction pattern would occupy but a very small portion of the field in the direction of y, so that we may without sensible error suppose the limits of η to be $\pm \infty$. We have then for the expression of the intensity

$$\frac{\rho_1}{\rho_0} \int_{-\infty}^{\infty} I d\eta = A^2 \frac{8a^2 b}{\rho_0^2 \rho_1 \lambda} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \left(\frac{p}{p \pm \alpha}\right)^2 \frac{\sin^2 \dfrac{N\pi}{\alpha}(p \mp \alpha)}{\left\{\dfrac{N\pi}{\alpha}(p \mp \alpha)\right\}^2} \quad \ldots\ldots\ldots (11),$$

the same law as for a luminous point when horizontal directions are alone considered.

69. The formulæ relating to a simple rectangular aperture are obtained from (9) and (11) by writing $\alpha = 0$, $N\pi/\alpha = a$ in these formulæ. We see then that in the case of a rectangular aperture the definition of the image of a vertical line is independent of the vertical length of the aperture. The

distribution of brightness* in the diffraction-pattern of the line is shown by the curve ABC representing the values of $u^{-2}\sin^2 u$ from $u=0$ to $u=2\pi$; the line OA is a line of symmetry, the part of the curve corresponding to negative values of u being similar to ABC.

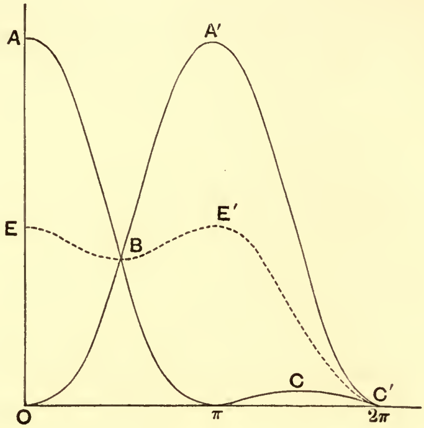


Fig. 21.

Suppose now that the subject of examination is a double line, the components of which have equal brightness and are at such an angular interval that the centre of the pattern due to the one falls on the first minimum of intensity in the pattern of the other. The curve of illumination for the second line will be $OA'C'$ and that representing half the combined brightness will be $E'BE$. At the point B midway between the central points of the two patterns, the intensity is ·8106 of that of the central points themselves, and this is considered to be about the limit at which there would be any decided appearance of resolution of the lines. But in the case considered the angle subtended by the components of the double line at the aperture is $\lambda/2a$, $2a$ being the horizontal aperture: hence, in order that a double line may be resolved, its components must subtend an angle exceeding that subtended by the wave-length of light at a distance equal to the horizontal aperture.

Let us consider the application of this result to the determination of the resolving power of a prism†. Let A_0B_0 be a plane wave-surface of the light before it falls upon the prism, AB the corresponding wave-surface of a definite part of the spectrum after the light has passed through the prism.

The path of any ray from the wave-surface A_0B_0 to A or B is determined by the condition that the optical distance $\int \mu ds$ is a minimum, and as AB is

* Lord Rayleigh, *Enc. Brit.* xxiv. 431; *Phil. Mag.* (5), viii. 261 (1879).

† Lord Rayleigh, *loc. cit.* p. 271.

supposed to be a wave-surface, this distance is the same for both points. Thus

$$\int \mu ds \text{ (for } A) = \int \mu ds \text{ (for } B) \quad \text{.......................(12).}$$

Now when light of a neighbouring part of the spectrum is considered, we may, though the path of the ray from A_0B_0 is changed, neglect this alteration in calculating the optical distance, since in virtue of the minimum

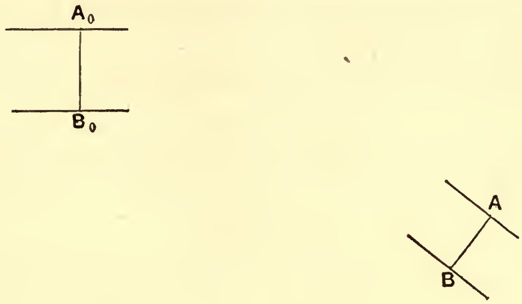


Fig. 22.

property it affects the result by quantities of the second order only in the change of refrangibility. Hence the optical distance from A_0B_0 to A is $\int (\mu + \delta\mu)\, ds$, the integration being along the path $A_0 \ldots A$, that from A_0B_0 to B is given by $\int (\mu + \delta\mu)\, ds$, the integration being along $B_0 \ldots B$. Thus from (12) the difference in the optical distances is

$$\int \delta\mu ds \text{ (along } B_0 \ldots B) - \int \delta\mu ds \text{ (along } A_0 \ldots A).$$

The new wave-surface is formed in such a position that the optical distance is constant and hence the dispersion, or the angle through which the wave-surface is turned in consequence of the change in the refrangibility, is the ratio of the above difference to AB.

If there be only one dispersive substance, $\int \delta\mu ds = \delta\mu \,.\, s$ where s is the thickness traversed by the ray: hence denoting by s_1 and s_2 the distances within the prism traversed by the extreme rays, the dispersion is represented by

$$\frac{s_2 - s_1}{a} \delta\mu$$

where a is the width of the emergent beam.

In general s_1 is small and s_2 is the aggregate thickness of the prisms at their thick ends: calling this t, the dispersion θ is given by

$$\theta = t\delta\mu / a \quad \text{...............................(13).}$$

But the condition for the resolution of a double line, the components of which subtend an angle θ, is that θ exceed λ/a. Hence in order that a double line with components having indices μ and $\mu+\delta\mu$ may be resolved, it is necessary that t should exceed the value given by

$$t=\lambda/\delta\mu \qquad \text{......(14).}$$

This expresses that the relative retardation $t\delta\mu$ of the extreme rays caused by the change of refrangibility is the same, λ, as that incurred on passing from the principal direction to that of the first minimum of illumination, when the refrangibility is unaltered.

If we assume Cauchy's formula $\mu = A + B\lambda^{-2}$, then

$$\delta\mu = -2B\lambda^{-3}\delta\lambda.$$

In the case of Chance's "extra-dense flint" the indices for C and the D line of lower period are

$$\mu_C = 1{\cdot}644866, \quad \mu_D = 1{\cdot}650388,$$

also $$\lambda_C = 6{\cdot}562 \times 10^{-5}\ \text{(cent.)}, \quad \lambda_D = 5{\cdot}89 \times 10^{-5}\ \text{(cent.)};$$

$$\therefore\ B = {\cdot}984 \times 10^{-10}$$

and $$t=\frac{\lambda^4}{2B\delta\lambda}=\frac{10^{10}\lambda^4}{1{\cdot}968\,\delta\lambda}.$$

For the soda-lines $\delta\lambda = {\cdot}006 \times 10^{-5}$ and thus the thickness necessary to resolve these lines is

$$t = 1{\cdot}02\ \text{(cent.)}.$$

The number of times the power of the spectroscope exceeds that required to resolve the D lines may be taken as its practical measure: thus in the case of an instrument with simple prisms of "extra-dense" glass, the power is expressed nearly by the number of centimetres of available thickness.

70. In order to increase the resolving power of a rectangular aperture, it is necessary to reduce the width of the central band, and this may be effected by the suppression of the secondary waves from the central part of the aperture. At the same time, since this has the result of increasing the brilliancy of the succeeding bright bands of the diffraction pattern, care must be taken not to carry this suppression of the central waves too far*.

As an example of this result, let us take the case in which the aperture is reduced to two narrow slits of width $2e$ at its edges: then if $2d$ be the distance between the centres of the slits, so that $d+e=a$, the formula obtained in § 66 for a number of similar and similarly situated apertures gives

$$I=\frac{64A^2b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\left(\frac{\sin pe}{pe}\right)^2\left(\frac{\sin qb}{qb}\right)^2\cos^2 pd \qquad \text{............(15),}$$

* Lord Rayleigh, *loc. cit.*

or in the case of a luminous line parallel to the slits

$$I = \frac{32A^2be^2}{\lambda\rho_0^2\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\left(\frac{\sin pe}{pe}\right)^2\cos^2 pd \text{ .... } (16).$$

Now d being large compared with e, the fluctuations of $(\sin pe)^2/(pe)^2$ are very slow compared with those of $\cos^2 pd$ and consequently the centre of the pattern consists of a number of equidistant fine bands of equal brightness, so that the arrangement is useless for the purposes of resolution.

Michelson* has however shown that by making the distance between the slits adjustable, the variation of the visibility of the bands affords a means of measuring the angular magnitudes of small sources of light and of resolving these sources when double.

If $\phi(x_1, y_1)$ be the intensity of illumination at the point (x_1, y_1) of the source, then at any point of the diffraction pattern the intensity will be

$$J = \iint I\phi(x_1, y_1)\,dx_1dy_1.$$

Now if the angular dimensions of the source be small compared with λ/b† and we confine our attention to the brightest part of the field, we may write

$$\left(\frac{\sin pe}{pe}\right)^2\left(\frac{\sin qb}{qb}\right)^2 = 1$$

throughout the range of integration, and we obtain

$$J = \frac{64b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\iint\cos^2 pd\,.\,\phi(x_1, y_1)\,dx_1dy_1 \text{ .... } (17).$$

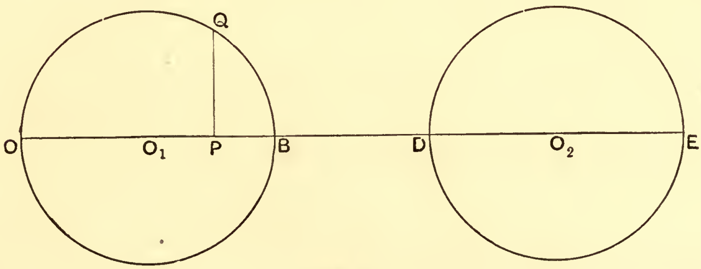


Fig. 23.

Suppose that we have two equal symmetrical sources of uniform illumination with their centres at O_1 and O_2 in the line of separation of the slits. Let

$$OO_1 = O_1B = DO_2 = O_2E = r,$$

$$O_1O_2 = 2s, \quad OP = x_1, \quad PQ = y_1,$$

* Michelson, *Phil. Mag.* (5), XXX. 1 (1890); XXXI. 256 (1891).

† For a discussion of other cases see Hamy, *Bull. Astron.* X. 489 (1893), XI. 48 (1894); *C. R.* CXXVII. 851 (1898). Filon, *Phil. Mag.* (5) XLVII. 441 (1899).

then integrating from O to B and from D to E

$$J=\frac{64A^2b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\left\{\int_0^{2r}f(x_1-r)\left\{1+\cos\frac{4\pi d}{\lambda}\left(\frac{x_1}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}dx_1\right.$$
$$\left.+\int_{2s}^{2s+2r}f(x_1-2s-r)\left\{1+\cos\frac{4\pi d}{\lambda}\left(\frac{x_1}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}dx_1\right\}.$$

In the first integral put $x_1-r=w$, and in the second $x_1-2s-r=w$, then

$$J=\frac{64A^2b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\left[\int_{-r}^{r}f(w)\left\{1+\cos\frac{4\pi d}{\lambda}\left(\frac{w}{\rho_1}+\frac{r}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}dw\right.$$
$$\left.+\int_{-r}^{r}f(w)\left\{1+\cos\frac{4\pi d}{\lambda}\left(\frac{w}{\rho_1}+\frac{r+2s}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}dw\right]\text{.........(18)}.$$

Expanding the first of these integrals we obtain

$$\int_{-r}^{r}f(w)\,dw+\cos\frac{4\pi d}{\lambda}\left(\frac{r}{\rho_1}+\frac{x_0}{\rho_0}\right)\int_{-r}^{r}f(w)\cos\frac{4\pi dw}{\lambda\rho_1}dw$$
$$-\sin\frac{4\pi d}{\lambda}\left(\frac{r}{\rho_1}+\frac{x_0}{\rho_0}\right)\int_{-r}^{r}f(w)\sin\frac{4\pi dw}{\lambda\rho_1}dw,$$

in which the first term is half the area of either constituent of the source and the last term is zero, since $f(w)$ is a symmetrical function. The same is true of the second integral. Writing then

$$\int_{-r}^{r}f(w)\,dw=\tfrac{1}{2}\Omega,\quad \int_{-r}^{r}f(w)\cos\frac{4\pi dw}{\lambda\rho_1}dw=\tfrac{1}{2}Q\Omega\quad\text{............(19)},$$

we obtain

$$J=\frac{64A^2b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\left[\Omega+\tfrac{1}{2}Q\Omega\left\{\cos\frac{4\pi d}{\lambda}\left(\frac{r}{\rho_1}+\frac{x_0}{\rho_0}\right)+\cos\frac{4\pi d}{\lambda}\left(\frac{r+2s}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}\right]$$
$$=\frac{64A^2b^2e^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\Omega\left\{1+Q\cos\frac{4\pi ds}{\lambda\rho_1}\cos\frac{4\pi d}{\lambda}\left(\frac{r+s}{\rho_1}+\frac{x_0}{\rho_0}\right)\right\}\quad\text{............(20)},$$

whence we have for the visibility of the fringes

$$V=Q\cos\frac{4\pi ds}{\lambda\rho_1}\quad\text{............................(21)}.$$

With a single source the visibility is Q.

Thus with two equal rectangles of height $2h$, we have

$$f(w)=h,\quad \Omega=4hr,$$

$$Q=\frac{1}{r}\int_0^r\cos\frac{4\pi dw}{\lambda\rho_1}dw=\left(\sin\frac{4\pi dr}{\lambda\rho_1}\right)\Big/\left(\frac{4\pi dr}{\lambda\rho_1}\right),$$

$$V=\frac{\sin\dfrac{4\pi dr}{\lambda\rho_1}}{\dfrac{4\pi dr}{\lambda\rho_1}}\cos\frac{4\pi ds}{\lambda\rho_1}.$$

Again, with two elliptic discs, having axes ϖ and σ parallel and perpendicular respectively to the direction of separation of the slits,

$$f(w) = \sigma\sqrt{1 - w^2/\varpi^2}, \quad \Omega = \pi\varpi\sigma,$$

$$Q = \frac{4}{\pi\varpi}\int_0^{\varpi}\sqrt{1 - w^2/\varpi^2}\cos\frac{4\pi dw}{\lambda\rho_1}\,dw = 2\frac{J_1\left(\dfrac{4\pi d\varpi}{\lambda\rho_1}\right)}{\dfrac{4\pi d\varpi}{\lambda\rho_1}},$$

and
$$V = 2\frac{J_1\left(\dfrac{4\pi d\varpi}{\lambda\rho_1}\right)}{\dfrac{4\pi d\varpi}{\lambda\rho_1}}\cos\frac{4\pi ds}{\lambda\rho_1}.$$

71. Before passing on to the consideration of a number of rectangular apertures in the diffraction screen, let us take the case in which we have only two, one of which is covered by a retarding plate bounded by parallel faces.

Let us suppose that the apertures are parallel to one another with their centres on a line perpendicular to their lengths: let $2l$ be the lengths, $2h$ the width of the uncovered, $2k$ that of the covered aperture, and $2g$ the opaque space between them. Then if δ be the retardation of phase introduced by the plate,

$$\phi_0(t) = -\frac{A}{\lambda}\frac{1}{\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\left\{\int_{-(g+2h)}^{-g}\int_{-l}^{l}\iota e^{\iota(px+qy)}\,dx\,dy + \int_g^{g+2k}\int_{-l}^{l}\iota e^{\iota(px+qy-\delta)}\,dx\,dy\right\}$$

$$= -\iota\frac{A}{\lambda}\frac{1}{\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}e^{\iota\kappa\omega t}\frac{2\sin ql}{q}\left\{e^{-\iota p(g+h)}\frac{2\sin ph}{p} + e^{\iota\{p(g+k)-\delta\}}\frac{2\sin pk}{p}\right\},$$

and the intensity is

$$I = \frac{A^2}{\lambda^2}\frac{1}{\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{16l^2}{p^2}\left(\frac{\sin ql}{ql}\right)^2[\sin^2 ph + \sin^2 pk$$
$$+ 2\sin ph\sin pk\cos\{\delta - p(2g+h+k)\}] \qquad \text{.........(22)};$$

and when the object examined is a long luminous line parallel to the length of the slits,

$$I = \frac{A^2 l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{2\lambda\rho_0}{\pi^2\xi^2}\left[\sin^2\frac{2\pi h}{\lambda\rho_0}\xi + \sin^2\frac{2\pi k}{\lambda\rho_0}\xi\right.$$
$$\left. + 2\sin\frac{2\pi h\xi}{\lambda\rho_0}\sin\frac{2\pi k\xi}{\lambda\rho_0}\cos\left\{\delta - \frac{2\pi\xi}{\lambda\rho_0}(2g+h+k)\right\}\right] \qquad \text{.........(23)}.$$

As a first application of this expression, let us suppose that the breadths of the interfering streams are equal and that the streams are contiguous; then $k = h$ and $g = 0$, whence

$$I = \frac{A^2 l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{8\lambda\rho_0}{\pi^2\xi^2}\sin^2\frac{2\pi h\xi}{\lambda\rho_0}\cos^2\left\{\frac{\delta}{2} - \frac{2\pi h\xi}{\lambda\rho_0}\right\} \qquad \text{............(24)},$$

and the minima are given by

$$\sin\frac{2\pi h\xi}{\lambda\rho_0} = 0 \text{ or } \xi = \pm n\lambda\frac{\rho_0}{2h},$$

and $$\cos\left(\frac{\delta}{2} - \frac{2\pi h\xi}{\lambda\rho_0}\right) = 0 \text{ or } \xi = \left\{\pm(2n+1)\frac{\lambda}{2} + \Delta\right\}\frac{\rho_0}{2h},$$

where Δ is the retardation introduced by the plate, measured in length in air.

Thus the 2nd, 4th, ... bands on each side of the central line occupy the same position as if there were no retarding plate, while the 1st, 3rd, ... bands on each side are displaced in the direction of the retarded stream. This result we shall have occasion to employ later.

The chief interest of the expression (23) lies in its application to the phenomenon of Talbot's bands. These are dark lines that are seen, when a tolerably pure spectrum is viewed with the naked eye or a telescope, half of the aperture being covered with a thin retarding plate. A peculiarity of these bands is that they are only observed when the plate is held on the side towards the blue end of the spectrum*.

Since the object examined is a line of white light, the constituents of which have been separated so that the different colours occupy different angular positions in the field of view, the aggregate illumination at any point M is found by integrating the expression for the intensity so as to include all the components that have their foci near enough to M to afford a sensible contribution to the illumination. We may thus with convenience regard M as origin, so that ξ is the coordinate relatively to M of the focal point corresponding to a component for which the retardation of phase is δ, and the required result is obtained by integrating with respect to ξ between $-\infty$ and $+\infty$.

A different value of λ and of δ corresponds to each value of ξ; but in the integration the variation of λ may be neglected, and regarding δ as a function of ξ, we may put

$$\delta = \delta_0 + \varpi\xi,$$

where δ_0 and ϖ are the values of δ and $d\delta/d\xi$ at the point M, ϖ being positive when the *blue* end of the spectrum is seen on the side on which the retarding plate is held.

Let us write for shortness

$$h_1 = 2\pi h/(\lambda\rho_0), \quad k_1 = 2\pi k/(\lambda\rho_0), \quad g_1 = \varpi - 2\pi(2g + h + k)/(\lambda\rho_0),$$

the expression for the intensity becomes

$$I = \frac{A^2 l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{2\lambda\rho_0}{\pi^2}\left[\int_{-\infty}^{\infty}\frac{\sin^2 h_1\xi}{\xi^2}d\xi + \int_{-\infty}^{\infty}\frac{\sin^2 k_1\xi}{\xi^2}d\xi\right.$$
$$\left. + 2\int_{-\infty}^{\infty}\sin h_1\xi \,.\, \sin k_1\xi \,.\, \cos(\delta_0 + g_1\xi)\frac{d\xi}{\xi^2}\right] \quad\text{.........(25).}$$

* Stokes, *Phil. Trans.* CXXXVIII. 227 (1848); *Math. and Phys. Papers*, II. 14. Lord Rayleigh, *Enc. Brit.* XXIV. 441. For the case of Talbct's lines with a circular aperture, see H. Struve, *Mém. de l'Acad. des Sc. de St Pétersbourg* (7), XXXI. No. 1 (1883).

The last term within the vinculum is equal to

$$2\cos\delta_0\int_{-\infty}^{\infty}\sin h_1\xi\,.\,\sin k_1\xi\,.\,\cos g_1\xi\,.\,\frac{d\xi}{\xi^2}=\cos\delta_0\,.\,w\ \text{(say)}.$$

But $2\sin h_1\xi\,.\,\sin k_1\xi\,.\,\cos g_1\xi=\sin^2\frac{h_1+k_1+g_1}{2}\xi+\sin^2\frac{h_1+k_1-g_1}{2}\xi$

$$-\sin^2\frac{h_1-k_1+g_1}{2}\xi-\sin^2\frac{h_1-k_1-g_1}{2}\xi,$$

and
$$\int_{-\infty}^{\infty}\frac{\sin^2\gamma\xi}{\xi^2}d\xi=\pi\sqrt{\gamma^2},$$

in which in every case the positive value of the square root is to be taken; hence

$$w=\pi\left\{\sqrt{\left(\frac{h_1+k_1+g_1}{2}\right)^2}+\sqrt{\left(\frac{h_1+k_1-g_1}{2}\right)^2}-\sqrt{\left(\frac{h_1-k_1+g_1}{2}\right)^2}-\sqrt{\left(\frac{h_1-k_1-g_1}{2}\right)^2}\right\}$$

$= 0$ if $g_1^2>(h_1+k_1)^2$,
$= \pi(h_1+k_1-\sqrt{g_1^2})$ if $(h_1+k_1)^2>g_1^2>(h_1\sim k_1)^2$,
$= 2\pi h_1$ or $2\pi k_1$, according as $h_1\lesseqgtr k_1$, if $(h_1\sim k_1)^2>g_1^2$.

Thus writing $g_1=2\pi g'/(\lambda\rho_0)$,

$$I=\frac{4A^2l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2(h+k)\ \text{.........................(26)},$$

when $g'^2>(h+k)^2$;

$$I=\frac{4A^2l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\{h+k+(h+k-\sqrt{g'^2})\cos\delta_0\}\ \text{...........(27)},$$

when g'^2 lies between $(h+k)^2$ and $(h\sim k)^2$;

$$\left.\begin{aligned}I&=\frac{4A^2l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\{h+k+2h\cos\delta_0\}\\ \text{or}\qquad &=\frac{4A^2l}{\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)^2\{h+k+2k\cos\delta_0\}\end{aligned}\right\}\ \text{..................(28)},$$

according as h or k is the smaller of the two, when $g'^2<(h\sim k)^2$.

Now $g'=\varpi\lambda\rho_0/(2\pi)-2g-h-k$, and it therefore follows that if ϖ be negative, there will be no bands, since in that case g' is negative, and numerically greater than $h+k$; but that if ϖ be positive, g' may be made to assume any value from $-(2g+h+k)$ to $+\infty$ by altering the thickness of the plate, since the value of ϖ varies as this thickness, and so long as it lies within certain limits, there will be bands visible in the spectrum.

Let T_0, T_1, T_2 be the values of the thickness of the plate when $g'=0$, $-(h\sim k)$, and $-(h+k)$ respectively; then if T be less than T_2 or greater than $2T_0-T_2$, there are no bands; for values of T between T_2 and T_1 or

between $2T_0 - T_1$ and $2T_0 - T_2$ there will be bands with visibility given by $(h + k - \sqrt{g'^2})/(h + k)$, and for values of T between T_1 and $2T_0 - T_1$, there will be bands with visibility $2h/(h + k)$ or $2k/(h + k)$ according as $h \lesseqgtr k$.

Now in passing from one band to the next, δ changes by 2π and ξ by e, where e is the distance between the bands, and for this small change of ξ we may regard the changes of δ and ξ as proportional: hence

$$e = 2\pi/\varpi,$$

but when $T = T_0$, which is called by Stokes the best thickness,

$$\varpi = \frac{2\pi}{\lambda\rho_0}(2g + h + k),$$

so that in this case

$$e = \lambda\rho_0/(2g + h + k).$$

The bands are thus spaced in this case exactly as those due to the interference of two streams of light of the colour considered, coming from a luminous line seen in focus and entering the object-glass through two very narrow slits parallel to the line and situated at the centres of the covered and uncovered apertures.

72. In considering the general properties of Fraunhofer's diffraction phenomena, the case of a number of equal, similar and similarly situated apertures in the diffraction screen was discussed, and it was pointed out that, when these are numerous and very close together, the pattern of a single aperture may be very considerably departed from.

The most important instance of such a series of apertures is afforded by an ordinary diffraction grating, formed by tracing a number of equidistant parallel lines on a glass plate. These lines, by diffusive reflection of the incident light, appear to act approximately as opaque intervals; and the transparent spaces being in this case rectangular, the intensity due to a luminous line parallel to the cuts on the grating is proportional to

$$\left.\begin{aligned} &n^2 \cos^2\delta \left(\frac{\sin \dfrac{np(a+d)}{2}}{n \sin \dfrac{p(a+d)}{2}}\right)^2 \left(\frac{\sin \dfrac{pa}{2}}{\dfrac{pa}{2}}\right)^2 \\ &p = \frac{2\pi}{\lambda}(\sin i + \sin\delta) \end{aligned}\right\} \text{..............(29)},$$

a being the width of the transparent, d that of the opaque spaces, and i, δ the angles of incidence and diffraction.

If a be very small compared with d, the last factor in (29) varies very slowly with δ compared with the last but one, and this passes through a large number of principal maxima before the last factor reaches its first

minimum. The pattern then will practically consist of the lateral spectra given by

$$\sin i + \sin \delta = m\lambda/(a+d),$$

of which several will be visible.

The first minimum of the mth spectrum occurs in a direction given by

$$\sin i + \sin \delta = (m + 1/n)\,\lambda/(a+d),$$

and in order that two lines of wave-lengths λ and $\lambda + \delta\lambda$ should just be resolved, the principal maximum for the latter must be in this direction. Hence the condition for resolution* is given by

$$(m + 1/n)\,\lambda = m\,(\lambda + \delta\lambda),$$

or

$$\delta\lambda/\lambda = 1/(mn) \qquad (30).$$

73. Closely allied in theory to the ordinary grating is Michelson's Echelon grating†, which is built up of a number of equally thick plates of glass arranged in a series of equal steps. Here there are no opaque intervals in the case of normal incidence, but the stream traversing any step is retarded in phase by an amount $2\pi(\mu - 1)\,t/\lambda$ with respect to that transmitted through the step below it, t denoting the thickness, μ the refractive index of the plates.

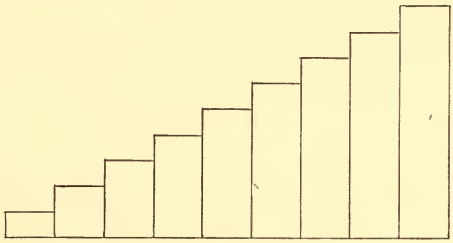


Fig. 24.

In this case, calling s the breadth of a step and assuming that the steps descend on the side of positive x, the intensity in a direction δ is by (5) proportional to

$$n^2 \cos^2 \delta \left\{ \frac{\sin n \frac{\pi}{\lambda}(s \sin \delta + \overline{\mu - 1}\,t)}{n \sin \frac{\pi}{\lambda}(s \sin \delta + \overline{\mu - 1}\,t)} \right\}^2 \left\{ \frac{\sin\left(\frac{\pi}{\lambda} s \sin \delta\right)}{\frac{\pi}{\lambda} s \sin \delta} \right\}^2 \qquad (31),$$

and the principal maxima are in directions given by

$$s \sin \delta + (\mu - 1)\,t = m\lambda \qquad (32).$$

* Lord Rayleigh, *Phil. Mag.* (4) XLVII. 199 (1874).

† *Amer. Journ. Sci.* (4) V. 215 (1898); *Astrophys. Journ.* VIII. 37 (1898); *J. de Phys.* (3) VIII. 305 (1899).

Now the last factor in (31) becomes almost insensible when $\sin\delta$ exceeds λ/s in absolute magnitude, and hence the light is concentrated in the two spectra, for which m lies between $(\mu-1)t/\lambda \pm 1$. If, however, the retardation introduced by a step be an exact multiple of λ—a result that may be obtained by a slight inclination of the echelon—there will be a single spectrum situated at the centre of the field.

A serious disadvantage of the echelon grating is the overlapping of the spectra of different orders, which renders it only suitable for use with light that is initially nearly monochromatic. Suppose that the incident light has wave-lengths between $\lambda \pm \delta\lambda$, then in order that there should be no overlapping, we must have

$$m(\lambda+\delta\lambda)-\left(\mu+\frac{d\mu}{d\lambda}\delta\lambda-1\right)t=(m+1)(\lambda-\delta\lambda)-\left(\mu-\frac{d\mu}{d\lambda}\delta\lambda-1\right)t,$$

or

$$\left(2m-2\frac{d\mu}{d\lambda}t+1\right)\delta\lambda=\lambda,$$

and writing for m its approximate value $m=(\mu-1)t/\lambda$, we obtain very approximately

$$\frac{\delta\lambda}{\lambda}=\frac{\lambda}{2(\mu-1-\lambda d\mu/d\lambda)t} \quad\ldots\ldots\ldots\ldots(33),$$

the factor $\mu-1-\lambda d\mu/d\lambda$ for most kinds of glass varying between 0·5 and 1·0.

The first minimum of the mth spectrum is in a direction given by

$$s\sin\delta=(m+1/n)\lambda-(\mu-1)t;$$

therefore if the lines λ and $\lambda+\delta\lambda$ be just resolved,

$$m(\lambda+\delta\lambda)-\left(\mu+\frac{d\mu}{d\lambda}\delta\lambda-1\right)t=(m+1/n)\lambda-(\mu-1)t,$$

or

$$\frac{\delta\lambda}{\lambda}=\frac{1}{n\left(m-\dfrac{d\mu}{d\lambda}t\right)}\doteqdot\frac{\lambda}{n\{(\mu-1)-\lambda d\mu/d\lambda\}t} \quad\ldots\ldots\ldots\ldots(34).$$

For the dispersion we have

$$s\cos\delta\frac{d\delta}{d\lambda}+\frac{d\mu}{d\lambda}t=m,$$

whence

$$\frac{d\delta}{d\lambda}\doteqdot\frac{(\mu-1-\lambda d\mu/d\lambda)t}{s\lambda} \quad\ldots\ldots\ldots\ldots(35).$$

74. Another method of treating the question of gratings, that is not without its advantages, is to deduce the effect of the ruling from the result of the theory of a "simple grating."* Any grating may be regarded as the superposition of a simple aperture and of a number of simple gratings; for whatever be the law of its ruling, its transmitting properties are expressed

* Schuster, *Phil. Mag.* (5) XXXVII. 509 (1894).

by writing the amplitude of the disturbance in the transmitted wave as a periodic function of x, and this by Fourier's theorem may be obtained as a series of terms, of which the first is constant and the others are simply periodic.

In the case of an ordinary grating with transparent intervals of width a separated by opaque spaces of width d, the transmitting property is a periodic function of period $a+d=\sigma$, and starting from the centre of one of the bright spaces, this function has a value unity from 0 to $a/2$ and from $\sigma - a/2$ to σ, and a value zero from $a/2$ to $\sigma - a/2$. Expressing this function then as a series of the form

$$b_0/2 + b_1 \cos \pi x/\sigma + b_2 \cos 2\pi x/\sigma + \dots,$$

we have

$$b_m = \frac{2}{\sigma}\int f(z) \cos \frac{m\pi z}{\sigma}\, d\sigma = \frac{2}{m\pi}(1+\cos m\pi) \sin \frac{m\pi a}{2\sigma},$$

and the series is

$$\frac{a}{\sigma} + \frac{2}{\pi}\sum_1 \frac{1}{m} \sin \frac{m\pi a}{\sigma} \cos \frac{2m\pi x}{\sigma}.$$

Thus considering only directions perpendicular to the lines of the grating, the amplitude of the disturbance in a direction δ is proportional to

$$\cos\delta \left\{ \frac{a}{\sigma}\frac{\sin pl}{pl} + \frac{2}{\pi}\sum_1 \frac{1}{m} \frac{p \sin \dfrac{m\pi a}{\sigma}}{p \pm \dfrac{2m\pi}{\sigma}} \frac{\sin \dfrac{n\sigma}{2m}\left(p \mp \dfrac{2m\pi}{\sigma}\right)}{\dfrac{n\sigma}{2m}\left(p \mp \dfrac{2m\pi}{\sigma}\right)} \right\} \dots (36),$$

where $2l$ is the total length of the grating and $p = 2\pi(\sin i + \sin\delta)/\lambda$.

Hence the positions of the lateral images are determined by

$$p = \pm 2m\pi/\sigma \quad \text{or} \quad \sin i + \sin\delta = \pm m\lambda/\sigma;$$

that is they are formed in directions such that the retardation between the secondary waves from the edges of the grating amounts to $mn\lambda$.

On either side of these spectra the illumination is distributed according to the same law as for the central image, vanishing for example when

$$\sigma(\sin i + \sin\delta) = \pm m\lambda \pm k\lambda/n \qquad k < m$$

or when the relative retardation amounts to $(mn \pm k)\lambda$.

If B_m denote the brightness of the mth lateral image, B_0 that of the central image and B that of the central image when the whole space of the grating is transparent, we have, since

$$\frac{\sin \dfrac{n\sigma}{2m}\left(p \mp \dfrac{2m\pi}{\sigma}\right)}{\dfrac{n\sigma}{2m}\left(p \mp \dfrac{2m\pi}{\sigma}\right)}$$

is very small except near the place $p = 2m\pi/\sigma$,

$$\frac{B_m}{B_0} = \frac{\cos^2\delta_m}{\cos^2 i}\left(\frac{\sin\dfrac{m\pi a}{\sigma}}{\dfrac{m\pi a}{\sigma}}\right)^2 < \left(\frac{\sin\dfrac{m\pi a}{\sigma}}{\dfrac{m\pi a}{\sigma}}\right)^2,$$

$$\frac{B_0}{B} = \left(\frac{a}{\sigma}\right)^2;$$

$$\therefore \quad \frac{B_m}{B} < \frac{1}{m^2\pi^2}\sin^2\frac{m\pi a}{\sigma}.$$

Hence under the most favourable circumstances, less than $1/m^2\pi^2$ of the original light can be obtained in the mth spectrum.

The mth spectrum will vanish if $\sin(m\pi a/\sigma) = 0$ or $ma/\sigma = m'$; whence it follows that if the ratio of the widths of the transparent and opaque spaces can be expressed as the ratio of two integers, the spectrum of the order equal to the sum of these integers is wanting.

75. It is at once clear from the above method of investigation, that any departure from regularity in the ruling of a grating, whether arising from variations in the hardness of the surface ruled or from irregularities in the screw with which the spacing is effected, will introduce other terms in the series representing its transmitting properties and give rise to additional spectra. These spectra are in general of less relative importance and are by reason of their faintness known as "ghosts."

So long as the defects in the ruling are very slight, their effect on the spectra escapes notice, but when, as may easily happen, there is a periodic variation in the spacing with each revolution of the screw, the ghosts may become relatively important.

As an instance of such periodicity* let us suppose a case in which the edges of the rth transparent interval are at distances from the centre of the grating given by

$$\sigma\left\{r \mp \frac{\alpha}{2} + 2\beta\sin(r \mp \tfrac{1}{2})\frac{2\pi}{\gamma}\right\} \quad\text{.....................}(37),$$

the width of the opaque spaces being $\sigma(1-\alpha)$. Then the function to be expanded has a period $\gamma\sigma$ and is equal to unity when x lies between

$$0 \quad\text{and}\quad \sigma\left(\frac{\alpha}{2} + 2\beta\sin\frac{\pi}{\gamma}\right),$$

or between

$$\sigma\left\{r - \frac{\alpha}{2} + 2\beta\sin(r-\tfrac{1}{2})\frac{2\pi}{\gamma}\right\} \text{ and } \sigma\left\{r + \frac{\alpha}{2} + 2\beta\sin(r+\tfrac{1}{2})\frac{2\pi}{\gamma}\right\},\ r < \gamma,$$

or between $\quad \sigma\left\{\gamma - \dfrac{\alpha}{2} + 2\beta\sin(\gamma - \tfrac{1}{2})\dfrac{2\pi}{\gamma}\right\}$ and $\gamma\sigma$,

and has otherwise the value zero.

* Peirce, *Amer. J. of Math.* II. 330 (1879).

Expressing this function by the series

$$b_0/2 + \Sigma b_m \cos(m\pi x/(\gamma\sigma)) \text{..........................} (38),$$

we have
$$b_0 = \frac{2}{\gamma\sigma} \times \text{sum of transparent intervals}$$
$$= \frac{2}{\gamma\sigma}\{\gamma\sigma - \gamma(1-\alpha)\sigma\} = 2\alpha,$$

$$b_m = \frac{2}{\gamma\sigma}\int f(z)\cos\frac{m\pi z}{\gamma\sigma}\,.\,dz$$
$$= \frac{2}{m\pi}\left[\sum_0^{\gamma-1}\sin\frac{m\pi}{\gamma}\left\{r + \frac{\alpha}{2} + 2\beta\sin(r+\tfrac{1}{2})\frac{2\pi}{\gamma}\right\}\right.$$
$$\left. - \sum_1^{\gamma}\sin\frac{m\pi}{\gamma}\left\{r - \frac{\alpha}{2} + 2\beta\sin(r-\tfrac{1}{2})\frac{2\pi}{\gamma}\right\}\right]$$
$$= \frac{2}{m\pi}(1+\cos m\pi)\sum_0^{\gamma-1}\sin\frac{m\pi}{\gamma}\left\{r + \frac{\alpha}{2} + 2\beta\sin(r+\tfrac{1}{2})\frac{2\pi}{\gamma}\right\}.$$

Now

$$e^{2\beta\sin x\,.\,i} = 1 + \beta(e^{xi} - e^{-xi}) + \frac{\beta^2}{\underline{|2}}(e^{xi} - e^{-xi})^2 + \dots,$$

and the nth term of the series is

$$\frac{\beta^n}{\underline{|n}}\sum_{s=1}^{s=\frac{n+1}{2}}(-1)^{s-1}\frac{\underline{|n}}{\underline{|s-1}\,\underline{|n-s+1}}\{e^{(n-2s+2)xi} - e^{-(n-2s+2)xi}\}$$

if n be odd and

$$\frac{\beta^n}{\underline{|n}}\left[\sum_{s=1}^{s=\frac{n}{2}}(-1)^{s-1}\frac{\underline{|n}}{\underline{|s-1}\,\underline{|n-s+1}}\{e^{(n-2s+2)xi} + e^{-(n-2s+2)xi}\}\right.$$
$$\left. + (-1)^{\frac{n}{2}}\frac{\underline{|n}}{\underline{|n/2}\,\underline{|n/2}}\right]$$

if n be even. Hence collecting the terms

$$e^{(a+2\beta\sin x)i} = \sum_{n=0}^{n=\infty}(-1)^n\frac{\beta^{2n}}{(\underline{|n})^2}e^{ai}$$
$$+\sum_{s=1}^{s=\infty}\{e^{(a+sx)i} + (-1)^s e^{(a-sx)i}\}\sum_{n=0}^{n=\infty}(-1)^n\frac{\beta^{2n+s}}{\underline{|n}\,\underline{|n+s}},$$

and

$$\sin(a+2\beta\sin x) = A_0\sin a + \sum_{s=1}^{s=\infty}A_s\{\sin(a+sx) + (-1)^s\sin(a-sx)\} \text{...} (39),$$

where
$$A_s = \sum_{n=0}^{n=\infty}(-1)^n\frac{\beta^{2n+s}}{\underline{|n}\,\underline{|n+s}} \text{..........................} (40).$$

From this result it follows that

$$b_m = \frac{2}{m\pi}(1+\cos m\pi)\left[\sum_{r=0}^{r=\gamma-1} A_0 \sin\left(\frac{\alpha}{2}+r\right)\frac{m\pi}{\gamma}\right.$$
$$+\sum_{s=1}^{s=\infty} A_s\left\{\sum_{r=0}^{r=\gamma-1}\sin\left[(m\alpha+2s)\frac{\pi}{2\gamma}+(m+2s)\frac{\pi}{\gamma}r\right]\right.$$
$$\left.\left.+(-1)^s\sum_{r=0}^{r=\gamma-1}\sin\left[(m\alpha-2s)\frac{\pi}{2\gamma}+(m-2s)\frac{\pi}{\gamma}r\right]\right\}\right]$$

$$=\frac{2}{m\pi}(1+\cos m\pi)\left[A_0\sin\frac{(\alpha-1+\gamma)\,m\pi}{2\gamma}\frac{\sin\frac{m\pi}{2}}{\sin\frac{m\pi}{2\gamma}}\right.$$
$$+\sum_{s=1}^{s=\infty}A_s\left\{\sin\frac{\{m(\alpha-1)+\gamma(m+2s)\}\pi}{2\gamma}\frac{\sin\frac{(m+2s)\pi}{2}}{\sin\frac{(m+2s)\pi}{2\gamma}}\right.$$
$$\left.\left.+(-1)^s\sin\frac{\{m(\alpha-1)+\gamma(m-2s)\}\pi}{2\gamma}\frac{\sin\frac{(m-2s)\pi}{2}}{\sin\frac{(m-2s)\pi}{2\gamma}}\right\}\right].$$

Hence the series is

$$\alpha+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\frac{1}{m}B_m\cos\frac{2m\pi x}{\gamma\sigma} \quad\text{........................(41)},$$

where

$$B_m=\tfrac{1}{2}\sin\frac{(\alpha-1)\,m\pi}{\gamma}\left[A_0\frac{\sin 2m\pi}{\sin\frac{m\pi}{\gamma}}+\sum_{s=1}^{s=\infty}A_s\frac{\sin 2(m+s)\pi}{\sin\frac{(m+s)\pi}{\gamma}}\right.$$
$$\left.+\sum_{s=1}^{s=\infty}(-1)^s A_s\frac{\sin 2(m-s)\pi}{\sin\frac{(m-s)\pi}{\gamma}}\right] \quad\text{.....................(42)},$$

with $\dfrac{\sin 2y\pi}{\sin(y\pi/\gamma)}=0$ unless $y=k\gamma$ when it becomes $=2\gamma\cos k\pi$,

and $$A_s=\sum_{n=0}^{n=\infty}(-1)^n\frac{(2m\pi\beta/\gamma)^{2n+s}}{\lfloor n \,\lfloor n+s} \quad\text{........................(43)}.$$

From this result it follows that the amplitude of the disturbance in the direction δ perpendicular to the lines of the grating is proportional to

$$\cos\delta\left\{\alpha\frac{\sin pl}{pl}+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\frac{1}{m}\frac{pB_m}{p\pm\frac{2m\pi}{\gamma\sigma}}\frac{\sin\frac{n\gamma\sigma}{2m}\left(p\mp\frac{2m\pi}{\gamma\sigma}\right)}{\frac{n\gamma\sigma}{2m}\left(p\mp\frac{2m\pi}{\gamma\sigma}\right)}\right\} \quad\text{......(44)},$$

$2l$ being the length of the grating, which is here supposed to contain an even number (n) of complete periods. The lateral images are determined by

$$\sin i + \sin \delta = \pm m\lambda/\gamma\sigma.$$

Now neglecting the cube and higher powers of the small quantity $(2m\pi\beta/\gamma)$ we have

$$A_0 = 1 - \left(\frac{2m\pi\beta}{\gamma}\right)^2, \quad A_1 = \frac{2m\pi\beta}{\gamma}, \quad A_2 = \tfrac{1}{2}\left(\frac{2m\pi\beta}{\gamma}\right)^2, \quad A_s\,(s > 2) = 0,$$

and

$$\begin{aligned} B_{k\gamma-2} &= \gamma A_2 \sin \{k\alpha - 2(\alpha - 1)/\gamma\}\,\pi, \\ B_{k\gamma-1} &= \gamma A_1 \sin \{k\alpha - (\alpha - 1)/\gamma\}\,\pi, \\ B_{k\gamma} &= \gamma A_0 \sin k\alpha\pi, \\ B_{k\gamma+1} &= -\gamma A_1 \sin \{k\alpha + (\alpha - 1)/\gamma\}\,\pi, \\ B_{k\gamma+2} &= \gamma A_2 \sin \{k\alpha + 2(\alpha - 1)/\gamma\}\,\pi, \\ B_{k\gamma\pm m'} &= 0 \qquad m' > 2. \end{aligned}$$

Hence, whenever $m = k\gamma$, that is in directions given by

$$\sin i + \sin \delta = \pm k\lambda/\sigma,$$

corresponding to the lateral images given by an ordinary grating of period σ equal to the mean interval of the transparent spaces, we have bright spectra and on either side of these a series of faint spectra or ghosts, that are the less conspicuous the further they are from the principal spectra.

76. Another peculiarity exhibited by certain gratings is that of exercising a converging or diverging influence on the spectra formed by them*. This has been attributed by Cornu to a regular variation in the spacing of the lines, and elementary reasoning shows that a gradual increase in the interval of a plane grating has the effect of a convex lens as regards the spectra on one side of the central image and acts as a concave lens with respect to the lateral images on the other side.

Let us suppose that the surface on which the lines are traced is curved, and that the lines are determined by the intersection of this surface with a series of parallel planes, one of which is the normal plane at the centre of the grating, and that the spacing is such that the distance of the kth plane from the centre of the series is represented by

$$s = \sigma k + \sigma' k^2 + \sigma'' k^3 \qquad (45).$$

When the striated surface is irregular, the spectral images are defective, but when it is approximately a surface of the second degree with a plane of symmetry parallel to the lines, the images may be very distinct and the

* Mascart, *Ann. de l'Éc. Norm. Sup.* I. 250 (1864). Merczyng, *C. R.* XCVII. 570 (1883). Rydberg, *Phil. Mag.* (5) XXXV. 190 (1893). Cornu, *Ass. Franç. pour l'avancement des Sc.* Nantes (1875), p. 376; *C. R.* LXXX. 645 (1875); CXVI. 1215, 1421; CXVII. 1032 (1893); *J. de Phys.* (3) II. 385, 441 (1893); *Séances de la Soc. Franç. de Phys.* (1893) 215, 223. Lord Rayleigh, *Enc. Brit.* XXIV. 438.

inevitable astigmatism, though considerable, is of little consequence. In considering the focal properties of such gratings the curvature parallel to the lines may be neglected and that normal to the lines has alone to be taken into account. This is equivalent to assuming that the surface and the incident wave are both cylindrical, the generating lines of these cylinders being parallel to the ruling of the grating. The problem is thus reduced to one of two dimensions.

Let QA, QP be two rays starting from a point Q and falling on the concave side of the striated surface AP, the centre of curvature of which is O.

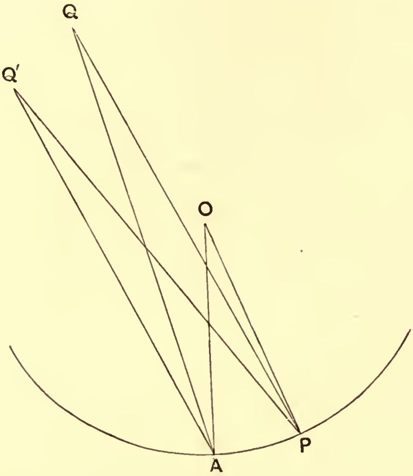


Fig. 25.

Let $QA = \rho$, $OA = a$, the angle $QAO = \alpha$ and the angle $AOP = \omega$, then $AP = 2a \sin \omega/2$ and the angle $QAP = \pi/2 + \alpha - \omega/2$. Hence

$$QP^2 = \rho^2 + 4a^2 \sin^2 \omega/2 - 4a\rho \sin \omega/2 \sin (\omega/2 - \alpha)$$
$$= (\rho + a \sin \alpha \sin \omega)^2 - a^2 \sin^2 \alpha \sin^2 \omega + 4a\,(a - \rho \cos \alpha) \sin^2 \omega/2.$$

Now as far as $\sin^4 \omega$

$$4 \sin^2 \omega/2 = \sin^2 \omega + (1/4) \sin^4 \omega$$

and to the same order

$$QP^2 = (\rho + a \sin \alpha \sin \omega)^2 + a \cos \alpha\,(a \cos \alpha - \rho) \sin^2 \omega + \tfrac{1}{4}\,a\,(a - \rho \cos \alpha) \sin^4 \omega$$

and
$$QP = \rho + a \sin \alpha \sin \omega + \tfrac{1}{2} \cos \alpha \left(\frac{\cos \alpha}{\rho} - \frac{1}{a}\right) a^2 \sin^2 \omega$$
$$- \tfrac{1}{2}\,\frac{\sin a \cos \alpha}{\rho} \left(\frac{\cos \alpha}{\rho} - \frac{1}{a}\right) a^3 \sin^3 \omega$$
$$+ \frac{1}{2}\left\{\frac{\sin^2 \alpha \cos \alpha}{\rho^2}\left(\frac{\cos \alpha}{\rho} - \frac{1}{a}\right) + \frac{1}{4a^2}\left(\frac{1}{\rho} - \frac{\cos \alpha}{a}\right) - \frac{1}{4}\,\frac{\cos^2 \alpha}{\rho}\left(\frac{\cos \alpha}{\rho} - \frac{1}{a}\right)^2\right\} a^4 \sin^4 \omega \quad \ldots (46).$$

Again let Q' be another point on the same side of the normal OA and let $Q'A=\rho'$, $Q'AO=\alpha'$; then $Q'P$ is obtained from QP by writing ρ' for ρ and α' for α.

Suppose now that A is a point on the central line of the grating and P a corresponding point on some other line, then Q' will be a focus of the diffracted light if

$$QP \pm Q'P = QA \pm Q'A + m\lambda \quad\text{.....................(47)},$$

m being a positive or negative integer, and the upper or lower sign being taken, according as the grating acts by reflection or transmission. Taking the former case we have

$$\begin{aligned}&(\sin\alpha+\sin\alpha')\,a\sin\omega+\frac{1}{2}\left(\frac{\cos^2\alpha}{\rho}+\frac{\cos^2\alpha'}{\rho'}-\frac{\cos\alpha}{a}-\frac{\cos\alpha'}{a}\right)a^2\sin^2\omega\\
&-\frac{1}{2}\left\{\frac{\sin\alpha\cos\alpha}{\rho}\left(\frac{\cos\alpha}{\rho}-\frac{1}{a}\right)+\frac{\sin\alpha'\cos\alpha'}{\rho'}\left(\frac{\cos\alpha'}{\rho'}-\frac{1}{a}\right)\right\}a^3\sin^3\omega\\
&+\frac{1}{2}\left\{\frac{\sin^2\alpha\cos\alpha}{\rho^2}\left(\frac{\cos\alpha}{\rho}-\frac{1}{a}\right)+\frac{1}{4a^2}\left(\frac{1}{\rho}-\frac{\cos\alpha}{a}\right)-\frac{1}{4}\frac{\cos^2\alpha}{\rho}\left(\frac{\cos\alpha}{\rho}-\frac{1}{a}\right)^2\right.\\
&\left.+\frac{\sin^2\alpha'\cos\alpha'}{\rho'^2}\left(\frac{\cos\alpha'}{\rho'}-\frac{1}{a}\right)+\frac{1}{4a^2}\left(\frac{1}{\rho'}-\frac{\cos\alpha'}{a}\right)-\frac{1}{4}\frac{\cos^2\alpha'}{\rho'}\left(\frac{\cos\alpha'}{\rho'}-\frac{1}{a}\right)^2\right\}a^4\sin^4\omega\\
&=m\lambda \quad\text{..(48).}\end{aligned}$$

Now if P be on the kth line

$$a\sin\omega=\sigma k+\sigma' k^2+\sigma'' k^3$$

and writing $b=\sigma^2/\sigma'$, $c=\sigma^3/\sigma''$, the equation becomes

$$\begin{aligned}&(\sin\alpha+\sin\alpha')\,\sigma k+\frac{1}{2}\left(\frac{\cos^2\alpha}{\rho}+\frac{\cos^2\alpha'}{\rho'}-\frac{\cos\alpha+\cos\alpha'}{a}+\frac{\sin\alpha+\sin\alpha'}{b}\right)\sigma^2k^2\\
&+\frac{1}{2}\left\{\frac{2}{b}\left(\frac{\cos^2\alpha}{\rho}+\frac{\cos^2\alpha'}{\rho'}-\frac{\cos\alpha+\cos\alpha'}{a}\right)-\frac{\sin\alpha}{\rho}\left(\frac{\cos^2\alpha}{\rho}-\frac{\cos\alpha}{a}\right)\right.\\
&\left.\quad-\frac{\sin\alpha'}{\rho'}\left(\frac{\cos^2\alpha'}{\rho'}-\frac{\cos\alpha'}{a}\right)+\frac{\sin\alpha+\sin\alpha'}{c}\right\}\sigma^3k^3+\ldots\ldots=n\lambda \quad\text{.........(49).}\end{aligned}$$

So long as σ^3k^3 is small, this equation will hold for all values of k, if

$$\sigma(\sin\alpha+\sin\alpha')=m\lambda \quad\text{...........................(50)},$$

and

$$\left.\begin{aligned}\frac{\cos^2\alpha}{\rho}-\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}&=\frac{1}{d}\\
\frac{\cos^2\alpha'}{\rho'}-\frac{\cos\alpha'}{a}+\frac{\sin\alpha'}{b}&=-\frac{1}{d}\end{aligned}\right\}\quad\text{.....................(51).}$$

We thus have two families of curves, called by Cornu "focal curves" and by Baily "diffraction curves,"* that are conjugate to one another and have

* *Phil. Mag.* (5) xv. 183 (1883).

the property, that if the source be at any point on one of the curves, the spectra lie on the conjugate curve. Writing

$$a/b = \tan(\phi + \psi), \quad (a-d)/(a+d) = \tan\phi \tan\psi, \quad K^{-2} = a^{-2} + b^{-2}$$

the equation to the curves may be put under the form

$$\rho = \frac{K}{2} \frac{\cos^2 \alpha}{\cos\left(\frac{\alpha}{2} + \phi\right) \cos\left(\frac{\alpha}{2} + \psi\right)} \quad \text{.....................(52).}$$

Among these curves there is one, the principal focal curve, that merits special attention, as it passes through both the source and the spectra, when the former is at any point upon it. This is the curve for which $d = \infty$ and its equation is

$$\frac{\cos^2 \alpha}{\rho} = \frac{\cos \alpha}{a} - \frac{\sin \alpha}{b} \quad \text{..........................(53),}$$

or

$$\rho = a \cos\phi \left\{ \frac{\sin^2 \phi}{\cos(\alpha + \phi)} + \cos(\alpha - \phi) \right\} \quad \text{...............(54),}$$

where $\tan\phi = a/b$. This latter form of the equation leads to an elegant geometrical construction of the curve*.

When the spacing is correct, $b = \infty$, and the principal focal curve becomes

$$\rho = a \cos \alpha \text{..................................(55);}$$

a circle on the radius of curvature of the grating as diameter. This is the arrangement usually adopted with curved gratings and it is clear from (48) that in this case the outstanding aberration is of the fourth order and equal to

$$\frac{a}{8} (\sin \alpha \tan \alpha + \sin \alpha' \tan \alpha') \sin^4 \omega \dagger.$$

If the grating be curved with a very small systematic error in the ruling, then ϕ in (54) is very small and we have sensibly

$$\rho = a \cos\phi \cos(\alpha - \phi),$$

a circle of diameter $a \cos\phi$ inclined at the small angle ϕ to the normal to the grating at its central point.

Finally when the grating is plane, $a = \infty$ and the principal focal curve is

$$\rho = -b \cot \alpha \cos \alpha \text{.............................(56),}$$

a cissoid of Diocles with its cusp at the centre of the grating and its asymptote at right angles to the plane of the grating at a distance b on the side on which the spacing increases.

77. We will now consider a case that is of primary importance in the study of diffraction, on account of its application to the theory of optical instruments, namely that in which the aperture is circular.

* Cornu, *J. de Phys.* (3) II. 391 (1893).

† Rowland, *Phil. Mag.* (5) XVI. 197 (1883). Glazebrook, *ibid.* (5) XV. 414; XVI. 377 (1883).

Let the centre of the aperture be taken as origin, its radius being R, and let us write

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta,$$
$$p = \sigma \cos \theta', \qquad q = \sigma \sin \theta',$$

so that

$$\sigma = \sqrt{p^2 + q^2} = 2\pi r/(\lambda \rho_0),$$

where r is the distance of the point of the pattern under consideration from the image of the luminous point. Then

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t} \frac{A}{\lambda \rho_1 \rho_0} \frac{\partial \rho_0}{\partial z} \int_0^R \int_{\theta'}^{2\pi+\theta'} e^{\iota\rho\sigma\cos(\theta-\theta')} \rho\, d\rho\, d\theta$$
$$= -\iota e^{\iota\kappa\omega t} \frac{A}{\lambda \rho_1 \rho_0} \frac{\partial \rho_0}{\partial z} \int_0^R \int_0^{2\pi} e^{\iota\rho\sigma\cos\theta} \rho\, d\rho\, d\theta,$$

or if $\rho\sigma = \zeta$

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t} \frac{A}{\lambda \rho_1 \rho_0} \frac{\partial \rho_0}{\partial z} \frac{1}{\sigma^2} \int_0^{R\sigma} \zeta\, d\zeta \int_0^{2\pi} e^{\iota\zeta\cos\theta}\, d\theta$$
$$= -\iota e^{\iota\kappa\omega t} \frac{A}{\lambda \rho_1 \rho_0} \frac{\partial \rho_0}{\partial z} \frac{2\pi}{\sigma^2} \int_0^{R\sigma} J_0(\zeta)\, \zeta\, d\zeta$$
$$= -\iota e^{\iota\kappa\omega t} \frac{A}{\lambda \rho_1 \rho_0} \frac{\partial \rho_0}{\partial z}\, 2\pi R^2 \frac{J_1(R\sigma)}{R\sigma}\text{*} \qquad \text{.....................(57)},$$

and the intensity is

$$I = \frac{A^2}{\lambda^2 \rho_1^2 \rho_0^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 4\pi^2 R^4 \left(\frac{J_1(R\sigma)}{R\sigma}\right)^2 \qquad \text{..................(58)}.$$

Thus the illumination vanishes in accordance with the roots of $J_1(\zeta) = 0$, and calling these $\zeta_1, \zeta_2, \ldots$ the radii of the dark rings in the diffraction pattern are

$$\frac{\rho_0 \lambda \zeta_1}{2\pi R}, \quad \frac{\rho_0 \lambda \zeta_2}{2\pi R}.$$

The values of the first six roots of $J_1(\zeta) = 0$ are

3·831706, 7·015587, 10·173468, 13·323692, 16·470630, 19·615858.

For the maxima we have

$$\frac{\partial}{\partial \zeta} \frac{J_1(\zeta)}{\zeta} = -\frac{1}{\zeta} J_2(\zeta) = 0,$$

and thus the maxima occur in correspondence with roots of $J_2(\zeta) = 0$, the first six of which are

0, 5·135630, 8·147236, 11·619857, 14·795938, 17·959820;

and since when ζ has one of these values

$$2J_1(\zeta)/\zeta = J_0(\zeta),$$

the intensity of the maxima is

$$I_{\text{max.}} = \frac{A^2}{\lambda^2 \rho_1^2 \rho_0^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \pi^2 R^4 J_0^2(R\sigma).$$

* For the properties of Bessel's and Struve's functions required in this and the succeeding sections see Appendix I.

The total illumination distributed over a circle of radius r is

$$2\pi \int_0^r I r dr = \frac{\lambda^2 \rho_0^2}{2\pi R^2} \int_0^\zeta I \zeta d\zeta = \frac{A^2}{\rho_1^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \pi R^2 2 \int_0^\zeta \frac{J_1^2(\zeta)}{\zeta} d\zeta$$

$$= \frac{A^2}{\rho_1^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \pi R^2 \{1 - J_0^2(\zeta) - J_1^2(\zeta)\},$$

since $J_0(0) = 0$, $J_1(0) = 1$.

If r and consequently ζ be infinite, $J_0(\zeta)$ and $J_1(\zeta)$ vanish, and thus the proportion of the whole illumination that is without a circle of radius r is $J_0^2(\zeta) + J_1^2(\zeta)$ and since for a dark ring $J_1(\zeta) = 0$, the fraction of the light that is outside any dark ring is $J_0^2(\zeta)$. The values of this fraction for the successive roots of $J_1(\zeta) = 0$ are ·161, ·090, ·062, ·047, ..., so that more than $\frac{9}{10}$ths of the whole light is inside the second dark ring*.

78. When the object under examination is a luminous line, the various elements of which are to be regarded as independent sources, the intensity may be determined by integrating the expression for the intensity due to a luminous point. In this way Struve† has obtained by the aid of properties of Bessel's functions an expression suitable for numerical calculation. Lord Rayleigh‡ has however shown that the problem may be solved more easily by a method due to Stokes§, in which the integration over the diffraction aperture is postponed until that with respect to the direction of the luminous line has been effected.

Since the intensity due to a luminous point is obtained by multiplying $\phi_0(t)$ by the conjugate expression, we have

$$I = \frac{A^2}{\lambda^2 \rho_0^2 \rho_1^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \iiiint e^{\iota \{p(x-x') + q(y-y')\}} dx dy dx' dy',$$

and the intensity due to a luminous line parallel to the axis of y and of breadth dx_1 is

$$J = dx_1 \int I dy_1 = \frac{A^2}{\rho_0^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 \frac{dp}{4\pi^2} \int_{-\infty}^{\infty} dq \iiiint e^{\iota \{p(x-x') + q(y-y')\}} dx dy dx' dy'.$$

In the present shape of the integral, the integration with respect to q must be reserved to the end, but if we introduce the factor Exp $(\mp \beta q)$, where the sign $-$ or $+$ is to be taken according as q is positive or negative, we shall evidently arrive at the same result as before, provided that in the end we

* Lord Rayleigh, *Phil. Mag.* (5) XI. 414 (1881); *Enc. Brit.* XXIV. 433.

† *Wied. Ann.* XVII. 1008 (1882); *Mém. de l'Acad. des Sc. de St Pétersbourg* (7) XXX. No. 8 (1882).

‡ *Enc. Brit.* XXIV. p. 433.

§ *Edin. Trans.* XX. 317 (1853).

suppose β to vanish without limit, and when this factor is introduced we may integrate with respect to q first. Thus

$$J = \mathrm{Lt}_{\beta=0} \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{4\pi^2}\iiiint\!\!\int_{-\infty}^{\infty} e^{\mp\beta q}\, e^{\iota\{p(x-x')+q(y-y')\}}\,dx\,dy\,dx'\,dy'\,dq$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{4\pi^2}\,\mathrm{Lt}_{\beta=0}\iiiint \frac{2\beta}{\beta^2+(y-y')^2}\, e^{\iota p(x-x')}\,dx\,dy\,dx'\,dy'.$$

Now $\mathrm{Lt}_{\beta=0}\int \dfrac{2\beta dy'}{\beta^2+(y-y')^2} = 0$, unless the range of integration for y' include the value y, in which case it is equal to 2π and therefore

$$\mathrm{Lt}_{\beta=0}\int \frac{2\beta dy'dy}{\beta^2+(y-y')^2} = 2\pi Y,$$

where Y is the common part of the ranges of integration for y' and y corresponding to any values of x' and x, and since the aperture is symmetrical with respect to the axis of y

$$J = \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{2\pi}\iint e^{\iota p(x-x')}\,Y\,dx\,dx'$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{2\pi}\int_0^R\!\!\int_0^R Y\left(e^{\iota px}+e^{-\iota px}\right)\left(e^{\iota px'}+e^{-\iota px'}\right)dx\,dx'$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{\pi}\, 2\int_0^R\!\!\int_0^R Y\cos px\cos px'\,dx\,dx'.$$

Now Y is the smaller of the two quantities $2\sqrt{R^2-x^2}$, $2\sqrt{R^2-x'^2}$ and therefore

$$J = \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{\pi}\, 4\left\{\int_0^R\!\!\int_0^x \sqrt{R^2-x^2}\cos px\cos px'\,dx\,dx' \right.$$
$$\left. + \int_0^R\!\!\int_x^R \sqrt{R^2-x'^2}\cos px\cos px'\,dx\,dx'\right\}$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{dp}{\pi}\, 8\int_0^R\!\!\int_0^x \sqrt{R^2-x^2}\cos px\cos px'\,dx\,dx'$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{4}{\pi}\frac{dp}{p}\int_0^R \sqrt{R^2-x^2}\sin 2px\,dx$$

$$= \frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{4R^2}{\pi}\frac{dp}{p}\int_0^{\frac{\pi}{2}} \sin(2pR\sin\theta)\cos^2\theta\,d\theta$$

$$= 4\frac{A^2}{\rho_0^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 R^2 d\zeta\,\frac{H_1(2\zeta)}{(2\zeta)^2} \qquad\qquad (59),$$

where $\zeta = pR = \dfrac{2\pi}{\lambda}\dfrac{R}{\rho_0}\xi$, ξ being the abscissa of the point considered relatively to the image of the luminous line.

The points of maximum and minimum illumination occur in accordance with the roots of

$$\frac{\partial}{\partial\zeta}\frac{H_1(2\zeta)}{(2\zeta)^2}=0,$$

or

$$3H_1(2\zeta)=2\zeta H_0(2\zeta),$$

which, when ζ is very large, become approximately the roots of the equation

$$\sin(2\zeta-\pi/4)=2/\sqrt{\pi\zeta}.$$

Since $H_1(2\zeta)$ is essentially positive, the intensity is nowhere zero.

79. Let us now examine the case, in which there are two parallel and equally luminous linear sources, the components of which subtend an angle at the aperture equal to that subtended by the wave-length of light at a distance equal to the diameter of the aperture. Since

$$\zeta=\frac{2\pi}{\lambda}\frac{R}{\rho_0}\xi \text{ and } \frac{\xi}{\rho_0}=\frac{\lambda}{2R}$$

the corresponding value of ζ is π. Writing

$$L(\zeta)=\frac{\pi}{2}\frac{H_1(2\zeta)}{(2\zeta)^2}=\frac{1}{1^2.3}-\frac{(2\zeta)^2}{1^2.3^2.5}+\frac{(2\zeta)^4}{1^2.3^2.5^2.7}-\dots$$

the intensity at the geometrical focus of either of the lines is proportional to

$$L(0)+L(\pi),$$

and that at the point midway between the geometrical images of the lines is

$$2L(\pi/2).$$

Now $\qquad L(0)=\cdot3333,\quad L(\pi/2)=\cdot1671,\quad L(\pi)=\cdot0164,$

so that the ratio of the intensity of illumination midway between the images to that at either image is

$$\frac{2L(\pi/2)}{L(0)+L(\pi)}=\cdot955.$$

But in order that the lines may be fairly resolved, this ratio should not, as we have seen, exceed the value ·8 approximately, and hence it follows that for resolution the angular interval between the lines must exceed that subtended by the wave-length of light at a distance equal to the diameter of the circular aperture.

80. If we now integrate (59) from ζ to ∞, we shall obtain the illumination due to an uniform luminous area bounded by a straight line parallel to the axis of y, at a point situated at a distance ξ from the geometrical image of the edge. This point will be without or within the geometrical image of the source according as ζ is positive or negative, and denoting the intensity of illumination by $I(\zeta)$, we have

$$I(+\zeta)+I(-\zeta)=1,$$

if the unit of intensity be such that the illumination due to an infinitely extended plane area be unity.

Hence
$$I(+\zeta) = C\int_\zeta^\infty \frac{H_1(2\zeta)}{(2\zeta)^2}\,d\zeta$$

with the condition that for $\zeta = 0$, $I(+\zeta) = \frac{1}{2}$. But

$$\int_0^\infty \frac{H_1(2\zeta)}{(2\zeta)^2}\,d\zeta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \int_0^\infty \frac{\sin(2\zeta\sin\theta)}{\zeta}\,d\zeta$$
$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{\pi}{8},$$

and from this it results that $C = 4/\pi$, and

$$I(+\zeta) = \frac{4}{\pi}\int_\zeta^\infty \frac{H_1(2\zeta)}{(2\zeta)^2}\,d\zeta = \frac{1}{2} - \frac{4}{\pi}\int_0^\zeta \frac{H_1(2\zeta)}{(2\zeta)^2}\,d\zeta$$
$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_0^\infty (-1)^s \frac{2s+3}{2s+1}\frac{(2\zeta)^{2s+1}}{(1.3\ldots 2s+3)^2}$$

using the ascending series for H_1.

For large values of the argument, it is more convenient to use the semi-convergent series for H_1 and this, retaining only the principal terms, gives

$$I(+\zeta) = \frac{2}{\pi^2}\left(\frac{1}{\zeta} + \frac{1}{12\zeta^3}\right) - \frac{1}{2\pi^{\frac{3}{2}}}\frac{\cos(2\zeta + \pi/4)}{\zeta^{\frac{5}{2}}}.$$

For very large values of ζ this reduces to

$$I(\zeta) = \frac{2}{\pi^2\zeta} = \frac{1}{\pi^3}\frac{\lambda\rho_0}{R\xi},$$

and thus at great distances from the geometrical image of the border of the radiant area the illumination is inversely proportional to the distance ξ and to the radius of the aperture.

81. The case of a diffraction aperture in the form of a sector of a circle is interesting on account of its application to the heliometer objective*. The fundamental formulæ of the problem were first given by Struve, but the case was first fully worked out by Bruns in a manner substantially the same as that given below.

Let 2β be the angle of the sector, then writing $R\sigma = \zeta$ we have

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{R^2}{\zeta^2}\int_0^\zeta z\,dz\int_{-\beta}^{\beta} e^{\iota z\cos(\theta-\theta')}\,d\theta$$
$$= -\iota e^{\iota\kappa\omega t}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{R^2}{\zeta^2}\int_0^\zeta z\,dz\left\{\int_0^{\beta-\theta'} e^{\iota z\cos\psi}\,d\psi + \int_0^{\beta+\theta'} e^{\iota z\cos\psi}\,d\psi\right\}.$$

* Struve, *Mém. de l'Acad. des Sc. de St Pétersbourg* (7) xxx. No. 8 (1882). Bruns, *Astron. Nachr.* CIV. 1 (1883). Straubel, Inaugural-Dissertation, Jena (1888).

Now $$\int_0^{\psi} e^{\iota z \cos\psi}\, d\psi = J_0(z)\,.\,\psi + 2\sum_1^{\infty} \frac{\iota^n}{n} \sin n\psi\,.\,J_n(z),$$

and $$\sin n(\beta - \theta') + \sin n(\beta + \theta') = 2 \sin n\beta \cos n\theta',$$

whence

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t} \frac{2A}{\lambda\rho_0\rho_1} \frac{\partial\rho_0}{\partial z} \frac{R^2}{\zeta^2} \int_0^{\zeta} \left\{\beta J_0(z) + 2\sum_1^{\infty} \frac{\iota^n}{n} \sin n\beta \cos n\theta' J_n(z)\right\} z\,dz.$$

But

$$\int_0^z zJ_0\,.\,dz = zJ_1, \quad \int_0^z zJ_n dz = 2zn \sum_{s=0}^{s=\infty} \frac{n+2s+1}{(n+2s)(n+2s+2)}\,.\,J_{n+2s+1},$$

therefore

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t} \frac{2A}{\lambda\rho_0\rho_1} \frac{\partial\rho_0}{\partial z} \frac{R^2}{\zeta} \left\{\beta J_1(\zeta) + 4\sum_{n=1}^{n=\infty} \iota^n \sin n\beta \cos n\theta' \sum_{s=0}^{s=\infty} \frac{n+2s+1}{(n+2s)(n+2s+2)} J_{n+2s+1}(\zeta)\right\}$$

$$= e^{\iota\kappa\omega t} \frac{2A}{\lambda\rho_0\rho_1} \frac{\partial\rho_0}{\partial z} \frac{R^2}{\zeta} \times \left[4\sum_{n=1}^{n=\infty} \frac{2n}{(2n-1)(2n+1)} J_{2n}(\zeta) \sum_{s=1}^{s=n} (-1)^{s-1} \sin(2s-1)\beta \cos(2s-1)\theta' - \iota\left\{\beta J_1(\zeta) + 4\sum_{n=1}^{n=\infty} \frac{2n+1}{2n(2n+2)} J_{2n+1}(\zeta) \sum_{s=1}^{s=n} (-1)^s \sin 2s\beta \cos 2s\theta'\right\}\right] \quad \ldots(60),$$

and the intensity of illumination is

$$I = \frac{4A^2}{\lambda^2\rho_0^2\rho_1^2} \left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{R^4}{\zeta^2} \left[\left\{\beta J_1(\zeta) + 4\sum_{n=1}^{n=\infty} \frac{2n+1}{2n(2n+2)} J_{2n+1}(\zeta) \sum_{s=1}^{s=n} (-1)^s \sin 2s\beta \cos 2s\theta'\right\}^2 + 16\left\{\sum_{n=1}^{n=\infty} \frac{2n}{(2n-1)(2n+1)} J_{2n}(\zeta) \sum_{s=1}^{s=n} (-1)^{s-1} \sin(2s-1)\beta \cos(2s-1)\theta'\right\}^2\right] \quad \ldots\ldots\ldots(61).$$

The diffraction pattern is thus symmetrical with respect to lines parallel and perpendicular to the bisector of the angle of the sector.

In the case of the heliometer objective, $\beta = \pi/2$, and since

$$\sum_1^n \cos(2s-1)\theta' = \sin 2n\theta'/(2\sin\theta'),$$

we have

$$I = \frac{4A^2}{\lambda^2\rho_0^2\rho_1^2} \left(\frac{\partial\rho_0}{\partial z}\right)^2 \frac{R^4}{\zeta^2} \left[\frac{\pi^2}{4} J_1^2(\zeta) + \frac{4}{\sin^2\theta'} \left\{\sum_1^{\infty} \frac{2n}{(2n-1)(2n+1)} \sin 2n\theta' J_{2n}(\zeta)\right\}^2\right] \quad \ldots\ldots\ldots(62).$$

When $\theta' = 0$ or π, the intensity may be simply expressed in terms of the Bessel's and Struve's functions J_1 and H_1: for in this case

$$\phi_0(t) = -\iota e^{\iota\kappa\omega t}\frac{2A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{R^2}{\zeta^2}\int_0^\zeta zdz\int_0^{\frac{\pi}{2}} e^{\pm\iota z\cos\psi}\,d\psi$$

$$= -\iota e^{\iota\kappa\omega t}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{\pi R^2}{\zeta^2}\int_0^\zeta (J_0 \pm \iota H_0)\,zdz$$

$$= -\iota e^{\iota\kappa\omega t}\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}\frac{\pi R^2}{\zeta}(J_1 \pm \iota H_1) \quad\text{......................(63)},$$

whence

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{\pi^2R^4}{\zeta^2}\{J_1^2(\zeta) + H_1^2(\zeta)\} \quad\text{...............(64)}.$$

When $\theta' = \pi/2$, the intensity is given by the simple expression

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{\pi^2R^4}{\zeta^2}\,.\,J_1^2(\zeta) \quad\text{......................(65)},$$

and when $\theta' = \pm\,\pi/4$

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2\frac{\pi^2R^4}{\zeta^2}\left[J_1^2(\zeta) + \frac{32}{\pi^2}\left\{\sum_1^\infty(-1)^{n-1}\frac{(4n-2)\,J_{4n-2}(\zeta)}{(4n-3)(4n-1)}\right\}^2\right] \cdots\text{(66)}.$$

CHAPTER VIII.

FRESNEL'S DIFFRACTION PHENOMENA.

82. IN the more general case of Fresnel's diffraction phenomena, the disturbance at the point (x_0, y_0, z_0) due to a radiant point at (x_1, y_1, z_1) is

$$\phi_0(t) = -\iota \frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z} e^{\iota(\kappa\omega t+\delta)} \iint e^{\iota\kappa\left\{x\left(\frac{x_1}{\rho_1}+\frac{x_0}{\rho_0}\right)+y\left(\frac{y_1}{\rho_1}+\frac{y_0}{\rho_0}\right)-\frac{x^2+y^2}{2}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)\right\}} dx\,dy,$$

or writing

$$\frac{2\pi}{\lambda}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)=k, \quad \frac{2\pi}{\lambda}\left(\frac{x_0}{\rho_0}+\frac{x_1}{\rho_1}\right)=l, \quad \frac{2\pi}{\lambda}\left(\frac{y_0}{\rho_0}+\frac{y_1}{\rho_1}\right)=m,$$

and remembering that

$$e^{\iota z} = \cos z + \iota \sin z = \sqrt{\frac{\pi z}{2}}\{J_{-\frac{1}{2}}(z) + \iota J_{\frac{1}{2}}(z)\},$$

$$\phi_0(t) = -\iota \frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z} e^{\iota(\kappa\omega t+\delta)} \frac{\pi}{2}\int (my)^{\frac{1}{2}}\{J_{-\frac{1}{2}}(my) + \iota J_{\frac{1}{2}}(my)\}e^{-\frac{1}{2}ky^2\iota}\,dy$$

$$\times \int (lx)^{\frac{1}{2}}\{J_{-\frac{1}{2}}(lx) + \iota J_{\frac{1}{2}}(lx)\}\, e^{-\frac{1}{2}kx^2\iota}\,dx \quad \text{..................(1)},$$

the integration being extended over the diffraction aperture*.

The expression for $\phi_0(t)$ thus depends upon integrals of the form

$$\gamma_\nu = \int (lx)^\nu J_{\nu-1}(lx)\, e^{-\frac{1}{2}kx^2\iota}\,dx \quad \text{......................(2)},$$

$$\sigma_\nu = \iota\int (lx)^{-\nu+1} J_\nu(lx)\, e^{-\frac{1}{2}kx^2\iota}\,dx \quad \text{......................(3)},$$

where ν is real and assumes in the case considered the value 1/2.

Now by successive integration by parts, using the formula

$$\int z^\nu J_{\nu-1}(z)\,dz = z^\nu J_\nu(z)$$

we find
$$\gamma_\nu = \frac{l^{2\nu-1}}{k^\nu}\{U_\nu(y, z) + \iota U_{\nu+1}(y, z)\}\, e^{-\frac{1}{2}y\iota} \quad \text{..................(4)},$$

* Lommel, *Abh. der K. Bayer. Akad. der Wissen.* xv. 233, 531 (1884–1886).

where $$U_\nu = \sum_0 (-1)^s (y/z)^{\nu+2s} J_{\nu+2s}(z) \quad\text{.........................(5)},$$

y being written for kx^2 and z for lx. In the same way from the formula

$$\frac{\partial}{\partial z}\{z^\nu J_\nu(z)\} = z^\nu J_{\nu-1}(z)$$

we obtain $$\gamma_\nu = \frac{l^{2\nu-1}}{k^\nu}\{V_{-\nu+2}(y, z) + \iota V_{-\nu+1}(y, z)\} e^{-\frac{1}{2}y\iota} \quad\text{..............(6)},$$

where $$V_\nu(y, z) = \sum_0 (-1)^s (y/z)^{-\nu-2s} J_{-\nu-2s}(z) \quad\text{..................(7)}.$$

Similarly from the formula

$$\frac{\partial}{\partial z}\{z^{-\nu} J_\nu(z)\} = -z^{-\nu} J_{\nu+1}(z)$$

we find $$\sigma_\nu = -\frac{l^{1-2\nu}}{k^{1-\nu}}\left\{U_\nu\left(\frac{z^2}{y}, z\right) + \iota U_{\nu+1}\left(\frac{z^2}{y}, z\right)\right\} e^{-\frac{1}{2}y\iota} \quad\text{...........(8)},$$

and from the formula

$$\int z^{-\nu+1} J_\nu(z)\, dz = -z^{-\nu+1} J_{\nu-1}(z)$$

we obtain

$$\sigma_\nu = -\frac{l^{1-2\nu}}{k^{1-\nu}}\left\{V_{-\nu+2}\left(\frac{z^2}{y}, z\right) + \iota V_{-\nu+1}\left(\frac{z^2}{y}, z\right)\right\} e^{-\frac{1}{2}y\iota} \quad\text{...........(9)},$$

where $U_\nu(z^2/y, z)$ and $V_\nu(z^2/y, z)$ are the series obtained from (5) and (7) respectively by writing therein z^2/y for y.

Since $$(lx)^{\nu+s} J_{\nu+s}(lx) = 0$$

for $x = 0$, if ν be positive, we have if $l > 0$

$$\int_0^r (lx)^\nu J_{\nu-1}(lx)\, e^{-\iota\frac{kx^2}{2}}\, dx = \frac{l^{2\nu-1}}{k^\nu}\{U_\nu(y, z) + \iota U_{\nu+1}(y, z)\} e^{-\iota\frac{y}{2}} \quad\text{...(10)},$$

where $y = kr^2$ and $z = lr$; and again since

$$(lx)^{\nu-s-1} J_{\nu-s-1}(lx) = 0$$

for $x = \infty$, if ν be less than unity and $l > 0$,

$$\int_r^\infty (lx)^\nu J_{\nu-1}(lx)\, e^{-\iota\frac{kx^2}{2}}\, dx = -\frac{l^{2\nu-1}}{k^\nu}\{V_{-\nu+2}(y, z) + \iota V_{-\nu+1}(y, z)\} e^{-\iota\frac{y}{2}} \quad\text{...(11)},$$

and if ν be not less than 1/2, these equations are also true when $l = 0$.

Hence if ν be less than unity and not less than 1/2,

$$\int_0^\infty (lx)^\nu J_{\nu-1}(lx)\, e^{-\iota\frac{kx^2}{2}}\, dx = \frac{l^{2\nu-1}}{k^\nu}[U_\nu(y, z) - V_{-\nu+2}(y, z)$$
$$+ \iota\{U_{\nu+1}(y, z) - V_{-\nu+1}(y, z)\}] e^{-\iota\frac{y}{2}} \quad\text{.........(12)}.$$

Now consider the integral

$$\int_0^\infty (lx)^\nu J_{\nu-1}(lx)\, e^{-(a+\frac{1}{2}k\iota)x^2}\, dx.$$

Since $$J_{\nu-1}(lx) = \sum_0^\infty (-1)^s \frac{(lx)^{\nu-1+2s}}{2^{\nu-1+2s}\,\Gamma(s+1)\,\Gamma(\nu+s)},$$

we have

$$\int_0^\infty (lx)^\nu J_{\nu-1}(lx)\, e^{-(a+\frac{1}{2}k\iota)x^2}\, dx$$

$$= \Sigma (-1)^s \frac{l^{2\nu-1+2s}}{2^{\nu-1+2s}\,\Gamma(s+1)\,\Gamma(\nu+s)} \int_0^\infty e^{-(a+\frac{1}{2}k\iota)x^2} x^{2\nu-1+2s}\, dx$$

$$= \Sigma (-1)^s \frac{l^{2\nu-1+2s}}{2^{\nu-1+2s}\,\Gamma(s+1)\,\Gamma(\nu+s)} \frac{\Gamma(\nu+s)}{2\,(a+\frac{1}{2}k\iota)^{\nu+s}}$$

$$= \frac{l^{2\nu-1}}{(2a+k\iota)^\nu} \Sigma (-1)^s \frac{1}{\Gamma(s+1)} \left(\frac{l^2}{4a+2k\iota}\right)^s$$

$$= \frac{l^{2\nu-1}}{(2a+k\iota)^\nu} e^{-\frac{l^2}{4a+2k\iota}}$$

$$= \frac{l^{2\nu-1}}{K^\nu} e^{-\frac{al^2}{K^2}} e^{\left(\frac{k}{2}\frac{l^2}{K^2}-\nu\phi\right)\iota}$$

where $K^2 = 4a^2 + k^2$, $\sin\phi = k/K$.

This equation holds for all positive values of ν, so long as a is positive, however small it may be: it also holds for $a = 0$ provided the integral on the left still retains a meaning, which is the case if $\nu \gtreqless \frac{1}{2}$ and $< \frac{3}{2}$. With this condition we then have

$$\int_0^\infty (lx)^\nu J_{\nu-1}(lx)\, e^{-\frac{1}{2}kx^2\iota}\, dx = \frac{l^{2\nu-1}}{k^\nu} e^{\left(\frac{l^2}{2k}-\nu\frac{\pi}{2}\right)\iota}$$

$$= \frac{l^{2\nu-1}}{k^\nu} e^{\left(\frac{z^2}{2y}-\nu\frac{\pi}{2}\right)\iota} \quad \text{..................(13)}$$

Again, from (8) and (9)

$$\iota \int_0^r (lx)^{1-\nu} J_\nu(lx)\, e^{-\frac{1}{2}kx^2\iota}\, dx = \frac{l^{1-2\nu}}{k^{1-\nu}} \left[U_\nu\left(\frac{l^2}{k}, 0\right) + \iota U_{\nu+1}\left(\frac{l^2}{k}, 0\right) - \left\{U_\nu\left(\frac{l^2}{k}, lr\right) + \iota U_{\nu+1}\left(\frac{l^2}{k}, lr\right)\right\} e^{-\frac{1}{2}kr^2\iota}\right]$$

$$= \frac{l^{1-2\nu}}{k^{1-\nu}} \left[V_{-\nu+2}\left(\frac{l^2}{k}, 0\right) + \iota V_{-\nu+1}\left(\frac{l^2}{k}, 0\right) - \left\{V_{-\nu+2}\left(\frac{l^2}{k}, lr\right) + \iota V_{-\nu+1}\left(\frac{l^2}{k}, lr\right)\right\} e^{-\frac{1}{2}kr^2\iota}\right] \text{...(14)},$$

and if $\nu > -\frac{1}{2}$ and $l \neq 0$,

$$[(lx)^{-\nu-2s} J_{\nu+2s}(lx)]_{x=\infty} = 0,$$

whence

$$\iota \int_r^\infty (lx)^{1-\nu} J_\nu(lx)\, e^{-\frac{1}{2}kx^2\iota}\, dx = \frac{l^{1-2\nu}}{k^{1-\nu}} \left\{U_\nu\left(\frac{l^2}{k}, lr\right) + \iota U_{\nu+1}\left(\frac{l^2}{k}, lr\right)\right\} e^{-\frac{1}{2}kr^2\iota} \text{...(15)},$$

and

$$\iota \int_0^\infty (lx)^{1-\nu} J_\nu(lx)\, e^{-\frac{1}{2}kx^2\iota}\, dx = \frac{l^{1-2\nu}}{k^{1-\nu}} \left\{U_\nu\left(\frac{l^2}{k}, 0\right) + \iota U_{\nu+1}\left(\frac{l^2}{k}, 0\right)\right\}^* \text{... (16)}.$$

* For the properties of Lommel's functions $U_\nu(y, z)$ and $V_\nu(y, z)$ see Appendix II.

83. Applying these formulae to different cases of diffraction, let us first consider that in which the aperture is a rectangle of width $2a$ in the direction of the axis of x and of length $2b$ in the direction of the axis of y.

Taking the origin at the centre of the rectangle, the limits of integration for x are $\pm a$ and for y are $\pm b$, and since for these limits the integrals $\sigma_{\frac{1}{2}}$ are zero, we have

$$\phi_0(t) = -\iota \frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right) e^{\iota(\kappa\omega t+\delta)}\, 2a\sqrt{\frac{\pi}{2u}}\,\{U_{\frac{1}{2}}(u, v) + \iota U_{\frac{3}{2}}(u, v)\}\, e^{-\iota\frac{u}{2}}$$

$$\times\, 2b\sqrt{\frac{\pi}{2u'}}\,\{U_{\frac{1}{2}}(u', v') + \iota U_{\frac{3}{2}}(u', v')\}\, e^{-\iota\frac{u'}{2}} \;\ldots (17),$$

where

$$\left.\begin{aligned} u &= \frac{2\pi}{\lambda}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)a^2, & u' &= \frac{2\pi}{\lambda}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)b^2 \\ v &= \frac{2\pi}{\lambda}\left(\frac{x_0}{\rho_0}+\frac{x_1}{\rho_1}\right)a, & v' &= \frac{2\pi}{\lambda}\left(\frac{y_0}{\rho_0}+\frac{y_1}{\rho_1}\right)b \end{aligned}\right\} \;\ldots\ldots\ldots\ldots (18),$$

and the intensity is

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 16a^2b^2 \times \frac{\pi}{2u}\{U_{\frac{1}{2}}^2(u, v) + U_{\frac{3}{2}}^2(u, v)\}$$

$$\times\, \frac{\pi}{2u'}\{U_{\frac{1}{2}}^2(u', v') + U_{\frac{3}{2}}^2(u', v')\} \;\ldots\ldots\ldots (19).$$

Now in Fraunhofer's special case of diffraction, $\rho_0^{-1} + \rho_1^{-1} = 0$, so that we obtain this case by writing $u = u' = 0$: but when $u = 0$,

$$\frac{\pi}{2u}\{U_{\frac{1}{2}}^2(u, v) + U_{\frac{3}{2}}^2(u, v)\} = \frac{\pi}{2v} J_{\frac{1}{2}}^2 v = \frac{\sin^2 v}{v^2},$$

and the expression for the intensity becomes that found in the last chapter, namely,

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 16a^2b^2 \frac{\sin^2 v}{v^2}\frac{\sin^2 v'^2}{v'^2}.$$

In the general expression (19) relating to Fresnel's phenomena, the last two terms have the form

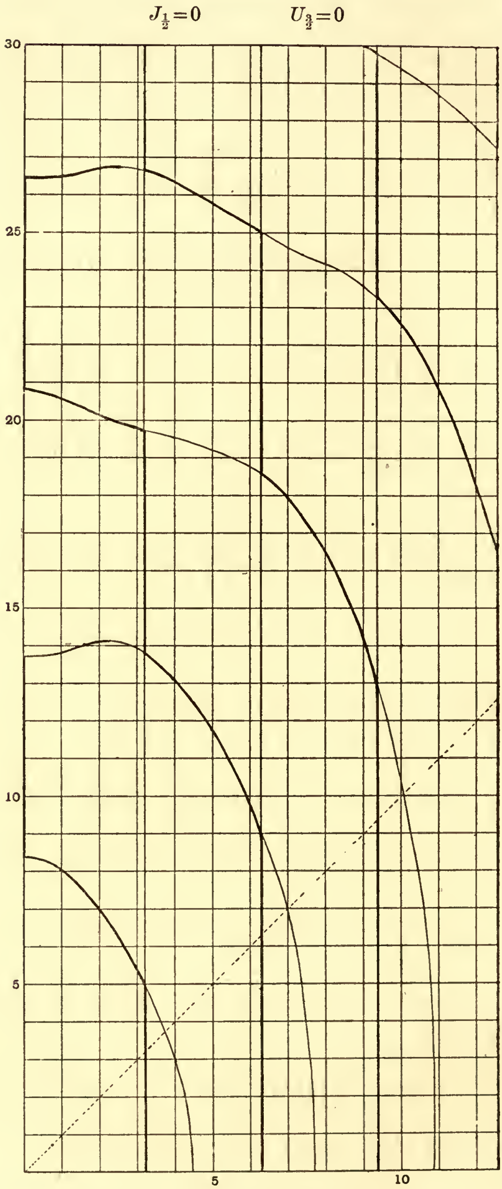
$$M = \frac{\pi}{2u}\{U_{\frac{1}{2}}^2(u, v) + U_{\frac{3}{2}}^2(u, v)\} \;\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and we have to examine the character of this function.

For a given position of the screen of observation, that is for a given value of u, M is a maximum or a minimum for values of v that make $\partial M/\partial v = 0$. But

$$\frac{\partial M}{\partial v} = \frac{\pi}{u}\left\{U_{\frac{1}{2}}\frac{\partial U_{\frac{1}{2}}}{\partial v} + U_{\frac{3}{2}}\frac{\partial U_{\frac{3}{2}}}{\partial v}\right\} = -\frac{\pi v}{u^2}U_{\frac{3}{2}}(U_{\frac{1}{2}} + U_{\frac{5}{2}}) = -\frac{\pi v^{\frac{1}{2}}}{u^{\frac{3}{2}}} J_{\frac{1}{2}}(v)\, U_{\frac{3}{2}}(u, v) \ldots (21);$$

PLATE I.*



The regions of the lines that give minima are indicated by heavy ruling: those that give maxima by light ruling.

* Plates I. to IV. are, with the permission of the Royal Bavarian Academy of Sciences, reduced from plates published with Lommel's papers.

hence M is a maximum or a minimum when either $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = \sqrt{\frac{2}{\pi}} \sin v = 0$, that is when $v = n\pi$ $(n = 0, 1, 2 \ldots)$, or when $u^{-\frac{3}{2}} U_{\frac{3}{2}}(u, v) = 0$. Now

$$v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = -\frac{\partial}{\partial v}\{v^{\frac{1}{2}} J_{-\frac{1}{2}}(v)\}, \text{ and } U_{\frac{3}{2}}(u, v) = -\frac{u}{v}\frac{\partial}{\partial v} U_{\frac{1}{2}}(u, v),$$

and therefore the maximum and minimum values of M occur in correspondence with either maximum or minimum values of $U_{\frac{1}{2}}(u, v)$ or of $v^{\frac{1}{2}} J_{-\frac{1}{2}}(v) = \sqrt{\frac{2}{\pi}} \cos v$. The intensity at the minima is never zero, since $U_{\frac{1}{2}}$ and $U_{\frac{3}{2}}$ do not vanish simultaneously, as may be seen at once from their expression in terms of Fresnel's integrals.

In order to distinguish between the values of v that give the maxima and the minima, we must form the expression $\partial^2 M/\partial v^2$: now

$$\frac{\partial^2 M}{\partial v^2} = -\frac{\pi}{u^{\frac{3}{2}}}\left\{v^{\frac{1}{2}} J_{\frac{1}{2}}(v)\frac{\partial U_{\frac{3}{2}}}{\partial v} + U_{\frac{3}{2}}\frac{\partial}{\partial v}\{v^{\frac{1}{2}} J_{\frac{1}{2}}(v)\}\right\}$$

$$= \frac{\pi}{u^{\frac{3}{2}}}\left\{\frac{v^{\frac{3}{2}}}{u} J_{\frac{1}{2}}(v) U_{\frac{5}{2}} - v^{\frac{1}{2}} J_{-\frac{1}{2}}(v) . U_{\frac{3}{2}}\right\} \quad \text{.....................(22).}$$

Hence the roots of $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = 0$ correspond to maxima or minima of M according as $v^{\frac{1}{2}} J_{-\frac{1}{2}} U_{\frac{3}{2}}$ or $\cos v . U_{\frac{3}{2}}$ is positive or negative, and the roots of $u^{-\frac{3}{2}} U_{\frac{3}{2}} = 0$ give the maxima or minima according as $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) U_{\frac{5}{2}}$ or $\sin v . U_{\frac{5}{2}}$ is negative or positive.

If, however, $J_{\frac{1}{2}}(v) = 0$ and $U_{\frac{3}{2}}(u, v) = 0$ simultaneously, $\partial^2 M/\partial v^2 = 0$, and in this case

$$\frac{\partial^3 M}{\partial v^3} = 2\pi\frac{v^{\frac{3}{2}}}{u^{\frac{5}{2}}} J_{-\frac{1}{2}} . U_{\frac{5}{2}} = -2\pi\frac{v^{\frac{1}{2}}}{u^{\frac{3}{2}}} J_{-\frac{1}{2}} U_{\frac{1}{2}} \quad \text{.................(23),}$$

which does not vanish unless $v = 0$, because for these values of v, $v^{\frac{1}{2}} J_{-\frac{1}{2}}$ and $U_{\frac{1}{2}}$ have their maximum and minimum values.

Again $\partial^2 M/\partial v^2 = 0$, if $U_{\frac{3}{2}}(u, v) = 0$ and $U_{\frac{5}{2}}(u, v) = 0$, while $\partial^3 M/\partial v^3$ does not vanish, but is equal to

$$-\pi v^{\frac{3}{2}} u^{-\frac{5}{2}} J_{\frac{1}{2}} U_{\frac{7}{2}} = -\pi v u^{-2} J_{\frac{1}{2}} J_{\frac{3}{2}} \quad \text{.....................(24).}$$

Hence the roots of $U_{\frac{3}{2}}(u, v) = 0$, other than $v = 0$, that satisfy either of the equations $J_{\frac{1}{2}}(v) = 0$, $U_{\frac{5}{2}}(u, v) = 0$ correspond to neither a maximum nor a minimum value of M.

Now when $v = 0$, that is at the centre of the pattern

$$\frac{\partial^4 M}{\partial v^4} = \frac{\pi}{u^{\frac{3}{2}}}\left[v^{\frac{1}{2}} J_{-\frac{1}{2}}\left(U_{\frac{3}{2}} + \frac{3}{u} U_{\frac{5}{2}}\right)\right]_{v=0} = \frac{\sqrt{2\pi}}{u^{\frac{3}{2}}}\left\{U_{\frac{3}{2}}(u, 0) + \frac{3}{u} U_{\frac{5}{2}}(u, 0)\right\} \ldots (25),$$

and when $U_{\frac{3}{2}}(u, 0) = 0$,

$$\frac{\partial^4 M}{\partial v^4} = \frac{3\sqrt{2\pi}}{u^{\frac{5}{2}}} U_{\frac{5}{2}}(u, 0).$$

But $\sqrt{\frac{\pi}{2u}}\, U_{\frac{5}{2}}(u, 0) = 1 - \sqrt{\frac{\pi}{2u}}\, U_{\frac{1}{2}}(u, 0) = 1 - \int_0^1 \cos \tfrac{1}{2}u\,(1-y^2)\,.\,dy,$

and therefore $U_{\frac{5}{2}}(u, 0)$ is positive and for these values of u, M is a minimum.

When, however, $v = 0$ and $U_{\frac{3}{2}}(u, 0) \neq 0$, then $\partial M/\partial v = 0$ and

$$\frac{\partial^2 M}{\partial v^2} = -\frac{\pi}{u^{\frac{3}{2}}}\,[v^{\frac{1}{2}} J_{-\frac{1}{2}} U_{\frac{3}{2}}] = -\frac{\sqrt{2\pi}}{u^{\frac{3}{2}}}\, U_{\frac{3}{2}}(u, 0),$$

and M is a maximum when u is between u_{2n} and u_{2n+1}, and a minimum when it is between u_{2n+1} and u_{2n+2} $(n = 0, 1, 2\ldots)$ where $u_{2n}, u_{2n+1}\ldots$ are the successive roots of $U_{\frac{3}{2}}(u, 0) = 0$.

The edge of the geometrical shadow of the screen is very approximately at the point for which $u = v$, the shadow lying on the side for which $u < v$. Now when $u < v$ we have

$$U_{\frac{3}{2}}(u, v) = -\frac{1}{2}\left[V_{\frac{3}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} + V_{\frac{3}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \sin v$$
$$-\frac{1}{2}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} - V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \cos v \ldots\ldots(26);$$

hence for points within the shadow that satisfy the equation $v_{\frac{1}{2}} J_{\frac{1}{2}}(v) = 0$, that is for which $v = n\pi$ $(n = 1, 2, \ldots)$,

$$U_{\frac{3}{2}}(u, v) = -\frac{1}{2}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} - V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \cos n\pi,$$

and $$v^{\frac{1}{2}} J_{-\frac{1}{2}}(v)\, U_{\frac{3}{2}}(u, v) = -\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} - V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right],$$

and this expression is essentially negative, so that within the shadow the roots of $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = 0$ give the minima. Further it follows from (26) that the equations $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = 0$ and $u^{-\frac{3}{2}} U_{\frac{3}{2}}(u, v) = 0$ have no common roots when $u < v$, and that only one root of the latter equation lies between two consecutive roots of the former, so that the roots of $u^{-\frac{3}{2}} U_{\frac{3}{2}}(u, v) = 0$ give the maxima within the shadow*.

84. When the diffraction is produced by an opaque rectangle of breadth $2a$ and length $2b$, we have to integrate for x from $-\infty$ to $-a$ and from a to ∞, and for y from $-\infty$ to $-b$ and from b to ∞. Hence

$$\phi_0(t) = -\iota \frac{A}{\lambda\rho_0\rho_1} \frac{\partial\rho_0}{\partial z} e^{\iota\kappa(\omega t+\delta)}\,.\,2a\sqrt{\frac{\pi}{2u}}\,\{V_{\frac{3}{2}}(u, v) + \iota V_{\frac{1}{2}}(u, v)\}\, e^{-\iota\frac{u}{2}}$$
$$\times\, 2b\sqrt{\frac{\pi}{2u'}}\,\{V_{\frac{3}{2}}(u', v') + \iota V_{\frac{1}{2}}(u', v')\}\, e^{-\iota\frac{u'}{2}} \ldots(27)$$

* See Plate I. for a graphical representation of the equations $J_{\frac{1}{2}}(v) = 0$, $U_{\frac{3}{2}}(u, v) = 0$.

and the intensity is

$$I = \frac{A^2}{\lambda^2 \rho_0^2 \rho_1^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 . 16 a^2 b^2 \times \frac{\pi}{2u} \{V_{\frac{3}{2}}^2 (u, v) + V_{\frac{1}{2}}^2 (u, v)\}$$
$$\times \frac{\pi}{2u'} \{V_{\frac{3}{2}}^2 (u', v') + V_{\frac{1}{2}}^2 (u', v')\} \quad \ldots (28).$$

In this case, since the values of $V_{\frac{1}{2}}^2 (u, 0)$ and $V_{\frac{3}{2}}^2 (u, 0)$ decrease continuously as u increases, the intensity at the centre of the geometrical shadow varies continuously as the screen of observation is moved towards or away from the opaque obstacle without passing through maxima and minima.

The function that we have here to consider is

$$M = \frac{\pi}{2u} \{V_{\frac{1}{2}}^2 (u, v) + V_{\frac{3}{2}}^2 (u, v)\}.$$

Now if u be constant

$$\frac{\partial M}{\partial v} = \frac{\pi}{u} \left\{V_{\frac{1}{2}} \frac{\partial V_{\frac{1}{2}}}{\partial v} + V_{\frac{3}{2}} \frac{\partial V_{\frac{3}{2}}}{\partial v}\right\} = -\frac{\pi v}{u^2} V_{\frac{1}{2}} (V_{-\frac{1}{2}} + V_{\frac{3}{2}}) .$$
$$= -\frac{\pi v^{\frac{1}{2}}}{u^{\frac{3}{2}}} J_{\frac{1}{2}} (v)\, V_{\frac{1}{2}} (u, v) \ldots\ldots\ldots (29).$$

Hence for a given value of u, the expression is a maximum or a minimum when, either

$$v^{\frac{1}{2}} J_{\frac{1}{2}} (v) = \sqrt{\frac{2}{\pi}} \sin v = 0, \text{ or } V_{\frac{1}{2}} (u, v) = 0,$$

and since $$v^{\frac{1}{2}} J_{\frac{1}{2}} (v) = -\frac{\partial}{\partial v} (v^{\frac{1}{2}} J_{-\frac{1}{2}} (v)), \qquad V_{\frac{1}{2}} (u, v) = -\frac{u}{v} \frac{\partial V_{\frac{3}{2}}}{\partial v}$$

the maxima and minima occur in correspondence with either maxima or minima values of either $v^{\frac{1}{2}} J_{-\frac{1}{2}} (v)$ or $V_{\frac{3}{2}} (u, v)$.

To distinguish between the maxima and minima we have

$$\frac{\partial^2 M}{\partial v^2} = -\frac{\pi}{u^{\frac{3}{2}}} \left\{v^{\frac{1}{2}} J_{\frac{1}{2}} \frac{\partial V_{\frac{1}{2}}}{\partial v} + V_{\frac{1}{2}} \frac{\partial (v^{\frac{1}{2}} J_{\frac{1}{2}})}{\partial v}\right\} = \frac{\pi}{u^{\frac{3}{2}}} \left\{\frac{v^{\frac{3}{2}}}{u} J_{\frac{1}{2}} V_{-\frac{1}{2}} - v^{\frac{1}{2}} J_{-\frac{1}{2}} V_{\frac{1}{2}}\right\} \ldots\ldots (30),$$

whence it follows that to the roots of $v^{\frac{1}{2}} J_{\frac{1}{2}} (v) = 0$ correspond maxima or minima values of M according as $J_{-\frac{1}{2}} V_{\frac{1}{2}}$ is positive or negative and that the roots of $V_{\frac{1}{2}} (u, v) = 0$ determine maxima or minima according as $J_{\frac{1}{2}} V_{-\frac{1}{2}}$ is negative or positive.

At the centre of the pattern, where $v = 0$, we have

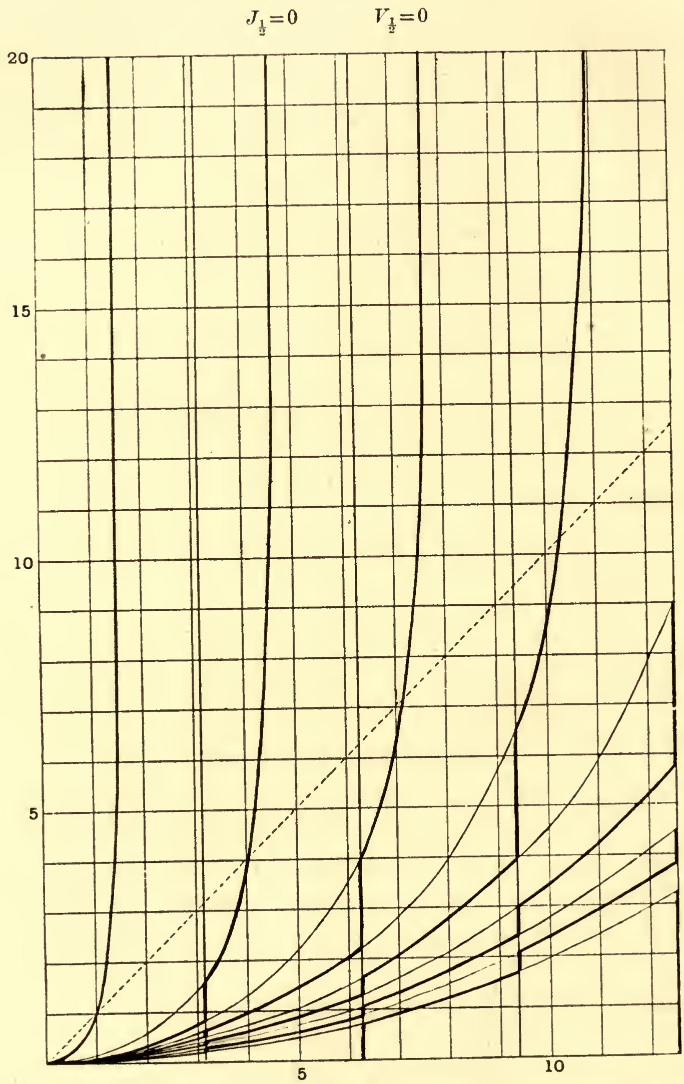
$$\frac{\partial^2 M}{\partial v^2} = -\frac{1}{u} \sqrt{\frac{2\pi}{u}} V_{\frac{1}{2}} (u, 0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

and since $V_{\frac{1}{2}} (u, 0)$ is always positive, M is a maximum at this point.

When $v^{\frac{1}{2}} J_{\frac{1}{2}} (v)$ and $V_{\frac{1}{2}} (u, v)$ vanish simultaneously, $\partial^2 M / \partial v^2 = 0$; in this case

$$\frac{\partial^3 M}{\partial v^3} = \frac{2\pi}{u^{\frac{5}{2}}} v^{\frac{3}{2}} J_{-\frac{1}{2}} (v)\, V_{-\frac{1}{2}} (u, v) = -\frac{2\pi}{u^{\frac{3}{2}}} v^{\frac{1}{2}} J_{-\frac{1}{2}} (v)\, V_{\frac{3}{2}} (u, v) \ldots\ldots (32)$$

PLATE II.



The regions of the lines that give minima are indicated by heavy ruling: those that give maxima by light ruling.

and this is not zero, since $J_{-\frac{1}{2}}(v)$ and $V_{\frac{3}{2}}(u, v)$ do not become zero at the same time as $J_{\frac{1}{2}}(v)$ and $V_{\frac{1}{2}}(u, v)$. To such values of v therefore correspond neither maxima nor minima of M.

These are the only exceptional cases, for a graphical representation (Plate II) of the equation $V_{\frac{1}{2}}(u, v) = 0$ shows that at all points of this curve du/dv is always positive: now

$$\frac{du}{dv} = \frac{2\frac{v}{u} V_{-\frac{1}{2}}}{V_{\frac{3}{2}} + \frac{v^2}{u^2} V_{-\frac{1}{2}}} \quad \text{.............................(33)},$$

and since $V_{\frac{3}{2}}$ does not vanish with $V_{\frac{1}{2}}$, $V_{-\frac{1}{2}}$ will not do so.

For points within the geometrical shadow $u > v$ and

$$V_{\frac{1}{2}}(u, v) = \frac{1}{2}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} + V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \cos v$$
$$+ \frac{1}{2}\left[V_{\frac{3}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} - V_{\frac{3}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \sin v \text{...(34)}.$$

We have then when $v = n\pi$

$$V_{\frac{1}{2}}(u, v) = \frac{1}{2}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} + V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \cos n\pi$$

and the expression $J_{-\frac{1}{2}} V_{\frac{1}{2}}$ that determines the maxima and the minima has the positive value

$$\frac{1}{\pi}\sqrt{\frac{1}{2n}}\left[V_{\frac{1}{2}}\left\{\frac{(u-v)^2}{u}, 0\right\} + V_{\frac{1}{2}}\left\{\frac{(u+v)^2}{u}, 0\right\}\right] \cos^2 n\pi.$$

Accordingly within the shadow the roots of $v^{\frac{1}{2}} J_{-\frac{1}{2}}(v) = 0$ give the maxima. Further it follows from (34) that within the shadow, the equations $v^{\frac{1}{2}} J_{\frac{1}{2}}(v) = 0$ and $V_{\frac{1}{2}}(u, v) = 0$ have no common root and that only one root of the latter occurs between two consecutive roots of the former equation, and accordingly the minima of M are given by the roots of $V_{\frac{1}{2}}(u, v) = 0$.

85. Let us now take the case of the diffraction caused by a train of waves passing an infinite screen bounded by a straight line. Taking this line as the axis of y, we have as the limits of integration 0 and ∞ for x and $-\infty$ and $+\infty$ for y, and consequently from (13) and (16) we have

$$\phi_0(t) = -\iota \frac{A}{\lambda \rho_0 \rho_1}\left(\frac{\partial \rho_0}{\partial z}\right) e^{\iota(\kappa\omega t + \delta)} \cdot 2\sqrt{\frac{\pi}{2k}}\, e^{\left(\frac{m^2}{2k} - \frac{\pi}{4}\right)\iota}$$
$$\times \sqrt{\frac{\pi}{2k}}\left\{U_{\frac{1}{2}}\left(\frac{l^2}{k}, 0\right) + \iota U_{\frac{3}{2}}\left(\frac{l^2}{k}, 0\right) + e^{\left(\frac{l^2}{2k} - \frac{\pi}{4}\right)\iota}\right\} \text{...(35)},$$

and the intensity is

$$I = \frac{A^2}{\lambda^2 \rho_0^2 \rho_1^2} \left(\frac{\partial \rho_0}{\partial z}\right)^2 4 \frac{\pi}{2k} \cdot \frac{\pi}{2k} \left[\left\{ \cos\left(\frac{l^2}{2k} - \frac{\pi}{4}\right) + U_{\frac{1}{2}}\left(\frac{l^2}{k}, 0\right) \right\}^2 + \left\{ \sin\left(\frac{l^2}{2k} - \frac{\pi}{4}\right) + U_{\frac{3}{2}}\left(\frac{l^2}{k}, 0\right) \right\}^2 \right] \ldots (36),$$

or in terms of a new unit of intensity, that is practically the intensity at a point on the screen due to the uninterrupted wave,

$$I = \left\{ \tfrac{1}{2} \cos\left(\frac{l^2}{2k} - \frac{\pi}{4}\right) + \tfrac{1}{2} U_{\frac{1}{2}}\left(\frac{l^2}{k}, 0\right) \right\}^2 + \left\{ \tfrac{1}{2} \sin\left(\frac{l^2}{2k} - \frac{\pi}{4}\right) + \tfrac{1}{2} U_{\frac{3}{2}}\left(\frac{l^2}{k}, 0\right) \right\}^2 \ldots (37).$$

This formula holds for all points, whether within or without the geometrical shadow of the screen, l being negative in the first case and positive in the second case, but it is convenient to alter the expression so that in both cases l is regarded as positive. Now, since

$$U_{\frac{1}{2}}\left(\frac{l^2}{k}, 0\right) = \frac{l}{\sqrt{2k}} \sum_0 (-1)^s \frac{(l^2/2k)^{2s}}{\Gamma(\frac{3}{2} + 2s)},$$

$$U_{\frac{3}{2}}\left(\frac{l^2}{k}, 0\right) = \frac{l}{\sqrt{2k}} \sum_0 (-1)^s \frac{(l^2/2k)^{2s+1}}{\Gamma(\frac{5}{2} + 2s)},$$

we have

$$U_{\frac{1}{2}}\left\{\frac{(-l)^2}{k}, 0\right\} = -U_{\frac{1}{2}}\left(\frac{l^2}{k}, 0\right), \qquad U_{\frac{3}{2}}\left\{\frac{(-l)^2}{k}, 0\right\} = -U_{\frac{3}{2}}\left(\frac{l^2}{k}, 0\right),$$

and writing $l^2/k = u$, the intensity at points outside the geometrical shadow is given by

$$\begin{aligned} I_1 &= \left\{ \tfrac{1}{2} \cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2} U_{\frac{1}{2}}(u, 0) \right\}^2 + \left\{ \tfrac{1}{2} \sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2} U_{\frac{3}{2}}(u, 0) \right\}^2 \\ &= \left\{ \cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2} V_{\frac{3}{2}}(u, 0) \right\}^2 + \left\{ \sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2} V_{\frac{1}{2}}(u, 0) \right\}^2 \ldots (38), \end{aligned}$$

while for points within the shadow we have

$$\begin{aligned} I_2 &= \left\{ \tfrac{1}{2} \cos\left(\frac{u}{2} - \frac{\pi}{4}\right) - \tfrac{1}{2} U_{\frac{1}{2}}(u, 0) \right\}^2 + \left\{ \tfrac{1}{2} \sin\left(\frac{u}{2} - \frac{\pi}{4}\right) - \tfrac{1}{2} U_{\frac{3}{2}}(u, 0) \right\}^2 \\ &= \tfrac{1}{4} V_{\frac{3}{2}}^2(u, 0) + \tfrac{1}{4} V_{\frac{1}{2}}^2(u, 0) \ldots\ldots (39), \end{aligned}$$

u in both cases being regarded as positive. At the edge of the shadow where $l = 0$, the intensity as given by the above expressions is

$$(I_1)_0 = (I_2)_0 = 1/4.$$

Now $V_{\frac{1}{2}}^2(u, 0)$ and $V_{\frac{3}{2}}^2(u, 0)$ both decrease continuously as u increases and only vanish when u is infinitely great: hence on moving into the geometrical shadow away from its edge we find a regular decrease of intensity. Outside the shadow the illumination is never nil: for that to be the case, we should require that

$$V_{\frac{3}{2}}(u, 0) = -2 \cos\left(\frac{u}{2} - \frac{\pi}{4}\right), \qquad V_{\frac{1}{2}}(u, 0) = -2 \sin\left(\frac{u}{2} - \frac{\pi}{4}\right),$$

or
$$V_{\frac{1}{2}}^2(u, 0) + V_{\frac{3}{2}}^2(u, 0) = 4,$$
and this is impossible, since $V_{\frac{1}{2}}(u, 0)$ and $V_{\frac{3}{2}}(u, 0)$ are both numerically less than $1/\sqrt{2}$.

In order to determine the maxima and minima in the space outside the geometrical shadow, let us write
$$P = \tfrac{1}{2}\sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{3}{2}}(u, 0), \qquad Q = \tfrac{1}{2}\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{1}{2}}(u, 0) \quad \text{...(40)},$$
then
$$\frac{dP}{du} = \tfrac{1}{4}\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{4}U_{\frac{1}{2}}(u, 0) = \tfrac{1}{2}Q,$$
and since
$$U_{-\frac{1}{2}}(u, 0) + U_{\frac{3}{2}}(u, 0) = u^{-\frac{1}{2}}\left[v^{\frac{1}{2}}J_{-\frac{1}{2}}(v)\right]_{v=0} = \sqrt{\frac{2}{\pi u}},$$
$$\begin{aligned}\frac{dQ}{du} &= -\tfrac{1}{4}\sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{4}U_{-\frac{1}{2}}(u, 0)\\ &= \frac{1}{2}\left\{\frac{1}{2}\sqrt{\frac{2}{\pi u}} - \tfrac{1}{2}U_{\frac{3}{2}}(u, 0) - \tfrac{1}{2}\sin\left(\frac{u}{2} - \frac{\pi}{4}\right)\right\} = \frac{1}{2}\left\{\frac{1}{2}\sqrt{\frac{2}{\pi u}} - P\right\}.\end{aligned}$$
Hence
$$\begin{aligned}\frac{dI_1}{du} &= 2P\frac{dP}{du} + 2Q\frac{dQ}{du} = \frac{1}{2}\sqrt{\frac{2}{\pi u}}\,Q\\ &= \frac{1}{2}\sqrt{\frac{2}{\pi u}}\left\{\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}V_{\frac{3}{2}}(u, 0)\right\} \quad \text{...(41)},\end{aligned}$$
and the maxima and minima occur in accordance with the roots of
$$\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}V_{\frac{3}{2}}(u, 0) = 0 \quad \text{..................(42)}.$$

Now $V_{\frac{3}{2}}(u, 0)$ is always negative and less than $1/\sqrt{2}$ in absolute value and therefore this equation is only satisfied if $\cos\left(\frac{u}{2} - \frac{\pi}{4}\right)$ is positive and less than $1/\sqrt{2}$, in other words when u lies between $(4n + 1)\pi$ and $(4n + 3/2)\pi$ or between $(4n + 7/2)\pi$ and $(4n + 4)\pi$, $(n = 0, 1, 2 \dots)$.

Since $\frac{1}{2}V_{\frac{3}{2}}(u, 0)$ is small and approaches the value zero as u increases, the roots of (42) are approximately the same as those of $\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) = 0$ which gives
$$u = \frac{4n + 3}{2}\pi, \qquad (n = 0, 1, 2, \dots) \quad \text{..................(43)},$$
a second approximation to their values being
$$u = \frac{4n + 3}{2}\pi + \cos n\pi\, V_{\frac{3}{2}}\left(\frac{4n + 3}{2}\pi, 0\right) \quad \text{...............(44)},$$
and since the intensity of the maxima and minima is
$$I = \left\{\sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}V_{\frac{1}{2}}(u, 0)\right\}^2,$$

it follows that the maxima occur when n is odd, and that the minima correspond to the even values of n*.

The locus of a band in space is given by

$$\frac{2\pi}{\lambda u}\left(\frac{x_0}{\rho_0}+\frac{x_1}{\rho_1}\right)^2=\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)$$

where u has the value corresponding to a maximum or a minimum. Since $\rho_0 \risingdotseq z_0$, this is approximately the hyperbola

$$\frac{2\pi}{\lambda u}(\rho_1 x_0 + x_1 z_0)^2 - \rho_1\left(z_0+\frac{\rho_1}{2}\right)^2+\frac{\rho_1^3}{4}=0,$$

having its vertices very nearly at the source of light and the edge of the diffracting screen.

86. It has been pointed out in Chapter II. that in the cases of interference therein considered the phenomenon is considerably modified by the effect of diffraction. As an instance of the disturbance thus produced, let us consider the case of Fresnel's biprism†, the acute angles of which are equal, and let us suppose that the source of light is a radiant point in the plane through the edge of the prism perpendicular to its flat face. We have then to deal with two correlated sources of light—the virtual images of the luminous point produced by the two halves of the prism—the streams from which pass through the corresponding parts of the prism.

Let the coordinates of these two sources be $(\pm c, 0, z_1)$: then since it is clear that the disturbance at the point (x_0, y_0, z_0) due to the source $(-c, 0, z_1)$ is the same as that at the point $(-x_0, y_0, z_0)$ due to the source $(c, 0, z_1)$, we have

$$\begin{aligned}\phi_0(t) = -\iota\frac{A}{\lambda\rho_0\rho_1}\frac{\partial\rho_0}{\partial z}e^{\iota(\kappa\omega t+\delta)}\,.\,2\sqrt{\frac{\pi}{2k}}e^{\left(\frac{m^2}{2k}-\frac{\pi}{4}\right)\iota} \\ \times\sqrt{\frac{\pi}{2k}}\left[U_{\frac{1}{2}}\left(\frac{l_1^2}{k},0\right)+\iota U_{\frac{3}{2}}\left(\frac{l_1^2}{k},0\right)+e^{\left(\frac{l_1^2}{2k}-\frac{\pi}{4}\right)\iota}\right. \\ \left.+U_{\frac{1}{2}}\left(\frac{l_2^2}{k},0\right)+\iota U_{\frac{3}{2}}\left(\frac{l_2^2}{k},0\right)+e^{\left(\frac{l_2^2}{2k}-\frac{\pi}{4}\right)\iota}\right]\end{aligned} \quad \text{...(45)},$$

where

$$\left.\begin{aligned} l_1 &= \frac{2\pi}{\lambda}\left(\frac{c}{\rho_1}+\frac{x_0}{\rho_0}\right), & l_2 &= \frac{2\pi}{\lambda}\left(\frac{c}{\rho_1}-\frac{x_0}{\rho_0}\right) \\ m &= \frac{2\pi}{\lambda}\frac{y_0}{\rho_0}, & k &= \frac{2\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)\end{aligned}\right\} \quad \text{...............(45)};$$

* The approximate value of u given by (44) only requires a small correction ϵ determined by

$$Q+\epsilon\frac{dQ}{du}=0, \quad \text{or } 2Q+\epsilon\left\{\frac{1}{2}\sqrt{\frac{2}{\pi u}}-P\right\}=0:$$

the corresponding intensity is $P^2(u+\epsilon)$, and

$$P(u+\epsilon)=P+\epsilon\frac{dP}{du}+\frac{\epsilon^2}{2}\frac{d^2P}{du^2}=P+\tfrac{1}{2}\epsilon Q+\tfrac{1}{4}\epsilon^2\frac{dQ}{du}=P+\tfrac{1}{4}\epsilon Q.$$

† Struve, *Wied. Ann.* xv. 49 (1882). Weber, *ibid.* VIII. 407 (1879).

and writing $u_1 = l_1^2/k$, $u_2 = l_2^2/k$ and adopting a new scale of intensities, we have as the intensity at the given point

$$I = \left\{\frac{1}{2}\cos\left(\frac{u_1}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{1}{2}}(u_1, 0) + \frac{1}{2}\cos\left(\frac{u_2}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{1}{2}}(u_2, 0)\right\}^2$$

$$+ \left\{\frac{1}{2}\sin\left(\frac{u_1}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{3}{2}}(u_1, 0) + \frac{1}{2}\sin\left(\frac{u_2}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{3}{2}}(u_2, 0)\right\}^2 \quad \text{...(46).}$$

To determine the maxima and minima of intensity we have to find the value of $\partial I/\partial x_0$ and since $\partial u_1/\partial x_0 = a\sqrt{u_1}$, $\partial u_2/\partial x_0 = -a\sqrt{u_2}$ where a is a constant, we have

$$\frac{\partial I}{\partial x_0} = 2(P_1 + P_2)\left(a\sqrt{u_1}\frac{\partial P_1}{\partial u_1} - a\sqrt{u_2}\frac{\partial P_2}{\partial u_2}\right)$$

$$+ 2(Q_1 + Q_2)\left(a\sqrt{u_1}\frac{\partial Q_1}{\partial u_1} - a\sqrt{u_2}\frac{\partial Q_2}{\partial u_2}\right) \quad \text{...(47),}$$

where

$$\left.\begin{aligned} P &= \frac{1}{2}\sin\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{3}{2}}(u, 0) \\ Q &= \frac{1}{2}\cos\left(\frac{u}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{1}{2}}(u, 0) \end{aligned}\right\} \quad \text{......(48),}$$

and since
$$\frac{\partial P}{\partial u} = \tfrac{1}{2}Q, \qquad \frac{\partial Q}{\partial u} = \frac{1}{2}\left\{\sqrt{\frac{1}{2\pi u}} - P\right\},$$

we obtain

$$\frac{\partial I}{\partial x_0} = (P_1 + P_2)(a\sqrt{u_1}Q_1 - a\sqrt{u_2}Q_2) + (Q_1 + Q_2)(a\sqrt{u_2}P_2 - a\sqrt{u_1}P_1)$$

$$= a(\sqrt{u_1} + \sqrt{u_2})(P_2Q_1 - P_1Q_2) \quad \text{......(49),}$$

and the maxima and minima occur, when

$$P_1/P_2 = Q_1/Q_2 \quad \text{......(50).}$$

Now within the part of the field common to the two streams, as determined by geometrical optics, l_1 and l_2 are both positive, while outside this region l_1 and l_2 have opposite signs, l_2 being negative on the side of positive x and positive on the side of negative x. Hence giving l_1 and l_2 their absolute values we have to write for the field common to the streams

$$\left.\begin{aligned} P_n &= \frac{1}{2}\sin\left(\frac{u_n}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{3}{2}}(u_n, 0) \\ Q_n &= \frac{1}{2}\cos\left(\frac{u_n}{2} - \frac{\pi}{4}\right) + \tfrac{1}{2}U_{\frac{1}{2}}(u_n, 0) \end{aligned}\right\} (n = 1, 2) \quad \text{......(51),}$$

and for the outer portions of the complex field

$$\left.\begin{aligned} P_1 &= \frac{1}{2}\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right) \pm \tfrac{1}{2}U_{\frac{3}{2}}(u_1, 0) \\ Q_1 &= \frac{1}{2}\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right) \pm \tfrac{1}{2}U_{\frac{1}{2}}(u_1, 0) \\ P_2 &= \frac{1}{2}\sin\left(\frac{u_2}{2}-\frac{\pi}{4}\right) \mp \tfrac{1}{2}U_{\frac{3}{2}}(u_2, 0) \\ Q_2 &= \frac{1}{2}\cos\left(\frac{u_2}{2}-\frac{\pi}{4}\right) \mp \tfrac{1}{2}U_{\frac{1}{2}}(u_2, 0) \end{aligned}\right\} \quad \text{..................(52)},$$

the upper or the lower signs being taken according as the region in question is on the side of positive or of negative x.

On the outer part of the screen therefore and on the side of positive x the maxima and minima are determined by

$$\frac{\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right) + \frac{1}{2}V_{\frac{3}{2}}(u_1, 0)}{\frac{1}{2}V_{\frac{3}{2}}(u_2, 0)} = \frac{\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right) + \frac{1}{2}V_{\frac{1}{2}}(u_1, 0)}{\frac{1}{2}V_{\frac{1}{2}}(u_2, 0)},$$

and as in the space considered u_1 has a very large value, we may write approximately

$$V_{\frac{3}{2}}(u_1, 0) = 0 \quad \text{and} \quad V_{\frac{1}{2}}(u_1, 0) = 0,$$

so that the position of the maxima and minima is given by

$$\tan\left(\frac{u_1}{2}-\frac{\pi}{4}\right) = \frac{V_{\frac{1}{2}}(u_2, 0)}{V_{\frac{3}{2}}(u_2, 0)} \quad \text{........................(53)}.$$

To the same approximation the intensity is

$$\begin{aligned} I &= \left\{\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right) + \tfrac{1}{2}V_{\frac{3}{2}}(u_2, 0)\right\}^2 + \left\{\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right) + \tfrac{1}{2}V_{\frac{1}{2}}(u_2, 0)\right\}^2 \\ &= 1 + V_{\frac{3}{2}}(u_2, 0)\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right) + V_{\frac{1}{2}}(u_2, 0)\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right) \\ &\qquad + \tfrac{1}{4}V_{\frac{3}{2}}^2(u_2, 0) + \tfrac{1}{4}V_{\frac{1}{2}}^2(u_2, 0) \text{........................(54)}, \end{aligned}$$

and the maxima and minima illuminations are

$$I = \{1 \pm \tfrac{1}{2}\sqrt{V_{\frac{3}{2}}^2(u_2, 0) + V_{\frac{1}{2}}^2(u_2, 0)}\}^2.$$

The further we recede from the edge of this part of the field, the smaller $V_{\frac{3}{2}}^2(u_2, 0)$ and $V_{\frac{1}{2}}^2(u_2, 0)$ become and the more nearly the intensity approaches the constant value unity. The maxima and minima become closer together and since $V_{\frac{3}{2}}(u_2, 0)$ decreases numerically with increasing u_2 more rapidly than does $V_{\frac{1}{2}}(u_2, 0)$, their position is defined by

$$u_1 = (4n-1)\pi/2, \quad (n = 1, 2, 3\ldots).$$

Within the space directly illuminated by both streams

$$I=\left\{\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right)+\cos\left(\frac{u_2}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{3}{2}}(u_1,0)+\tfrac{1}{2}V_{\frac{3}{2}}(u_2,0)\right\}^2$$

$$+\left\{\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right)+\sin\left(\frac{u_2}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{1}{2}}(u_1,0)+\tfrac{1}{2}V_{\frac{1}{2}}(u_2,0)\right\}^2$$

$$=4\cos^2\frac{u_1-u_2}{4}+\tfrac{1}{4}\{V_{\frac{3}{2}}(u_1,0)+V_{\frac{3}{2}}(u_2,0)\}^2+\tfrac{1}{4}\{V_{\frac{1}{2}}(u_1,0)+V_{\frac{1}{2}}(u_2,0)\}^2$$

$$+2\cos\frac{u_1-u_2}{4}\cos\frac{u_1+u_2-\pi}{4}\{V_{\frac{3}{2}}(u_1,0)+V_{\frac{3}{2}}(u_2,0)\}$$

$$+2\cos\frac{u_1-u_2}{4}\sin\frac{u_1+u_2-\pi}{4}\{V_{\frac{1}{2}}(u_1,0)+V_{\frac{1}{2}}(u_2,0)\}\quad\ldots\ldots(55),$$

and the condition for the maxima and minima is

$$\frac{\cos\left(\frac{u_1}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{3}{2}}(u_1,0)}{\cos\left(\frac{u_2}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{3}{2}}(u_2,0)}=\frac{\sin\left(\frac{u_1}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{1}{2}}(u_1,0)}{\sin\left(\frac{u_2}{2}-\frac{\pi}{4}\right)+\tfrac{1}{2}V_{\frac{1}{2}}(u_2,0)}\quad\ldots\ldots\ldots(56),$$

or

$$\left[\cos\frac{u_1-u_2}{4}+\tfrac{1}{4}\{V_{\frac{1}{2}}(u_1,0)+V_{\frac{1}{2}}(u_2,0)\}\sin\frac{u_1+u_2-\pi}{4}\right.$$
$$\left.+\tfrac{1}{4}\{V_{\frac{3}{2}}(u_1,0)+V_{\frac{3}{2}}(u_2,0)\}\cos\frac{u_1+u_2-\pi}{4}\right]$$
$$\times\left[\sin\frac{u_1-u_2}{4}+\tfrac{1}{4}\{V_{\frac{1}{2}}(u_1,0)-V_{\frac{1}{2}}(u_2,0)\}\cos\frac{u_1+u_2-\pi}{4}\right.$$
$$\left.-\tfrac{1}{4}\{V_{\frac{3}{2}}(u_1,0)-V_{\frac{3}{2}}(u_2,0)\}\sin\frac{u_1+u_2-\pi}{4}\right]$$
$$+\tfrac{1}{16}\left[\{V_{\frac{3}{2}}(u_2,0)\,V_{\frac{1}{2}}(u_2,0)-V_{\frac{3}{2}}(u_1,0)\,V_{\frac{1}{2}}(u_1,0)\}\sin\frac{u_1+u_2}{2}\right.$$
$$+\tfrac{1}{2}\{V_{\frac{1}{2}}^2(u_1,0)-V_{\frac{1}{2}}^2(u_2,0)-V_{\frac{3}{2}}^2(u_1,0)+V_{\frac{3}{2}}^2(u_2,0)\}\cos\frac{u_1+u_2}{2}$$
$$\left.+V_{\frac{1}{2}}(u_1,0)\,V_{\frac{3}{2}}(u_2,0)-V_{\frac{1}{2}}(u_2,0)\,V_{\frac{3}{2}}(u_1,0)\right]=0,$$

and omitting the small term in the last vinculum, this gives the two equations

$$\cos\frac{u_1-u_2}{4}=-\tfrac{1}{4}\{V_{\frac{1}{2}}(u_1,0)+V_{\frac{1}{2}}(u_2,0)\}\sin\frac{u_1+u_2-\pi}{4}$$
$$-\tfrac{1}{4}\{V_{\frac{3}{2}}(u_1,0)+V_{\frac{3}{2}}(u_2,0)\}\cos\frac{u_1+u_2-\pi}{4}\quad\ldots\ldots\ldots\ldots(57),$$

$$\sin\frac{u_1-u_2}{4}=-\tfrac{1}{4}\{V_{\frac{1}{2}}(u_1,0)-V_{\frac{1}{2}}(u_2,0)\}\cos\frac{u_1+u_2-\pi}{4}$$
$$+\tfrac{1}{4}\{V_{\frac{3}{2}}(u_1,0)-V_{\frac{3}{2}}(u_2,0)\}\sin\frac{u_1+u_2-\pi}{4}\quad\ldots\ldots\ldots\ldots(58),$$

of which the first gives the minima, as is easily seen from the expression for the intensity.

As a first approximation we may neglect the effect of diffraction, this gives for the minima

$$\cos\frac{u_1-u_2}{4}=0 \quad \text{or } u_1-u_2=(4n-2)\pi, \quad (n=1, 2, \ldots)$$

and for the maxima

$$\sin\frac{u_1-u_2}{4}=0 \quad \text{or } u_1-u_2=(4n-4)\pi, \quad (n=1, 2, \ldots)$$

and since
$$u_1-u_2=\frac{2\pi}{\lambda}\frac{4cx_0}{\rho_1+\rho_0}$$

we have for the position of the minima

$$x_0=(2n-1)\frac{\rho_1+\rho_0}{c}\frac{\lambda}{4} \qquad \text{(59)},$$

and for the position of the maxima

$$x_0=(2n-2)\frac{\rho_1+\rho_0}{c}\frac{\lambda}{4} \qquad \text{(60)}.$$

Let the right-hand side of (57) have the value $A_n\pi/2$ when x_0 is given by (59) and let the right-hand side of (58) have the value $B_n\pi/2$ when x_0 is given by (60), then to a second approximation the positions of the minima are given by

$$x_0=\{(2n-1)+\cos n\pi\,.\,A_n\}\frac{\rho_1+\rho_0}{c}\frac{\lambda}{4},$$

and those of the maxima by

$$x_0=\{(2n-2)-\cos n\pi\,.\,B_n\}\frac{\rho_1+\rho_0}{c}\frac{\lambda}{4}.$$

87. When the boundary of the diffracting aperture or screen is a circle, it becomes more convenient to employ polar coordinates with the pole at the centre of circle. In this case, supposing that the radiant point is on the axis of the circle that limits the transparent portion of the screen, we have

$$x_1=y_1=0, \quad x_0=\sigma\cos\theta', \quad y_0=\sigma\sin\theta', \quad x=\rho\cos\theta, \quad y=\rho\sin\theta,$$

and writing
$$\frac{2\pi}{\lambda}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)=k, \qquad \frac{2\pi}{\lambda}\frac{\sigma}{\rho_0}=l,$$

the disturbance is represented by

$$\phi_0(t)=-\iota\frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)e^{\iota(\kappa\omega t+\delta)}\iint_{\theta'}^{2\pi+\theta'}e^{\iota\left\{l\rho\cos(\theta-\theta')-k\frac{\rho^2}{2}\right\}}\rho\,d\rho\,d\theta$$

$$=-\iota\frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)e^{\iota(\kappa\omega t+\delta)}\int e^{-\iota k\frac{\rho^2}{2}}\rho\,d\rho\int_0^{2\pi}e^{\iota l\rho\cos\theta}\,d\theta$$

$$=-\iota\frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right)e^{\iota(\kappa\omega t+\delta)}\frac{2\pi}{l}\int e^{-\iota k\frac{\rho^2}{2}}(l\rho)\,J_0(l\rho)\,d\rho \qquad \text{(61)},$$

the integration being from 0 to r in the case of a circular aperture and r to ∞ for an opaque circular disc, r being in each case the radius of the circle.

Hence in the case of the aperture

$$\phi_0(t) = -\iota \frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right) e^{\iota(\kappa\omega t+\delta)} \cdot \pi r^2 \frac{2}{u}\{U_1(u, v) + \iota U_2(u, v)\} e^{-\iota\frac{u}{2}} \quad\text{.....(62)},$$

where $u = kr^2$ and $v = lr$, and

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 (\pi r^2)^2 \left(\frac{2}{u}\right)^2 \{U_1^2(u, v) + U_2^2(u, v)\} \quad\text{.........(63)};$$

while in the case of the disc

$$\phi_0(t) = \iota \frac{A}{\lambda\rho_0\rho_1}\left(\frac{\partial\rho_0}{\partial z}\right) e^{\iota(\kappa\omega t+\delta)} \cdot \pi r^2 \frac{2}{u}\{V_1(u, v) + \iota V_0(u, v)\} e^{-\iota\frac{u}{2}} \quad\text{.....(64)},$$

and

$$I = \frac{A^2}{\lambda^2\rho_0^2\rho_1^2}\left(\frac{\partial\rho_0}{\partial z}\right)^2 (\pi r^2)^2 \left(\frac{2}{u}\right)^2 \{V_1^2(u, v) + V_0^2(u, v)\} \quad\text{.........(65)}.$$

88. Taking first the case of a circular aperture, the expression that we have to discuss is

$$M = \left(\frac{2}{u}\right)^2 \{U_1^2(u, v) + U_2^2(u, v)\} \quad\text{..................(66)}.$$

From this we obtain at once Fraunhofer's special case by writing $u = 0$ and this gives

$$M = \{2J_1(v)/v\}^2$$

the same expression as that already considered.

In Fresnel's general case, when $v = 0$, that is at the centre of the pattern

$$M_0 = \left(\frac{2}{u}\right)^2 \{U_1^2(u, 0) + U_2^2(u, 0)\} = \left(\frac{\sin u/4}{u/4}\right)^2.$$

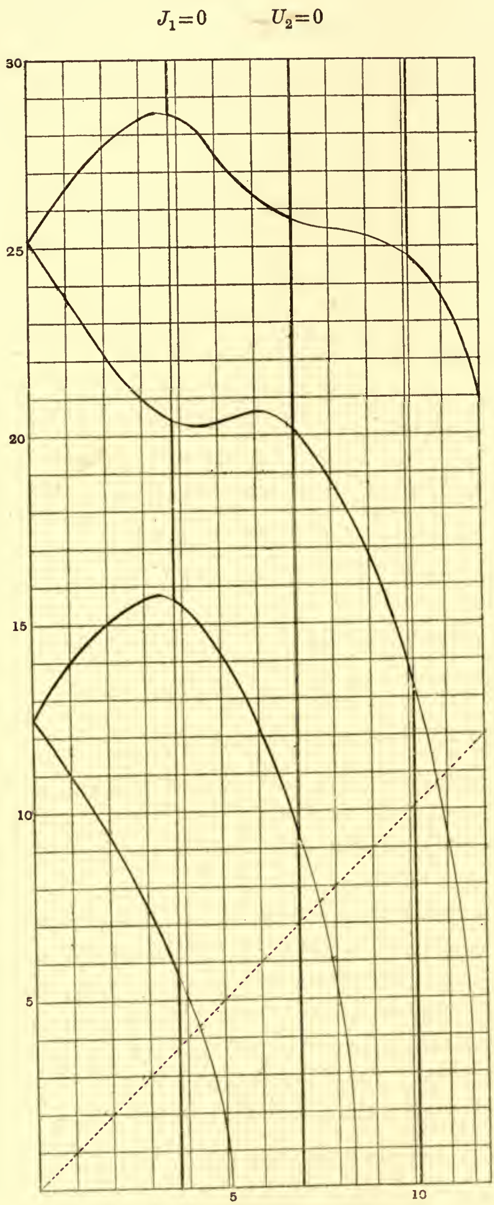
This expression we have already discussed; the maxima occur in correspondence with the roots of $\tan(u/4) = u/4$ and the minima are given by $u = 4m\pi$ ($m = 1, 2, \ldots$), the illumination then being zero. These minima, that occur at the centre of the pattern for certain positions of the screen of observation, and the minima in Fraunhofer's special case are the only ones that are perfectly black; for $U_1(u, v)$ and $U_2(u, v)$ only vanish together when $u = 4m\pi$, $v = 0$, and when $u = 0$, $J_1(v) = 0$.

For a given value of u, the maxima and minima are given by

$$\frac{\partial M}{\partial v} = \left(\frac{2}{u}\right)^2 2\left(U_1\frac{\partial U_1}{\partial v} + U_2\frac{\partial U_2}{\partial v}\right) = -\left(\frac{2}{u}\right)^2 \frac{2v}{u} U_2(U_1 + U_3)$$

$$= -2\left(\frac{2}{u}\right)^2 J_1(v)\, U_2(u, v) = 0 \quad\text{.....................................(67)}.$$

PLATE III.



The regions of the lines that give minima are indicated by heavy ruling: those that give maxima by light ruling.

Hence a maximum or a minimum of intensity occurs when either

$$J_1(v) = -\frac{d}{dv} J_0(v) = 0 \quad \text{or} \quad U_2(u, v) = -\frac{u}{v}\frac{\partial}{\partial v} U_1(u, v) = 0,$$

and therefore the illumination is either a maximum or a minimum for values of v that give a maximum or a minimum value to either $J_0(v)$ or $U_1(u, v)$.

To determine the values that represent the maxima and the minima respectively, we have

$$\frac{\partial^2 M}{\partial v^2} = -2\left(\frac{2}{u}\right)^2 \left\{U_2 \frac{\partial J_1}{\partial v} + J_1 \frac{\partial U_2}{\partial v}\right\}$$

$$= 2\left(\frac{2}{u}\right)^2 \left\{U_2\left(\frac{1}{v} J_1 - J_0\right) + J_1 \frac{v}{u} U_3\right\} \quad\text{.....................(68),}$$

and therefore the roots of $J_1(v) = 0$ correspond to maxima or minima of intensity according as $J_0(v)\, U_2(u, v)$ is positive or negative, and the roots of $U_2(u, v) = 0$ give maxima or minima according as $J_1(v)\, U_3(u, v)$ is negative or positive. At the centre of the pattern where $v = 0$, we have $[J_0(v)\, U_2(u, v)]_{v=0} = 2 \sin^2(u/4)$ and thus the centre of the pattern is a maximum of intensity, unless the position of the screen be such that $u = 4m\pi$, in which case, as we have seen, the illumination is zero and the central point is a minimum of intensity.

The second differential coefficient of M with respect to v is, however, zero, when $J_1 = 0$ and $U_2 = 0$ simultaneously, while in that case

$$\frac{\partial^3 M}{\partial v^3} = 2\left(\frac{2}{u}\right)^2 \frac{2v}{u} J_0(v)\, U_3(u, v) = -2\left(\frac{2}{u}\right)^2 \frac{2v}{u} J_0(v)\, U_1(u, v) \quad\text{......(69).}$$

This does not vanish, except in the case already mentioned when $u = 4m\pi$, $v = 0$, and therefore at such points the illumination is neither a maximum nor a minimum.

Again if $U_2(u, v) = 0$ and $U_3(u, v) = 0$, we have $\partial^2 M/\partial v^2 = 0$ and

$$\frac{\partial^3 M}{\partial v^3} = -2\left(\frac{2}{u}\right)^2 \left(\frac{v}{u}\right)^2 J_1(v)\, U_4(u, v)$$

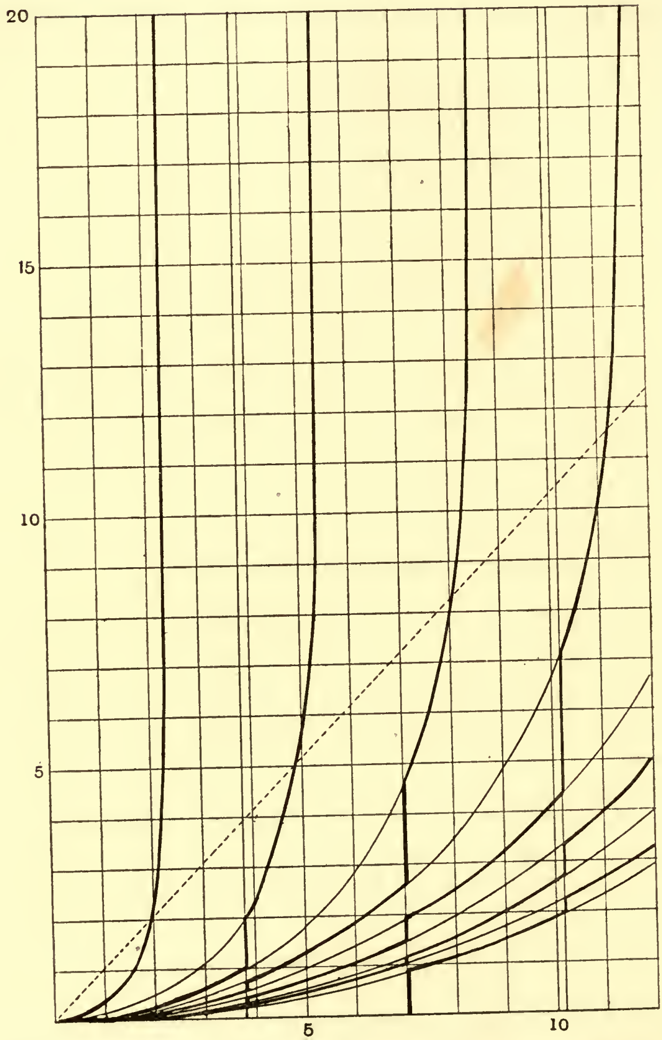
$$= -2\left(\frac{2}{u}\right)^2 J_1(v)\, J_2(v) \quad\text{..............................(70),}$$

and therefore these values of v give neither maxima nor minima. These exceptional cases are distinct from the former, since $U_1(u, v) = 0$ if both $J_1(v)$ and $U_3(u, v)$ are zero, and U_1, U_2 only vanish simultaneously when $u = 4m\pi$ and $v = 0$.

Within the geometrical shadow, that is at points for which $u < v$, $J_1(v) = 0$ and $U_2(u, v) = 0$ have no common roots; for a graphical representation

PLATE IV.

$J_1=0 \qquad V_0=0$



The regions of the lines that give minima are indicated by heavy ruling: those that give maxima by light ruling.

(Plate III.) of the equation $U_2(u, v) = 0$ shows that at such points the slope of the curve is such that du/dv is negative. Now

$$\frac{du}{dv} = 2\frac{v}{u}\frac{U_3}{U_1 + \left(\frac{v}{u}\right)^2 U_3}$$

and if $J_1(v) = 0$, $U_3 = -U_1$, wherefore

$$\frac{du}{dv} = -2\frac{v/u}{1-(v/u)^2}$$

which is positive so long as $v > u$. Hence at points within the shadow for which $J_1(v) = 0$, the sign of $J_0(v)\,U_2(u, v)$ is the same as when u is very small, but it is then negative and consequently the roots of $J_1(v) = 0$, with the exception of $v = 0$, give the minima. Further since $U_2(u, v)$ continuously approaches the value $(u^2/v^2)\,J_2(v)$ as u decreases, and the roots of $J_2(v) = 0$ separate those of $J_1(v) = 0$, it follows that for points within the shadow only one root of $U_2(u, v) = 0$ lies between two consecutive roots of $J_1(v) = 0$ and hence the roots of $U_2(u, v) = 0$ give the maxima.

89. It now remains to consider the case in which the diffraction is due to a stream of light passing the edge of an opaque circular disc. In this case, as we have already seen, the illumination depends upon the expression

$$M = \left(\frac{2}{u}\right)^2 (V_0^2 + V_1^2) \qquad \text{.........................(71).}$$

At the centre of the pattern, where $v = 0$, we have $V_0 = 1$, $V_1 = 0$ and $M_0 = (2/u)^2$; consequently at this point the illumination is practically the same as it would be at the same point if the disc were removed.

For a given value of u we have

$$\frac{\partial M}{\partial v} = 2\left(\frac{2}{u}\right)^2 \left\{V_0\frac{\partial V_0}{\partial v} + V_1\frac{\partial V_1}{\partial v}\right\}$$

$$= -2\left(\frac{2}{u}\right)^2 \frac{v}{u} V_0(V_{-1} + V_1) = -2\left(\frac{2}{u}\right)^2 V_0(u, v)\,J_1(v) \qquad \text{......(72).}$$

Thus the maxima and minima of intensity occur in accordance with the roots of $J_1(v) = 0$ and $V_0(u, v) = 0$, or since $J_1 = -\partial J_0/\partial v$, $V_0 = -(u/v)\,\partial V_1/\partial v$ the values of v that give either a maximum or a minimum of intensity are those that make $J_0(v)$ and $V_1(u, v)$ either a maximum or a minimum. Now

$$\frac{\partial^2 M}{\partial v^2} = -2\left(\frac{2}{u}\right)^2 \left\{J_1\frac{\partial V_0}{\partial v} + V_0\frac{\partial J_1}{\partial v}\right\}$$

$$= 2\left(\frac{2}{u}\right)^2 \left\{V_0\left(\frac{1}{v}J_1 - J_0\right) + \frac{v}{u}J_1 V_{-1}\right\} \qquad \text{.................(73),}$$

and we see that if $J_1(v) = 0$, the intensity is a maximum or a minimum according as $J_0 V_0$ is positive or negative, and if $V_0(u, v) = 0$, maxima or minima occur according as $(v/u)\,J_1 V_{-1}$ is negative or positive.

When however both $J_1(v)=0$ and $V_0(u, v)=0$, the value of $\partial^2 M/\partial v^2$ is zero but not that of $\partial^3 M/\partial v^3$; for in that case

$$\frac{\partial^3 M}{\partial v^3} = -2\left(\frac{2}{u}\right)^2 \frac{2v}{u} J_0 V_1 \quad \text{.........................(74).}$$

Hence to such values of v neither maxima nor minima correspond. These are the only exceptional cases, for V_0 and V_{-1} cannot simultaneously vanish. That this is the case is at once clear from a graphical representation (Plate IV.) of the equation $V_0(u, v)=0$; for it will be seen that the tangent to this curve always makes an acute angle with the axis of v, but

$$\frac{du}{dv} = \frac{2(v/u) V_{-1}}{V_1 + (v/u)^2 V_{-1}},$$

and since this is always positive, V_{-1} cannot vanish.

Further none of the cases in which $J_1(v)=0$ and $V_0(u, v)=0$ have common roots occur at points in the geometrical shadow, where $u > v$; for if $J_1(v)=0$ then $V_{-1} = -V_1$ and

$$\frac{du}{dv} = -\frac{2v/u}{1 - v^2/u^2},$$

which is negative when $u > v$. Also as u increases $V_0(u, v)$ continuously approaches the value $J_0(v)$ and since the roots of $J_0(v)=0$ and $J_1(v)=0$ occur alternately, it follows that for points within the shadow, one and only one root of $V_0(u, v)=0$ occurs between two consecutive roots of $J_1(v)=0$. But when $v=0$ the intensity is a maximum, and consequently within the shadow the roots of $J_1(v)=0$ give the maxima and those of $V_0(u, v)=0$ determine the minima of intensity.

CHAPTER IX.

MORE ACCURATE INVESTIGATION OF THE PROBLEM OF DIFFRACTION*.

90. THOUGH Huygens' principle is in itself exact, the method in which it has been applied to the explanation of diffraction in the previous chapters is open to serious objection. In order to obtain the expression for the polarisation-vector at any point, it is necessary to know the values of $\phi(t)$ and $\phi_n(t)$ at the different points of the diffracting screen, and it has been assumed that on the illuminated side of a perfectly black screen these quantities have the same value as when the screen is removed, while on the remaining portion their value is zero. The surface conditions are thus obtained by neglecting the effect of diffraction, or in other words it is first assumed that the wave-length of light is infinitesimal in order to arrive at results that are afterwards applied to the case in which it is finite, and this is done in spite of the fact that the results are then inadmissible, as they involve discontinuities, which are expressly excluded in the deduction of Huygens' principle.

That this faulty method of procedure leads to final formulæ that agree very closely with observed phenomena, at any rate as regards the positions of the maxima and minima of intensity, may be attributed to the fact that the measurements are made at distances from the diffracting screen that are large compared with the wave-length, in which case the errors due to the imperfection of the method are only a small fraction of the width of the fringes.

91. An absolutely black body is defined as one that neither transmits nor reflects the light incident upon it, and it is difficult to represent the action of such a body by any ordinary surface conditions. There is however a strong analogy between the effect of a thin, absolutely black screen and that of a branch cut in a Riemann's multiple space, one part of which represents the physical space; for this branch cut acts, as it were, the part of an

* Sommerfeld, *Gött. Nachr.* (1) 338 (1894), (1) 267 (1895); *Math. Ann.* XLVII. 317 (1896). Poincaré, *Acta Math.* XVI. 297 (1892). Macdonald, *Electric Waves*, p. 386, Camb. 1902.

open door, through which a stream of light can leave the physical space and spread into another infinite region without any portion of it returning. It seems probable then that an infinitely thin screen of absolute blackness may be regarded at any rate approximately as a branch cut in a multiple space, provided it be such that the light only passes out of the physical portion of the space.

The problem is however still indefinite; for to determine completely a Riemann's multiple space we require the form, position and order of its branch curves, and while these curves are given by the edge of the infinitely thin obstacle, their order remains arbitrary and depends upon the kind of blackness that is to be attributed to the screen. In fact black bodies, though alike in possessing the property of neither reflecting nor transmitting light, may differ physically in the way in which they affect a stream of light in their immediate vicinity.

While the multiple space is characterised by its branch curves, the form of the cut bounded by them is arbitrary, and all cuts are equivalent from the point of view of the problem of diffraction, provided they fulfil the condition that the light passes always *out* of the physical space. Thus the possibility is afforded of representing the diffraction due to a massive black body by regarding it as the part of the physical space between two branch cuts through the same branch curve. In this case a second arbitrary element is introduced, namely the line on the surface of the body that is to be taken as the branch curve, and to the various positions of this curve there correspond black bodies that are to be regarded as physically different*.

92. It is however necessary to specialise the investigation by assuming that the polarisation-vector is independent of one of the coordinates, say z: the multiple space then becomes a Riemann's surface with a branch point, where the branch curve cuts the plane of xy. This is equivalent to the assumption that the screen is limited by a straight line and that the source is a luminous line parallel to the edge of the screen.

The components of the polarisation-vector then satisfy three differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} = \omega^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

which in the case of monochromatic light becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0 \qquad \text{.............................(1)},$$

where $\kappa = 2\pi/\lambda$.

In the special case in which $\kappa = 0$, the determination of the functions required for the solution of the problem may be effected by the theory of

* Voigt, *Gött. Nachr.* (1) 1 (1899).

complex algebraical functions, but these methods cannot be directly transferred to the present more general case. It is however possible to deduce from a solution of $\nabla^2 u = 0$ with certain properties, a solution of equation (1) with corresponding properties.

93. Starting from a function $f(z)$ of the complex variable z, we refer the z-plane to a sphere of unit radius by stereographic projection. The centre of the sphere is taken at the zero-point of the z-plane and this plane being the equatorial plane, the centre of projection is the south pole of the sphere. Then if the centre of the sphere be the origin of a system of rectangular coordinates ξ, η, ζ, of which ξ and η coincide with the real and the imaginary axes of the z-plane, we obtain the function $f\left(\dfrac{\xi + \iota\eta}{1+\zeta}\right)$ and from this the solid spherical harmonic of degree 0

$$f\left(\frac{\xi + \iota\eta}{\rho+\zeta}\right), \qquad \rho^2 = \xi^2 + \eta^2 + \zeta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

whence we may deduce spherical harmonics of different degrees by multiplying by ρ^{-1} and differentiating m times with respect to any axis.

Thus taking the ζ-axis as the axis of differentiation and introducing a suitable numerical factor, we arrive at the spherical harmonic of degree $-(m+1)$

$$\frac{(-1)^{m+1}}{\underline{|m}} \frac{\partial^m}{\partial \zeta^m} \left\{ \frac{1}{\rho} f\left(\frac{\xi + \iota\eta}{\rho+\zeta}\right)\right\},$$

which by Cauchy's theorem may be written

$$\frac{1}{2\pi\iota} \int f\left(\frac{\xi + \iota\eta}{z+\rho}\right) \frac{1}{(\zeta - z)^{m+1}} \frac{dz}{\rho}, \qquad \rho^2 = \xi^2 + \eta^2 + z^2 \quad \ldots\ldots\ldots (3),$$

wherein z denotes a complex variable, that is taken by a closed path round the point $z = \zeta$ in the plane of the variable z, so as to leave this point always on the left hand.

This process however introduces a branch point $\rho = 0$, or $z = \pm \iota \sqrt{\xi^2 + \eta^2}$ that does not belong to the original function, but this may be removed by adding to (3) a second integral

$$-\frac{1}{2\pi\iota} \int f\left(\frac{\xi + \iota\eta}{z-\rho}\right) \frac{1}{(\zeta - z)^{m+1}} \frac{\partial z}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

obtained by the above process from stereographic projection of the reflection of the z-plane at its zero-point, the north pole of the sphere being now the centre of projection.

In order to pass from these spherical harmonics to a solution of equation (1) we write

$$\xi = \kappa x/m, \qquad \eta = \kappa y/m, \qquad \xi^2 + \eta^2 + \zeta^2 = 1,$$

and regard x and y as finite quantities, while m is increased to infinity. To facilitate this transformation, let us introduce polar coordinates by the equations

$$r^2 = x^2 + y^2, \qquad e^{\iota\phi} = (x + \iota y)/r.$$

The expression for ζ then becomes

$$\zeta = \{1 - \kappa^2(x^2 + y^2)/m^2\}^{\frac{1}{2}} = 1 - \kappa^2 r^2/(2m^2) - \ldots,$$

and writing

$$z = \iota\sqrt{\xi^2 + \eta^2}\,.\cos\alpha = \iota\kappa r\cos\alpha/m,$$

we have

$$\rho = \kappa r \sin\alpha/m, \qquad z \pm \rho = \frac{\kappa r}{m} e^{\mp\alpha\iota + \frac{\pi}{2}\iota}, \qquad \frac{dz}{\rho} = -\iota d\alpha,$$

$$\frac{\xi + \iota\eta}{z \pm \rho} = e^{\iota(\phi \pm \alpha - \pi/2)}, \quad \mathrm{Lt}_{m=\infty}(\zeta - z)^{-m-1} = e^{\iota\kappa r\cos\alpha},$$

and there results from the sum of (3) and (4)

$$u = \frac{1}{2\pi}\int\{f(e^{\iota(\phi - \alpha - \pi/2)}) - f(e^{\iota(\phi + \alpha - \pi/2)})\}\, e^{\iota\kappa r\cos\alpha}\, d\alpha \quad \ldots\ldots\ldots(5).$$

The closed path in the z-plane becomes in the α-plane a path in which the initial and final points may differ by 2π, and if the path extend to infinity, it must be so determined that the integral retains a meaning.

94. Writing now $f = -\dfrac{1/n}{1-(z/z')^{1/n}}$, where z' is a point on the unit circle, we obtain

$$u = -\frac{1}{2\pi n}\int\left\{\frac{1}{1 - e^{\frac{\iota}{n}(\phi - \phi' - \alpha)}} - \frac{1}{1 - e^{\frac{\iota}{n}(\phi - \phi' + \alpha)}}\right\} e^{\iota\kappa r\cos\alpha}\, d\alpha$$

$$= \frac{1}{2\pi n\iota}\int \frac{\sin\dfrac{\alpha}{n}}{\cos\dfrac{\alpha}{n} - \cos\dfrac{\phi - \phi'}{n}} e^{\iota\kappa r\cos\alpha}\, d\alpha \ldots\ldots\ldots\ldots\ldots(6),$$

if we write

$$z' = e^{\iota(\phi' - \pi/2)} \quad (\phi' \text{ real}).$$

As regards the path of integration it is to be remarked that in the z-plane we only obtain a closed path if we pass round $z = \infty$ and $\rho = 0$ both in the same direction, and that we must approach infinity by a path along which the real part of $\iota\kappa r\cos\alpha$ is negative. The regions for which this is the case on the α-plane are denoted by shading, and the path of integration has to start at ∞ in strip I. and to end at ∞ in strip II., the points $\alpha = \pm(\phi - \phi')$ lying outside the region enclosed by the path.

When $n = 1$ we have a solution that is everywhere finite in the simple plane and there results

$$u_0 = \frac{1}{2\pi\iota}\int \frac{\sin\alpha}{\cos\alpha - \cos(\phi - \phi')} e^{\iota\kappa r\cos\alpha} d\alpha.$$

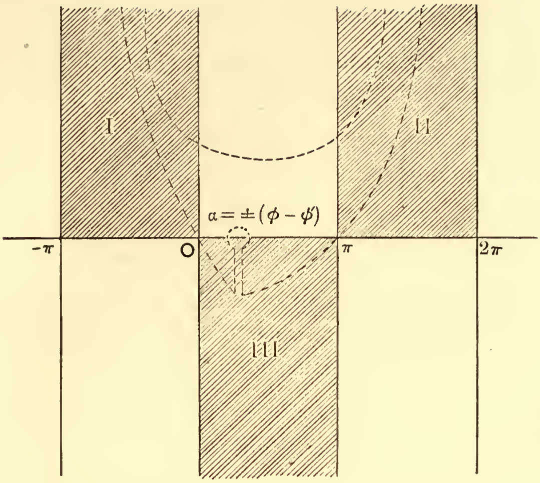


Fig. 26.

Regarding $\cos\alpha$ as the integrand, the corresponding path of integration is closed and includes the point $\cos\alpha=\cos(\phi-\phi')$, on which it may be contracted, so that by Cauchy's theorem

$$u_0=e^{\iota\kappa r\cos(\phi-\phi')}=e^{\iota\kappa(x\cos\phi'+y\sin\phi')}\ldots\ldots\ldots\ldots(7).$$

95. Returning to the general case, we have $\mathrm{Lt}_{r=\infty}u=0$, provided we can deform the path of integration, so that it entirely lies in the shaded strips, since in these $\mathrm{Exp}(\iota\kappa r\cos\alpha)$ becomes continually smaller as r increases. Now this is always possible, provided that the discontinuity in the denominator $\alpha=\pm(\phi-\phi')$ does not lie between 0 and π, that is if $|\phi-\phi'|>\pi$: in other cases we have to exclude the point of discontinuity by a loop, as indicated by the dotted line in the figure, and then by Cauchy's theorem the integral over the loop is $\mathrm{Exp}\{\iota\kappa r\cos(\phi-\phi')\}=u_0$, and the remaining parts vanish. Hence

$$\mathrm{Lt}_{r=\infty}\,u=0\ \text{ if }\ |\phi-\phi'|>\pi,\qquad \mathrm{Lt}_{r=\infty}\,u=u_0\ \text{ if }\ |\phi-\phi'|<\pi.$$

Let u_s denote the value of u at the point $(r,\phi+2(s-1)\pi)$, then

$$\sum_1^n u_s=\frac{1}{2\pi\iota}\int\sum_{s=1}^{s=n}\left\{\cos\frac{\alpha}{n}-\cos\frac{\phi-\phi'+2(s-1)\pi}{n}\right\}^{-1}\sin\frac{\alpha}{n}e^{\iota\kappa r\cos\alpha}\frac{d\alpha}{n};$$

but $$\cos\alpha-\cos(\phi-\phi')=2^{n-1}\prod_{s=1}^{s=n}\left\{\cos\frac{\alpha}{n}-\cos\frac{\phi-\phi'+2(s-1)\pi}{n}\right\},$$

and taking the logarithmic differential of each side

$$\frac{\sin\alpha}{\cos\alpha-\cos(\phi-\phi')}=\frac{1}{n}\sin\frac{\alpha}{n}\sum_{s=1}^{s=n}\left\{\cos\frac{\alpha}{n}-\cos\frac{\phi-\phi'+2(s-1)\pi}{n}\right\}^{-1},$$

whence $$\sum_1^n u_s=\frac{1}{2\pi\iota}\int\frac{\sin\alpha}{\cos\alpha-\cos(\phi-\phi')}e^{\iota\kappa r\cos\alpha}\,d\alpha=u_0\ldots\ldots\ldots\ldots(8).$$

Hence the solution u has the following properties:—

(*a*) it satisfies the differential equation (1), as do all functions derived by the method that has been employed;

(*b*) it is finite everywhere at a finite distance, since the path of integration has been so determined that it is finite for all values of r and ϕ;

(*c*) it is single-valued on a Riemann's surface of n sheets with a winding point of the $(n-1)$th order at the zero-point;

(*d*) denoting by the first sheet of this surface the assemblage of points for which $|\phi - \phi'| < \pi$, the function is equal to $u_0 = \text{Exp}\{\iota\kappa r \cos(\phi - \phi')\}$ at infinity on the first sheet and vanishes at infinity on the remaining sheets;

(*e*) the sum of the values of u at the points on the 1st, 2nd ... nth sheets that lie above one another is equal to u_0.

Thus we may take u to represent the disturbance due to plane waves of light incident in the direction $\phi = \phi'$ on the branch cut of the Riemann's surface of n sheets.

96. Let us now take the case of $n = 2$ and in order to follow the course of the multiform solution and to obtain results suitable for numerical calculation, let us transform (6) into an integral with a real path of integration.

Writing $(\phi - \phi') = \psi$ and assuming provisionally that $|\psi| < \pi$ we have

$$u_1 + u_2 = u_0 = e^{\iota\kappa r \cos\psi},$$

$$u_1 - u_2 = \frac{\cos\frac{\psi}{2}}{\pi\iota}\int \frac{e^{\iota\kappa r\cos\alpha}}{\cos\alpha - \cos\psi}\sin\frac{\alpha}{2}\,d\alpha,$$

and if
$$X = \frac{u_1 - u_2}{u_0} = \frac{\cos\frac{\psi}{2}}{\pi\iota}\int \frac{e^{\iota\kappa r(\cos\alpha - \cos\psi)}}{\cos\alpha - \cos\psi}\sin\frac{\alpha}{2}\,d\alpha,$$

then
$$\frac{\partial X}{\partial r} = \frac{\kappa}{\pi}\cos\frac{\psi}{2}\,e^{-2\iota\kappa r\cos^2\frac{\psi}{2}}\int e^{2\iota\kappa r\cos^2\frac{\alpha}{2}}\sin\frac{\alpha}{2}\,d\alpha.$$

Taking $\cos\frac{\alpha}{2}$ as the integrand and deforming the path of integration in the α-plane so that it becomes the imaginary axis in the $\cos(\alpha/2)$-plane, we have

$$\int e^{2\iota\kappa r\cos^2\frac{\alpha}{2}}\sin\frac{\alpha}{2}\,d\alpha = \sqrt{\frac{2\pi}{\kappa r}}\,e^{\iota\frac{\pi}{4}},$$

and
$$\frac{\partial X}{\partial r} = \sqrt{\frac{2\kappa}{\pi r}}\,e^{\iota\frac{\pi}{4}}\cos\frac{\psi}{2}\,e^{-2\iota\kappa r\cos^2\frac{\psi}{2}} = \frac{\partial}{\partial r}\left\{\frac{2}{\sqrt{\pi}}\,e^{\iota\frac{\pi}{4}}\int_0^{\sqrt{2\kappa r}\cos\frac{\psi}{2}} e^{-\iota\tau^2}\,d\tau\right\};$$

whence integrating between 0 and r we obtain, since $X=0$ when $r=0$

$$X=\frac{2}{\sqrt{\pi}}e^{\iota\frac{\pi}{4}}\int_0^\sigma e^{-\iota\tau^2}d\tau, \qquad \sigma=\sqrt{2\kappa r}\cos\frac{\psi}{2},$$

and
$$u_1-u_2=\frac{2}{\sqrt{\pi}}e^{\iota\left(\kappa r\cos\psi+\frac{\pi}{4}\right)}\int_0^\sigma e^{-\iota\tau^2}d\tau.$$

We may also write

$$u_1+u_2=\frac{2}{\sqrt{\pi}}e^{\iota\left(\kappa r\cos\psi+\frac{\pi}{4}\right)}\int_{-\infty}^0 e^{-\iota\tau^2}d\tau,$$

and from these two equations the values of u_1 and u_2 are obtained.

Removing now the restriction that $|\psi|<\pi$, we have

$$u=\frac{1}{\sqrt{\pi}}e^{\iota\left(\kappa r\cos\psi+\frac{\pi}{4}\right)}\int_{-\infty}^\sigma e^{-\iota\tau^2}d\tau, \qquad \sigma=\sqrt{2\kappa r}\cos\frac{\psi}{2}\text{..........(9)},$$

which holds for both sheets, σ being positive in the first and negative in the second sheet.

Now we have (Appendix II., equation 28)

$$V_{\frac{3}{2}}(2\sigma^2, 0)+\iota V_{\frac{1}{2}}(2\sigma^2, 0)=-2\frac{\sigma}{\sqrt{\pi}}\int_1^\infty e^{\iota\sigma^2(1-u^2)}du$$

$$=-\frac{2}{\sqrt{\pi}}e^{\iota\sigma^2}\int_0^\infty e^{-\iota\tau^2}d\tau+\frac{2}{\sqrt{\pi}}e^{\iota\sigma^2}\int_0^\sigma e^{-\iota\tau^2}d\tau,$$

$$\therefore\ \frac{2}{\sqrt{\pi}}\int_0^\sigma e^{-\iota\tau^2}d\tau=e^{-\iota\pi/4}+e^{-\iota\sigma^2}\{V_{\frac{3}{2}}(2\sigma^2, 0)+\iota V_{\frac{1}{2}}(2\sigma^2, 0)\}.$$

Hence for $\sigma>0$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^\sigma e^{-\iota\tau^2}d\tau=e^{-\iota\pi/4}+\tfrac{1}{2}e^{-\iota\sigma^2}\{V_{\frac{3}{2}}(2\sigma^2, 0)+\iota V_{\frac{1}{2}}(2\sigma^2, 0)\}\text{......(10)},$$

and for $\sigma<0$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^\sigma e^{-\iota\tau^2}d\tau=-\tfrac{1}{2}e^{-\iota\sigma^2}\{V_{\frac{3}{2}}(2\sigma^2, 0)+\iota V_{\frac{1}{2}}(2\sigma^2, 0)\}\text{.........(10').}$$

97. The application of these results to the problem of diffraction follows at once.

Suppose that plane waves of monochromatic light parallel to the axis of z are incident on an infinitely thin black screen that occupies the positive half of the plane of xz, the normals to the waves making an angle ϕ' with the plane of the screen. If we confine our attention to the plane of xy and assume that the action of the screen may be assimilated to that of a branch cut in a Riemann's surface of two sheets, the polarisation-vector may be represented by

$$u=(A/\sqrt{\pi})\,e^{\iota\{\kappa\omega t+\kappa r\cos(\phi-\phi')+\pi/4\}}\int_{-\infty}^\sigma e^{-\iota\tau^2}d\tau\text{............(11)},$$

wherein $$\sigma = \sqrt{2\kappa r} \,.\, \cos(\phi - \phi')/2,$$

and A is a constant, ϕ being measured from the branch cut and $0 < \phi < 2\pi$ in the physical sheet, $-2\pi < \phi < 0$ in the auxiliary sheet.

Now in the physical sheet we recognise two portions that are separated by the line $\phi = \phi' + \pi$, the edge of the geometrical shadow: in the first of these regions, that is outside the shadow, $|\phi - \phi'| < \pi$ and consequently

$$u = A \,.\, e^{\iota\{\kappa\omega t + \kappa r \,.\, \cos(\phi - \phi')\}} + \frac{A}{2}\{V_{\frac{3}{2}}(2\sigma^2, 0) + \iota V_{\frac{1}{2}}(2\sigma^2, 0)\}\, e^{\iota(\kappa\omega t - \kappa r + \pi/4)} \quad \text{....(12)},$$

while within the shadow $|\phi - \phi'| > \pi$, and hence

$$u = -\frac{A}{2}\{V_{\frac{3}{2}}(2\sigma^2, 0) + \iota V_{\frac{1}{2}}(2\sigma^2, 0)\}\, e^{\iota(\kappa\omega t - \kappa r + \pi/4)} \quad \text{...........(12')}.$$

Since $V_{\frac{3}{2}}(2\sigma^2, 0)$ and $V_{\frac{1}{2}}(2\sigma^2, 0)$ vary but slowly in comparison with the exponential $e^{-\iota\kappa r}$, we may say that the disturbance outside the shadow is approximately the same as that which results from a superposition of the incident waves and of cylindrical waves emanating from the edge of the screen, while within the shadow the disturbance is that due to the latter waves alone.

From the expressions (12), (12') we obtain for the intensity within the geometrical shadow

$$I = \frac{A^2}{4}\{V_{\frac{3}{2}}^2(2\sigma^2, 0) + V_{\frac{1}{2}}^2(2\sigma^2, 0)\} \quad \text{...................(13)},$$

and for the intensity outside the shadow

$$I = A^2\left[\left\{\tfrac{1}{2}V_{\frac{3}{2}}(2\sigma^2, 0) + \cos\left(\sigma^2 - \frac{\pi}{4}\right)\right\}^2 + \left\{\tfrac{1}{2}V_{\frac{1}{2}}(2\sigma^2, 0) + \sin\left(\sigma^2 - \frac{\pi}{4}\right)\right\}^2\right] \quad \text{...(13')},$$

expressions that have the same form as those obtained for the values of the intensity by the approximate method (Chap. VIII. (38), (39)).

In order to compare the results obtained by the two methods, let us suppose that the incident waves are parallel to the screen, then in the formulæ of Chapter VIII., we have to write

$$k = \kappa/r, \qquad l = \kappa \sin\theta,$$

where θ is the angle that the direction considered makes with the edge of the geometrical shadow, and the approximate method has

$$l^2/k = \kappa r \sin^2\theta,$$

in the place of

$$2\sigma^2 = 4\kappa r \cos^2\frac{\phi - \pi/2}{2} = 4\kappa r \sin^2\frac{\theta}{2},$$

given by the present investigation. Near the edge of the shadow, the expressions become identical, for as far as terms of the fourth order they both become $\kappa r\theta^2$.

98. The above investigation may also be applied to the case, in which the diffracting screen, instead of being absolutely black, is perfectly reflecting. Such a screen may be approximately realised by the employment of a highly polished silver sheet.

Taking again the case in which plane waves of light are incident on the screen, we may reduce the general problem of diffraction in which the polarisation-vector has any direction with respect to the plane of incidence into the two simpler cases, in which the vector is respectively perpendicular and parallel to this plane.

Now when two homogeneous media are separated by the plane of yz the general surface conditions are, as we shall see in the next chapter, that $\omega^2 v$, $\omega^2 w$, ϖ_2 and ϖ_3 are continuous across the interface. The plane of incidence being that of xy and the polarisation-vector being parallel to the axis of z, we have in the case of monochromatic light in the first medium

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \kappa^2 w = 0,$$

and in the second medium

$$\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \kappa'^2 w' = 0;$$

but if i be the angle of incidence $\partial^2 w/\partial y^2 = -\kappa^2 \sin^2 i \, . \, w$ and in the first medium

$$\frac{\partial^2 w}{\partial x^2} + \kappa^2 \cos^2 i \, . \, w = 0,$$

while in the second medium

$$\frac{\partial^2 w'}{\partial x^2} = (\kappa^2 \sin^2 i - \kappa'^2)\, w' = \beta^2 w' \text{ (say)},$$

which gives $$w' = f(y)\, e^{\beta x} + f_1(y)\, e^{-\beta x}.$$

Suppose that the second medium is on the side of negative x; then since w' does not become infinite with x, we must have $f_1(y) = 0$, if β be chosen so that its real part is positive: hence

$$w' = f(y)\, e^{\beta x} \quad \text{and} \quad \varpi_2' = -\omega'^2 \frac{\partial w'}{\partial x} = -\beta \omega'^2 w',$$

and since ϖ_2 and $\omega^2 w$ are continuous across the interface, we must have in the first medium just outside the interface

$$\varpi_2 = -\beta \omega^2 w,$$

and since when the reflecting power is very great, κ' and consequently β is very great, we have in the case of perfect reflection $w = 0$.

When the polarisation-vector is in the plane of incidence, the auxiliary

or light-vector ϖ is parallel to the axis of z and we obtain as in the former case

$$\varpi_3' = \phi(y)\, e^{\beta x}, \qquad \partial\varpi_3'/\partial x = \beta\varpi_3'.$$

But ϖ_3 is continuous and $\partial\varpi_3/\partial x = \partial v/\partial t$; $\partial\varpi_3/\partial x$ is therefore discontinuous and its values at two points close to the interface in the first and second medium respectively are as $\omega'^2 : \omega^2$ or as $\kappa^2 : \kappa'^2$. Hence in the first medium at the interface

$$\frac{\partial\varpi_3}{\partial x} = \frac{\kappa^2}{\kappa'^2}\beta\varpi_3,$$

and in the case of perfect reflection this reduces to $\partial\varpi_3/\partial x = 0$*.

Thus when the polarisation-vector is perpendicular to the plane of incidence, we have

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \kappa^2 w = 0, \text{ and } w = 0 \text{ when } x = 0,\ x > 0,$$

and when the polarisation-vector is in the plane of incidence

$$\frac{\partial^2\varpi_3}{\partial x^2} + \frac{\partial^2\varpi_3}{\partial y^2} + \kappa^2\varpi_3 = 0, \text{ and } \frac{\partial\varpi_3}{\partial x} = 0 \text{ when } x = 0,\ x > 0.$$

Supposing again that the direction of incidence of the waves is defined by $\phi = \phi'$, and denoting the function u in § 97 by $u(\phi')$ in order to indicate its dependence upon the angle ϕ', we obtain the solutions of the two problems by writing

$$w = A\,\{u(\phi') - u(-\phi')\}\, e^{\iota\kappa\omega t},$$

$$\varpi_3 = A\,\{u(\phi') + u(-\phi')\}\, e^{\iota\kappa\omega t},$$

since from symmetry we then have for $\phi = 0$, $w = 0$ and $\partial\varpi_3/\partial\phi = 0$.

The diffraction phenomena may therefore be regarded as due to the action of waves incident in the direction ϕ' in the physical sheet together with waves incident in the direction $-\phi'$ in the auxiliary sheet of a Riemann's surface of two sheets.

Now in the physical sheet of the surface we recognise three distinct regions, separated by the lines for which $\phi = \pi - \phi'$ and $\phi = \pi + \phi'$; in the first of these, extending from $\phi = 0$ to $\phi = \pi - \phi'$, we have $\phi - \phi' < \pi$ and $\phi + \phi' < \pi$ and consequently from (10)

$$\left.\begin{matrix} w \\ \varpi_3 \end{matrix}\right\} = Ae^{\iota\{\kappa\omega t + \kappa r\cos(\phi-\phi')\}} \mp Ae^{\iota\{\kappa\omega t + \kappa r\cos(\phi+\phi')\}}$$

$$+ \frac{A}{2}\left[V_{\frac{3}{2}}(2\sigma^2, 0) \mp V_{\frac{3}{2}}(2\sigma_1^2, 0) + \iota\,\{V_{\frac{1}{2}}(2\sigma^2, 0) \mp V_{\frac{1}{2}}(2\sigma_1^2, 0)\}\right] e^{\iota(\kappa\omega t - \kappa r + \pi/4)},$$

where $\qquad \sigma^2 = 2\kappa r \,.\, \cos^2(\phi - \phi')/2, \quad \sigma_1^2 = 2\kappa r \,.\, \cos^2(\phi + \phi')/2\,;$

* Poincaré, *loc. cit.*

the second region extends from $\phi = \pi - \phi'$ to $\phi = \pi + \phi'$ and therein $\phi - \phi' < \pi$, $\phi + \phi' > \pi$, whence from (10) and (10′)

$$\left.\begin{matrix} w \\ \varpi_3 \end{matrix}\right\} = Ae^{\iota\{\kappa\omega t + \kappa r \cos(\phi - \phi')\}}$$

$$+\frac{A}{2}[V_{\frac{3}{2}}(2\sigma^2, 0) \pm V_{\frac{3}{2}}(2\sigma_1^2, 0) + \iota\{V_{\frac{1}{2}}(2\sigma^2, 0) \pm V_{\frac{1}{2}}(2\sigma_1^2, 0)\}]\, e^{\iota(\kappa\omega t - \kappa r + \pi/4)},$$

while in the third region occupying the remainder of the physical sheet, $\phi - \phi' > \pi$, $\phi + \phi' > \pi$ and

$$\left.\begin{matrix} w \\ \varpi_3 \end{matrix}\right\} = -\frac{A}{2}[V_{\frac{3}{2}}(2\sigma^2, 0) \mp V_{\frac{3}{2}}(2\sigma_1^2, 0) + \iota\{V_{\frac{1}{2}}(2\sigma^2, 0) \mp V_{\frac{1}{2}}(2\sigma_1^2, 0)\}]\, e^{\iota(\kappa\omega t - \kappa r + \pi/4)}.$$

We may thus represent the effect of the screen approximately by a train of waves emanating from its edge, which we may call waves of diffraction: in the first region the disturbance is due to the interference of the incident and reflected waves and the waves of diffraction; in the second region we have the superposition of the incident waves and the waves of diffraction; while in the third region the waves of diffraction alone are operative.

For the intensity in the three regions we have

$$I_1 = A^2\left[\left\{\tfrac{1}{2}V_{\frac{3}{2}}(2\sigma^2, 0) + \cos\left(\sigma^2 - \frac{\pi}{4}\right)\right\} \mp \left\{\tfrac{1}{2}V_{\frac{3}{2}}(2\sigma_1^2, 0) + \cos\left(\sigma_1^2 - \frac{\pi}{4}\right)\right\}\right]^2$$

$$+ A^2\left[\left\{\tfrac{1}{2}V_{\frac{1}{2}}(2\sigma^2, 0) + \sin\left(\sigma^2 - \frac{\pi}{4}\right)\right\} \mp \left\{\tfrac{1}{2}V_{\frac{1}{2}}(2\sigma_1^2, 0) + \sin\left(\sigma_1^2 - \frac{\pi}{4}\right)\right\}\right]^2,$$

$$I_2 = A^2\left[\tfrac{1}{2}V_{\frac{3}{2}}(2\sigma^2, 0) + \cos\left(\sigma^2 - \frac{\pi}{4}\right) \pm \tfrac{1}{2}V_{\frac{3}{2}}(2\sigma_1^2, 0)\right]^2$$

$$+ A^2\left[\tfrac{1}{2}V_{\frac{1}{2}}(2\sigma^2, 0) + \sin\left(\sigma^2 - \frac{\pi}{4}\right) \pm \tfrac{1}{2}V_{\frac{1}{2}}(2\sigma_1^2, 0)\right]^2,$$

$$I_3 = \frac{A^2}{4}[\{V_{\frac{3}{2}}(2\sigma^2, 0) \mp V_{\frac{3}{2}}(2\sigma_1, 0)\}^2 + \{V_{\frac{1}{2}}(2\sigma^2, 0) \mp V_{\frac{1}{2}}(2\sigma_1^2, 0)\}^2],$$

the upper and lower signs referring to the cases in which the polarisation-vector is perpendicular and parallel to the plane of incidence respectively.

CHAPTER X.

REFLECTION AND REFRACTION AT THE SURFACE OF ISOTROPIC MEDIA.

99. WE have seen that the characteristic equations of the polarisation-vector d are for free space

$$\dot{d} = -\operatorname{curl} \varpi, \quad \dot{\varpi} = \operatorname{curl} e \quad \text{.......................(1)},$$

where ϖ is an auxiliary vector, that we may call the light-vector to distinguish it from the polarisation-vector d, and the vector e is defined by its components

$$(e_1, e_2, e_3) = \tfrac{1}{2}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)(\omega^2 d^2) \quad \text{....................(2).}$$

These equations were deduced from the principle of interference combined with the assumption that a train of waves is propagated with a speed that is independent of the intensity of the light and of the direction of the waves. We may therefore, when dispersion is neglected, extend their application to the case of any transparent isotropic medium, provided we regard ω no longer as an universal constant, but as a function of the period of the waves under consideration.

The interfacial conditions that must be satisfied at the passage from one isotropic medium to another follow at once from the above differential equations, if we assume that the transition takes place by a rapid, but continuous change of the properties of the one medium into those of the other and that the differential equations still hold within the region where the variation occurs. For taking the interface as the plane of yz, the equations give that $\partial\varpi_2/\partial x$, $\partial\varpi_3/\partial x$, $\partial e_2/\partial x$, $\partial e_3/\partial x$ remain finite, and that consequently ϖ_2, ϖ_3, e_2, e_3 must be continuous across the interface $x = 0$. To these we may add two further conditions: for since the curl of a vector has no divergence anywhere, div $d = 0$, and div $\varpi = 0$, and hence u and ϖ_1 must also be continuous. These last two conditions are not however independent of the four former.

100. These boundary conditions lead at once to the geometrical laws of reflection and refraction; for since they hold for all values of t, y, and z, and these variables occur in the expressions for the polarisation-vectors of the

incident, reflected and refracted waves only in the combinations of the form $lx + my + nz + st$, it follows that s, m, and n must have the same value for each of the streams, m and n being in general complex quantities in the case of unhomogeneous waves.

We see then that the periodicity is the same for the three streams, and when the waves are homogeneous, taking the axis of y perpendicular to the plane of incidence, $m = 0$ for all the waves, and

$$\sin i/\lambda = \sin i_1/\lambda = \sin r/\lambda',$$

where i, i_1, r are the angles of incidence, reflection and refraction, and λ, λ' are the wave-lengths of the light in the two media.

Hence the reflected and refracted wave-normals are in the plane of incidence; the angle of reflection is equal to the angle of incidence; and the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction, this being the ratio of the wave-lengths of the incident and the refracted light.

101. Since the vectors d, ϖ and e are connected by purely geometrical relations, we may in discussing the problem of reflection and refraction employ which we please as representative of the streams of light, and as the calculations are rather simpler with the light-vector ϖ, we shall in this chapter adopt this vector in our investigations.

Let the plane of incidence be that of xz and let the normal to the planes of constant amplitude of the incident waves be in this plane: then we may represent the light-vector ϖ by the expression

$$\varpi = \bar{A} \exp \{\iota (\bar{l}x + \bar{n}z + st)\} \quad \text{.........................(3)},$$

defined by the direction-cosines $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, bars (—) being placed over the letters to denote that the corresponding quantities are in the general case complex.

Since $\operatorname{div} \varpi = 0$ we have $\bar{\alpha}\bar{l} + \bar{\gamma}\bar{n} = 0$, and the axial components of the vector may be written

$$(\varpi_1, \varpi_2, \varpi_3) = (\bar{n}, \bar{k}, -\bar{l}) \bar{D} \exp \{\iota (\bar{l}x + \bar{n}z + st)\} \quad \text{............(4)},$$

where, $\bar{\phi}$ being a complex angle defining the vector with respect to the plane of incidence and ν being the coefficient of extinction,

$$\bar{k} = 2\pi \tan \bar{\phi} \sqrt{1 - \nu^2}/\lambda, \quad \bar{D} = \lambda \cos \bar{\phi} \bar{A}/(2\pi \sqrt{1 - \nu^2}) \quad \text{.........(5)}.$$

Hence if $\bar{F}$, $\bar{G}$ be the components of the complex amplitude perpendicular and parallel to the plane of incidence

$$\bar{F} = \bar{k}\bar{D}, \quad \bar{G} = 2\pi \bar{D} \sqrt{(1 - \nu^2)}/\lambda = \bar{D} \sqrt{\bar{l}^2 + \bar{n}^2} \quad \text{............(6)}.$$

Substituting the values (4) in the equations (1), we obtain

$$s^2 = \omega^2 (\bar{l}^2 + \bar{n}^2),$$

which determines $\bar{l}$ when s and $\bar{n}$ are given. There are then two values $\pm\bar{l}$ corresponding to waves of given period, the traces of which on the interface move at a given rate: one of these is a wave approaching the surface and the other is a wave leaving it.

Similarly if accented letters refer to the second medium, we have

$$s^2 = \omega'^2(\bar{l}'^2 + \bar{n}^2);$$

but as in the present investigation there is no question of an incident wave in this medium, we require only one of the two values of $\pm\bar{l}'$ thus determined, and this will be the one with the positive sign, if we regard the value $+\bar{l}$ as referring to the incident wave in the first medium.

We have then the following expressions for the system of waves:

Incident wave

$$(\varpi_1, \varpi_2, \varpi_3) = (\bar{n}, \bar{k}, -\bar{l})\bar{D}\exp\{\iota(\bar{l}x + \bar{n}z + st)\}.$$

Reflected wave

$$(\varpi_1, \varpi_2, \varpi_3) = (\bar{n}, \bar{k}_1, \bar{l})\bar{D}_1\exp\{\iota(-\bar{l}x + \bar{n}z + st)\}.$$

Refracted wave

$$(\varpi_1, \varpi_2, \varpi_3) = (\bar{n}, \bar{k}', -\bar{l}')\bar{D}'\exp\{\iota(\bar{l}'x + \bar{n}z + st)\}.$$

Introducing now the boundary conditions, the continuity of ϖ_2 or of u, when $x = 0$, gives

$$\bar{k}\bar{D} + \bar{k}_1\bar{D}_1 = \bar{k}'\bar{D}' \quad\ldots\ldots\ldots\ldots(7);$$

the continuity of e_3 requires that

$$\omega^2\bar{l}(\bar{k}\bar{D} - \bar{k}_1\bar{D}_1) = \omega'^2\bar{l}'\bar{k}'\bar{D}' \quad\ldots\ldots\ldots\ldots(8);$$

the continuity of ϖ_3 leads to the relation

$$\bar{l}(\bar{D} - \bar{D}_1) = \bar{l}'\bar{D}' \quad\ldots\ldots\ldots\ldots(9);$$

and from the continuity of ϖ_1 or of e_2 we have

$$\bar{D} + \bar{D}_1 = \bar{D}' \quad\ldots\ldots\ldots\ldots(10).$$

Introducing the components parallel and perpendicular to the plane of incidence these equations become

$$\bar{F} + \bar{F}_1 = \bar{F}',$$

$$\{\bar{l}/(\bar{l}^2 + \bar{n}^2)\}(\bar{F} - \bar{F}_1) = \{\bar{l}'/(\bar{l}'^2 + \bar{n}^2)\}\bar{F}',$$

$$\{\bar{l}/(\bar{l}^2 + \bar{n}^2)^{\frac{1}{2}}\}(\bar{G} - \bar{G}_1) = \{\bar{l}'/(\bar{l}'^2 + \bar{n}^2)^{\frac{1}{2}}\}\bar{G}',$$

$$\{\bar{n}/(\bar{l}^2 + \bar{n}^2)^{\frac{1}{2}}\}(\bar{G} + \bar{G}_1) = \{\bar{n}/(\bar{l}'^2 + \bar{n}^2)^{\frac{1}{2}}\}\bar{G}',$$

or if $\bar{i}$, $\bar{r}$ be the complex angles of incidence and refraction defined by

$$\bar{n} = 2\pi\sin\bar{i}\sqrt{1-\nu^2}/\lambda = 2\pi\sin\bar{r}\sqrt{1-\nu'^2}/\lambda',$$

we have $$\bar{F} + \bar{F}_1 = \bar{F}' \quad\text{.............................}(7'),$$

$$(\bar{F} - \bar{F}_1)\sin\bar{i}\cos\bar{i} = \bar{F}'\sin\bar{r}\cos\bar{r} \quad\text{.................}(8'),$$

$$(\bar{G} - \bar{G}_1)\cos\bar{i} = \bar{G}'\cos\bar{r} \quad\text{..........................}(9'),$$

$$(\bar{G} + \bar{G}_1)\sin\bar{i} = \bar{G}'\sin\bar{r} \quad\text{.......................}(10');$$

whence $$\frac{\bar{F}}{\sin(\bar{i}+\bar{r})\cos(\bar{i}-\bar{r})} = \frac{\bar{F}_1}{\cos(\bar{i}+\bar{r})\sin(\bar{i}-\bar{r})} = \frac{\bar{F}'}{\sin 2\bar{i}} \quad\text{......}(11),$$

$$\frac{\bar{G}}{\sin(\bar{i}+\bar{r})} = \frac{\bar{G}_1}{-\sin(\bar{i}-\bar{r})} = \frac{\bar{G}'}{\sin 2\bar{i}} \quad\text{......}(12).$$

These equations determine completely the specification of the reflected and the refracted streams.

102. When the incident waves are of constant amplitude and the second medium is the more highly refracting, i and r are in all cases real, and the reflected and the refracted waves are also of constant amplitude. Writing

$$\bar{G}/\bar{F} = (G/F)\,e^{\iota\Delta}, \quad \bar{G}_1/\bar{F}_1 = (G_1/F_1)\,e^{\iota\Delta_1}, \quad \bar{G}'/\bar{F}' = (G'/F')\,e^{\iota\Delta'},$$

we obtain $$\frac{G}{F}\cos(i-r) = -\frac{G_1}{F_1}\cos(i+r) = \frac{G'}{F'},$$

$$\Delta = \Delta_1 = \Delta'.$$

From these equations the elliptic constants of the reflected and the refracted streams may be obtained in terms of those of the incident stream, and we see that reflection and refraction introduces no new difference of phase between the components parallel and perpendicular to the plane of incidence, other than that of $\pm\pi$ implied by a change in the sign of the amplitude of the vibrations.

If the ratio $\bar{G}/\bar{F}$ be real, the incident, reflected and refracted streams are all plane polarised, and the azimuths ϕ, ϕ_1, ϕ' of the light-vectors with respect to the plane of incidence are connected by

$$\cot\phi' = -\cot\phi_1 \,.\, \cos(i+r) = \cot\phi \,.\, \cos(i-r).$$

In interpreting this result, it must be noticed that, in accordance with the specification of the three streams adopted above, the vectors are regarded as positive, when in each case the components in and perpendicular to the plane of incidence are related to one another and to the direction from which the stream travels in the same way as the axes of x, y and z. Thus positive values of F_1/F and F'/F mean that the directions of the incident, reflected and refracted light-vectors are the same: on the other hand a positive value of G_1/G signifies at normal incidence that the directions of the reflected and the incident light-vectors are opposite, at grazing incidence that they are identical.

103. When the incident stream consists of common light, we may in accordance with what has been shown in Chapter II, represent it by two components of equal intensity, that are polarised in planes parallel and perpendicular respectively to the plane of incidence, and from the results obtained in § 101, as also from considerations of symmetry, the reflection and refraction of these components may be treated separately.

Let us represent these components by

$$\Sigma F_n \exp\{\iota(l_n x + n_n z + s_n t + a_n)\} \text{ or } \Sigma G_n \exp\{\iota(l_n x + n_n z + s_n t + b_n)\},$$

according as the vector ϖ is perpendicular or parallel to the plane of incidence with the condition $\Sigma F_n^2 = \Sigma G_n^2 = L$, $2L$ being the intensity of the light. Then in the reflected stream the components become

$$\Sigma F_n \frac{\tan(i - r_n)}{\tan(i + r_n)} \exp\{\iota(-l_n x + n_n y + s_n t + a_n)\},$$

and
$$-\Sigma G_n \frac{\sin(i - r_n)}{\sin(i + r_n)} \exp\{\iota(-l_n x + n_n y + s_n t + b_n)\},$$

and in the refracted stream

$$\Sigma F_n \frac{\sin 2i}{\sin(i + r_n)\cos(i - r_n)} \exp\{\iota(l_n' x + n_n y + s_n t + a_n)\},$$

and
$$\Sigma G_n \frac{\sin 2i}{\sin(i + r_n)} \exp\{\iota(l_n' x + n_n z + s_n t + b_n)\}.$$

If now the incident light be practically monochromatic, we may neglect the change in the values of r_n in passing from one constituent of the streams to another and we see that in general the reflected light is partially plane polarised, having a polarised part with its light-vector ϖ parallel to the plane of incidence and of intensity

$$\frac{\sin^2(i-r)}{\sin^2(i+r)}\left\{1 - \frac{\cos^2(i+r)}{\cos^2(i-r)}\right\} L,$$

and a part consisting of common light of intensity

$$2\frac{\tan^2(i-r)}{\tan^2(i+r)} L.$$

Similarly the refracted light may be regarded as made up of a stream of common light and a stream of polarised light with its light-vector perpendicular to the plane of incidence, the intensities of these streams being in terms of a new unit

$$2\frac{\sin^2 2i}{\sin^2(i+r)} L \quad \text{and} \quad \frac{\sin^2 2i}{\sin^2(i+r)}\left\{\frac{1}{\cos^2(i-r)} - 1\right\} L.$$

At the particular angle of incidence given by $i + r = \pi/2$, the intensity of the common light in the reflected stream is zero, and the whole of the

reflected light is plane polarised with the light-vector ϖ in the plane of incidence. Now this phenomenon was first observed by Malus* in 1808, when viewing the light reflected from the windows of the Luxembourg palace through a doubly refracting prism. The angle of incidence at which this occurs, Malus called the polarising angle and he stated, as a definition of the plane of polarisation, that the reflected light is then polarised in the plane of incidence. It follows then, in accordance with what we have assumed, that the light-vector ϖ is in, and the polarisation-vector d is perpendicular to, the plane of polarisation†.

Since the polarising angle I is determined by the condition $I+R=\pi/2$, we have by Snell's law that $I=\tan^{-1}\mu$, a result that was found experimentally by Brewster in 1815‡.

104. Returning now to the case in which the incident stream is plane polarised, let θ, θ_1, θ' be the azimuths of the planes of polarisation of the incident, reflected and refracted streams respectively, measured from the plane of incidence. Then writing for shortness

$$f=\tan(i-r)/\tan(i+r), \qquad f'=\sin 2i/\{\sin(i+r)\cos(i-r)\},$$

$$g=-\sin(i-r)/\sin(i+r), \qquad g'=\sin 2i/\sin(i+r),$$

we have

$$A_1\sin\theta_1=fA\sin\theta, \quad A'\sin\theta'=f'A\sin\theta,$$

$$A_1\cos\theta_1=gA\cos\theta, \quad A'\cos\theta'=g'A\cos\theta,$$

whence

$$\frac{1}{A_1^2}=\frac{1}{A^2}\left\{\frac{\cos^2\theta_1}{g^2}+\frac{\sin^2\theta_1}{f^2}\right\}, \quad \frac{1}{A'^2}=\frac{1}{A^2}\left\{\frac{\cos^2\theta'}{g'^2}+\frac{\sin^2\theta'}{f'^2}\right\}.$$

Also

$$\tan\theta_1=(f/g)\tan\theta, \quad \tan\theta'=(f'/g')\tan\theta;$$

* *Mém. de la Soc. d'Arceuil*, II. 149 (1809).

† The question of the direction of the vibrations in polarised light has been much discussed both theoretically and experimentally. It must be remembered that in all cases we have to deal with two vectors, one parallel and the other perpendicular to the plane of polarisation, and in considering experimental determinations of the direction of vibrations we have first to decide with which of these vectors the phenomenon, that we observe, is connected. Cf. Babinet, *C. R.* XXIX. 514 (1849): *Pogg. Ann.* LXXVIII. 580 (1849). Haidinger, *Wien. Ber.* VIII. 52 (1852); XII. 685 (1854); XV. 6. 86 (1855). Ångström, *Pogg. Ann.* XC. 582 (1853). Stokes, *Camb. Phil. Trans.* IX. 35 (1856): *Phil. Mag.* (4) XIII. 159 (1857); XVIII. 426 (1859). Holtzmann, *Pogg. Ann.* XCIX. 446 (1856). Eisenlohr, *Pogg. Ann.* CIV. 337 (1858). Lorenz, *Pogg. Ann.* CXI. 315 (1860); CXIV. 238 (1861). Fizeau, *Ann. de Ch. et de Phys.* (3) LVII. 385 (1859). Quincke, *Berl. Monatsber.* (1862) 714: *Pogg. Ann.* CXVIII. 445 (1863). Lord Rayleigh, *Phil. Mag.* (4) XLI. 107, 447 (1871); XLII. 81 (1871). Rowland, *Phil. Mag.* (5) XVII. 413 (1884). Carvallo, *Thèse; Ann. de l'École Norm. supplément pour* 1890: *J. de Phys.* (2) IX. 257 (1890): *C. R.* CXII. 431 (1891). Wiener, *Wied. Ann.* XL. 203 (1890): *Ann. de Ch. et de Phys.* (6) XXIII. 387 (1891). Drude, *Wied. Ann.* XLI. 154 (1890); XLIII. 177 (1891); XLVIII. 119 (1893). Lommel, *Wied. Ann.* XLIV. 311 (1891). Cornu, *C. R.* CXII. 186, 365 (1891). Poincaré, *C. R.* CXII. 325, 456 (1891). Berthelot, *C. R.* CXII. 329 (1891). Potier, *C. R.* CXII. 383 (1891): *J. de Phys.* (2) X. 101 (1891). Drude and Nernst, *Gött. Nachr.* (1891) 346: *Wied. Ann.* XLV. 460 (1892).

‡ *Phil. Trans.* CV. 125 (1815).

hence when θ is varied, the angle of incidence remaining constant, we have

$$\sec^2\theta_1 d\theta_1 = (f/g)\sec^2\theta d\theta, \quad \sec^2\theta' d\theta' = (f'/g')\sec^2\theta d\theta,$$

or

$$A_1^2\, d\theta_1 = fgA^2 d\theta, \qquad A'^2 d\theta = f'g'A^2 d\theta.$$

Thus if the intensity of the incident light and the angle of incidence remain unaltered, while the polarisation of the incident stream varies, the amplitudes of the vibrations of the vectors of the reflected and the refracted streams may each be represented by the radius-vector of an ellipse, and the area described by this radius-vector is in a constant ratio to the area described by a vector representing the amplitude of the vibrations of the vector of the incident stream*.

Now θ_1 is always numerically less than θ, and the rotation R_1 of the plane of polarisation, measured from the primitive plane towards the plane of incidence, is for the reflected stream given by

$$\tan R_1 = \tan(\theta - \theta_1) = \tan\theta\,\frac{\cos(i-r) + \cos(i+r)}{\cos(i-r) - \cos(i+r)\tan^2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta + \tan i \tan r}.$$

When the angle of incidence remains constant, this rotation increases with the azimuth θ of the primitive plane of polarisation: while if θ be constant, it is a maximum and equal to 2θ at normal incidence.

On the other hand θ' is greater than θ and the rotation R' of the plane of polarisation of the refracted stream away from the plane of incidence is given by

$$\tan R' = \tan(\theta' - \theta) = \tan\theta\,\frac{1 - \cos(i-r)}{\cos(i-r) + \tan^2\theta};$$

the angle of incidence remaining constant, this is a maximum when

$$\tan^2\theta = \cos(i-r),$$

and the rotation then is

$$\tan^{-1}\left\{\frac{1-\cos(i-r)}{2\sqrt{\cos(i-r)}}\right\}.$$

When θ is constant, the rotation continuously increases from zero at normal incidence to the value

$$\tan^{-1}\left\{\frac{(\mu - 1)\tan\theta}{1 + \mu\tan^2\theta}\right\}$$

at grazing incidence, μ being the relative refractive index of the second medium.

* Cornu, *Ann. de Ch. et de Phys.* (4) XI. 326 (1867).

105. Among the methods that are employed for producing polarised light, we may mention that of transmitting a stream of common light through a pile of plates, and though the polarisation of the emergent light is by no means perfect, polarimeters in former years were frequently made with this form of polariser. The employment of a pile of plates as a polariser is now somewhat unusual, but it is a problem of considerable interest to determine how the degree of polarisation of the transmitted light is related to the number of plates, and what are the intensities of the reflected and the transmitted streams*.

Let us suppose that the plates are all of the same material and thickness and are placed parallel to one another, the plates themselves and the interposed layers of air being sufficiently thick to prevent the colours of thin plates.

There will then be no regular interference, and as we have seen in § 40 we have only to deal with intensities: whence taking the intensity of the incident light as unity, the intensities of the streams reflected from and transmitted through a single plate are

$$R_1 = \rho + \frac{(1-\rho)^2 \rho g^2}{1-\rho^2 g^2}, \quad T_1 = \frac{(1-\rho)^2 g}{1-\rho^2 g^2} \quad \ldots\ldots\ldots\ldots\ldots(13),$$

where ρ is the intensity of the light reflected at the first surface of the plate, and 1 to g the proportion in which the intensity of the light is reduced by absorption in a single transit through the plate.

Denoting by R_p and T_p the intensities of the reflected and the transmitted streams in the case of a pile of p plates, let us now determine in terms of R_p, T_p, R_1, T_1, the values of R_{p+1} and T_{p+1} in the case of $(p+1)$ plates. This pile may be considered as made up of a group of p plates to which a new plate has been added, and from this mode of regarding the pile, it follows that the reflected light will consist of that reflected from the group of p plates, together with that which has traversed the group and has been reflected once, twice, ... from the single plate. Hence since there is supposed to be no regular interference between the streams,

$$R_{p+1} = R_p + T_p^2 R_1 (1 + R_1 R_p + R_1^2 R_p^2 + \ldots) = R_p + \frac{T_p^2 R_1}{1 - R_1 R_p} \quad \ldots(14).$$

In the same way

$$T_{p+1} = T_1 T_p (1 + R_1 R_p + R_1^2 R_p^2 + \ldots) = \frac{T_1 T_p}{1 - R_1 R_p} \quad \ldots\ldots\ldots(15).$$

But we may regard the pile from another point of view and suppose that the single plate is placed before instead of behind the group of p plates. Hence

* Stokes, *Proc. R. S.* XI. 545 (1862): *Phil. Mag.* (4) XXIV. 480 (1862). Kirchhoff, *Vorles. über Math. Optik*, p. 166.

R_{p+1} and T_{p+1} must remain unchanged in value when the suffixes (1) and (p) are interchanged, so that

$$R_{p+1} = R_p + \frac{T_p^2 R_1}{1 - R_1 R_p} = R_1 + \frac{T_1^2 R_p}{1 - R_1 R_p} \quad \text{............(16);}$$

whence multiplying by $(1 - R_1 R_p)/(R_1 R_p)$, we obtain the relation

$$R_p + \frac{1 - T_p^2}{R_p} = R_1 + \frac{1 - T_1^2}{R_1} \quad \text{.....................(17).}$$

Let $$\frac{1 + R_1^2 - T_1^2}{2R_1} = \cos\alpha, \quad \frac{1 + T_1^2 - R_1^2}{2T_1} = \cos\beta \quad \text{.........(18);}$$

then since R_1 and T_1 are essentially positive

$$\frac{R_1}{\sin\beta} = \frac{T_1}{\sin\alpha} = \frac{1}{\sin(\alpha + \beta)} \quad \text{.......................(19).}$$

Now from (16)

$$R_p R_{p+1} - \frac{1}{R_1} R_{p+1} + \frac{T_1^2 - R_1^2}{R_1} R_p + 1 = 0;$$

let $$R_p = S_{p+1}/S_p + 1/R_1,$$

then $$S_{p+2} + \frac{T_1^2 - R_1^2 + 1}{R_1} S_{p+1} + \frac{T_1^2}{R_1^2} S_p = 0,$$

or $$S_{p+2} + 2\frac{\sin\alpha\cos\beta}{\sin\beta} . S_{p+1} + \frac{\sin^2\alpha}{\sin^2\beta} S_p = 0,$$

the solution of which is

$$S_p = \left(-\frac{\sin\alpha}{\sin\beta}\right)^p (M\cos p\beta + N\sin p\beta),$$

where M and N are constants. Hence

$$R_p = -\frac{\sin\alpha}{\sin\beta} \cdot \frac{M\cos(p+1)\beta + N\sin(p+1)\beta}{M\cos p\beta + N\sin p\beta} + \frac{\sin(\alpha+\beta)}{\sin\beta};$$

but $R_0 = 0$, $R_1 = \sin\beta/\sin(\alpha+\beta)$, whence $M = \sin\alpha$, $N = \cos\alpha$, and

$$R_p = \frac{\sin p\beta}{\sin(\alpha + p\beta)}.$$

Also from (17)

$$T_p^2 = 1 + R_p^2 - 2R_p\cos\alpha = \frac{\sin^2\alpha}{\sin^2(\alpha + p\beta)},$$

and hence $$\frac{R_p}{\sin p\beta} = \frac{T_p}{\sin\alpha} = \frac{1}{\sin(\alpha + p\beta)} \quad \text{.....................(20),}$$

the constants being determined from (18).

106. This method fails, if $\alpha = 0$; we then have

$$(1 - R_1)^2 - T_1^2 = 0, \quad \text{or} \quad (1 - R_1 - T_1)(1 - R_1 + T_1) = 0;$$

and since R_1 and T_1 are proper fractions, this relation gives

$$R_1 + T_1 = 1,$$

which expresses the fact that the plates are perfectly transparent.

In this case we may proceed as follows: the complete transparency of the plates gives that $R_p + T_p = 1$, and therefore

$$T_{p+1} = \frac{T_1 T_p}{1 - R_1 R_p} = \frac{T_1 T_p}{T_1 + R_1 T_p}.$$

This equation gives

$$\frac{1}{T_{p+1}} = \frac{1}{T_p} + \frac{R_1}{T_1},$$

and hence
$$\frac{1}{T_p} = C + p\frac{R_1}{T_1},$$

holding for all values of p. Writing then $p = 1$, we find that $C = 1$, and therefore

$$T_p = \frac{T_1}{T_1 + pR_1},$$

and
$$R_p = 1 - T_p = \frac{pR_1}{T_1 + pR_1},$$

or introducing the values of R_1 and T_1 from (13) and writing $g = 1$,

$$T_p = \frac{1-\rho}{1+(2p-1)\rho}, \quad R_p = \frac{2p\rho}{1+(2p-1)\rho} \quad\text{.............(21).}$$

When the number of plates is infinite, the intensity of the reflected light is unity, which explains the brilliantly white appearance in reflected light of a finely divided substance, that is transparent in mass.

107. Supposing still that the plates of the pile are perfectly transparent, we may now determine the degree of polarisation of the transmitted light, when common light is incident upon it.

Replacing the incident stream by two components of intensity L polarised in planes parallel and perpendicular to the plane of incidence, the intensities of the corresponding transmitted streams will be

$$G'^2 = L\frac{1-g^2}{1-g^2+2pg^2}, \quad F'^2 = L\frac{1-f^2}{1-f^2+2pf^2},$$

where $f = \tan(i-r)/\tan(i+r), \quad g = -\sin(i-r)/\sin(i+r).$

Let $i - r = \rho$, $i + r = \sigma$, then since $\sin i = \mu \sin r$

$$\frac{di}{\tan i} = \frac{dr}{\tan r} = \frac{d\rho}{\tan i - \tan r} = \frac{d\sigma}{\tan i + \tan r} = \cos i \cos r\, d\omega \text{ (say)},$$

whence
$$d\rho = \sin\rho\, d\omega, \qquad d\sigma = \sin\sigma\, d\omega \quad\text{.....................(22).}$$

Also
$$f^2 = \frac{\tan^2\rho}{\tan^2\sigma} = g^2\frac{\cos^2\sigma}{\cos^2\rho},$$

$$1 - f^2 = \frac{\tan^2\sigma - \tan^2\rho}{\tan^2\sigma} = \frac{\sin^2\sigma - \sin^2\rho}{\sin^2\sigma\cos^2\rho} = \frac{1-g^2}{\cos^2\rho},$$

$$\therefore\ \frac{f^2}{1-f^2} = \frac{g^2}{1-g^2}\cos^2\sigma = H^2\cos^2\sigma\ \text{(say)},$$

and
$$G'^2 = \frac{L}{1+2pH^2}, \qquad F'^2 = \frac{L}{1+2pH^2\cos^2\sigma}.$$

The intensity of the transmitted light is then

$$G'^2 + F'^2 = 2L\frac{1+p(1+\cos^2\sigma)H^2}{(1+2pH^2)(1+2pH^2\cos^2\sigma)},$$

that of the polarised part is

$$F'^2 - G'^2 = 2L\frac{pH^2\sin^2\sigma}{(1+2pH^2)(1+2pH^2\cos^2\sigma)},$$

and the measure of the polarisation is

$$\chi = \frac{F'^2 - G'^2}{F'^2 + G'^2} = \frac{p}{H^{-2}\operatorname{cosec}^2\sigma + p(2\operatorname{cosec}^2\sigma - 1)}$$

$$= \frac{p}{\operatorname{cosec}^2\rho + (2p-1)\operatorname{cosec}^2\sigma - p} \quad \text{............(23)},$$

since
$$H^{-2} = \frac{1-g^2}{g^2} = \frac{\operatorname{cosec}^2\rho}{\operatorname{cosec}^2\sigma} - 1.$$

Hence χ will be a maximum, if $\operatorname{cosec}^2\rho + (2p-1)\operatorname{cosec}^2\sigma - p$ is a minimum which gives by (22)

$$\operatorname{cosec}^2\rho\cos\rho + (2p-1)\operatorname{cosec}^2\sigma\cos\sigma = 0,$$

whence
$$2p - 1 = -\frac{\sin^2\sigma\cos\rho}{\cos\sigma\sin^2\rho} = -\frac{\tan\sigma}{\tan\rho}\cdot\frac{\sin\sigma}{\sin\rho} = \frac{1}{fg},$$

from which we see that $\cos\sigma$ is negative, so that $i + r > \pi/2$ or i is greater than the polarising angle. Substituting for p in (23), we find

$$\chi = \frac{2\mu^2}{(1+\mu^2)\sin^2 i} - 1.$$

When $p = \infty$, $i + r = \pi/2$ or $i = I$, and since $\sin^2 I = \mu^2/(1+\mu^2)$, $\chi = 1$; hence as the number of plates is indefinitely increased, the angle of incidence, at which the maximum polarisation occurs, approaches indefinitely to the polarising angle and the polarisation tends to become more and more perfect.

108. We have seen that when the incident waves are homogeneous, the coefficients of reflection and refraction are real for all angles of incidence,

provided the second medium is more highly refracting than the first; but this is no longer the case if the first medium refract more powerfully, for then the law of refraction for homogeneous waves ceases to be true, when the angle of incidence exceeds the critical angle $\sin^{-1}\mu$, μ being as before the relative refractive index of the second medium.

Now since we have in general

$$\bar{l}' = \frac{2\pi}{\lambda'}\sqrt{1-\nu^2}\,.\,\cos\bar{r} = \frac{2\pi}{\lambda'}(\cos r + \iota\nu\cos R),$$

$$\bar{n}' = \frac{2\pi}{\lambda'}\sqrt{1-\nu^2}\,.\,\sin\bar{r} = \frac{2\pi}{\lambda'}(\sin r + \iota\nu\sin R),$$

where $$\cos r\cos R + \sin r\sin R = 0,$$

and since in the case under consideration $\bar{n}'$ is real, because the incident waves are homogeneous, it follows that $\sin R = 0$, $\cos R = \pm 1$ and therefore $\cos r = 0$, $\sin r = 1$, a negative value of $\sin r$ being clearly foreign to the case.

Hence $$\sin\bar{r} = 1/\sqrt{1-\nu^2}, \qquad \cos\bar{r} = -\iota\nu/\sqrt{1-\nu^2},$$

the negative value of $\cos R$ being taken, because the second medium being on the side of negative x the positive value would correspond to a stream increasing indefinitely in intensity with the distance from the surface.

To determine ν, we have from the equality of the values of n for the two media, the generalised law of refraction

$$\sin i/\lambda = \sqrt{1-\nu^2}\,.\,\sin\bar{r}/\lambda' = 1/\lambda',$$

whence if Ω' be the propagational speed of the unhomogeneous waves of given period

$$\sin i/\omega = 1/\Omega' = 1/(\omega'\sqrt{1-\nu^2}),$$

giving $$\sqrt{1-\nu^2} = \omega/(\omega'\sin i) = \mu/\sin i,$$

where μ is the relative refractive index for homogeneous waves of the same frequency. Hence

$$\nu = \sqrt{1-\mu^2/\sin^2 i}, \quad \sin\bar{r} = \sin i/\mu, \quad \cos\bar{r} = -\iota\sqrt{\sin^2 i - \mu^2}/\mu \;\ldots(24).$$

Substituting these values in the expressions for the coefficients of reflection, we find that

$$\bar{g} = -\frac{\sin(i-\bar{r})}{\sin(i+\bar{r})} = \frac{\cos i + \iota\sqrt{\sin^2 i - \mu^2}}{\cos i - \iota\sqrt{\sin^2 i - \mu^2}} = e^{\iota a}\;\ldots\ldots\ldots(25),$$

$$\bar{f} = \frac{\tan(i-\bar{r})}{\tan(i+\bar{r})} = \frac{\mu^2\cos i + \iota\sqrt{\sin^2 i - \mu^2}}{\mu^2\cos i - \iota\sqrt{\sin^2 i - \mu^2}} = e^{\iota b}\;\ldots\ldots(26),$$

where $$\tan\frac{b}{2} = \frac{1}{\mu^2}\tan\frac{a}{2} = \frac{1}{\mu^2}\frac{\sqrt{\sin^2 i - \mu^2}}{\cos i}\;\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

Thus the amplitudes of the vibrations of the vectors for the streams polarised in planes parallel and perpendicular to the plane of incidence are unaltered by reflection, while the phases of the vibrations are accelerated by a and b respectively.

Hence if the incident light be polarised in any azimuth with respect to the plane of incidence, the reflected stream will be in general elliptically polarised and of the same intensity as the incident stream, the component polarised in a plane perpendicular to the plane of incidence being accelerated in phase relatively to that polarised in the plane of incidence by an amount Δ, given by

$$\tan\frac{\Delta}{2} = \tan\frac{b-a}{2} = \frac{(1-\mu^2)\tan\dfrac{a}{2}}{\mu^2+\tan^2\dfrac{a}{2}} = \frac{\cos i\sqrt{\sin^2 i-\mu^2}}{\sin^2 i} \quad\text{......(28)},$$

which is zero, when $i=\pi/2$ and when $i=\sin^{-1}\mu$, that is at grazing incidence and at the critical angle. Further since

$$\tan\frac{\Delta}{2} = \cot i\sqrt{1-\mu^2-\mu^2\cot^2 i},$$

$\tan(\Delta/2)$ is a maximum, when $\cot^2 i = (1-\mu^2)/(2\mu^2)$ or $\sin^2 i = 2\mu^2/(1+\mu^2)$ and its value then is $(1-\mu^2)/(2\mu)$ or $\cot(2\tan^{-1}\mu)$, whence $\Delta = \pi - 4\tan^{-1}\mu$.

Let us now determine the refractive index required to give a prescribed difference of phase. Solving (28) for $\sin^2 i$ we obtain

$$2\sin^2 i = (\mu^2+1)\cos^2\frac{\Delta}{2} \pm \cos\frac{\Delta}{2}\sqrt{(\mu^2+1)^2\cos^2\frac{\Delta}{2}-4\mu^2}.$$

Now the expression under the radical is

$$\cos^2\frac{\Delta}{2}\left(\mu-\tan\frac{\pi-\Delta}{4}\right)\left(\mu+\tan\frac{\pi-\Delta}{4}\right)\left(\mu-\cot\frac{\pi-\Delta}{4}\right)\left(\mu+\cot\frac{\pi-\Delta}{4}\right),$$

and hence for $\sin^2 i$ to be real, the value of μ must not lie between $\tan\{(\pi-\Delta)/4\}$ and $\cot\{(\pi-\Delta)/4\}$ and since $\cot\{(\pi-\Delta)/4\}$ is greater than unity, the maximum value of μ is $\tan\{(\pi-\Delta)/4\}$.

Thus Δ increases from 0 to $\pi - 4\tan^{-1}\mu$, as i increases from $\sin^{-1}\mu$ to $\sin^{-1}\{\sqrt{2}\,.\,\mu/\sqrt{1+\mu^2}\}$ and then decreases to 0 as i increases to $\pi/2$, and for a given value of Δ to be possible, μ must not exceed the value $\tan\{(\pi-\Delta)/4\}$.

Thus if $\Delta=\pi/2$, μ must be less than $\tan(\pi/8)$ or $\sqrt{2}-1$, and taking air as the second medium, the index of the substance must exceed $\sqrt{2}+1$ or 2·414, that is the substance must be at least as highly refracting as a diamond.

If $\Delta = \pi/4$, μ must be less than $\tan(3\pi/16)$, or the index of the substance must be greater than $\cot(3\pi/16) = 1{\cdot}4966$. When this is the case, it is possible, as with Fresnel's rhomb, to convert by two reflections a stream polarised at 45° to the plane of incidence into a circularly polarised stream.

Turning now to the coefficients of refraction we have

$$\bar{g}' = \frac{\sin 2i}{\sin(i+\bar{r})} = \frac{2\mu\cos i}{\sqrt{1-\mu^2}} e^{\iota a/2} \qquad \text{.......(29)},$$

$$\bar{f}' = \frac{\sin 2i}{\sin(i+\bar{r})\cos(i-\bar{r})} = \frac{2\mu^2\cos i}{\sqrt{\mu^4\cos^2 i + \sin^2 i - \mu^2}} \,.\, e^{\iota b/2} \qquad \text{.....(30)}.$$

Thus when the light-vector of the incident stream is perpendicular to the plane of incidence, that of the refracted stream is in the same direction and is represented symbolically by

$$\varpi_2' = \frac{2\mu^2\cos i}{\sqrt{\mu^4\cos^2 i + \sin^2 i - \mu^2}} F e^{\frac{2\pi}{\lambda'}\nu x} \,.\, e^{\iota\frac{2\pi}{\lambda'}(\omega' t + z + b/2)} \qquad \text{..........(31)};$$

on the other hand when the light-vector is in the plane of incidence, that of the refracted stream is

$$\frac{2\mu\cos i}{\sqrt{1-\mu^2}} G e^{\frac{2\pi}{\lambda'}\nu x} \,.\, e^{\iota\frac{2\pi}{\lambda'}(\omega' t + z + a/2)}$$

defined by the complex direction-cosines $\sin\bar{r}$, 0, $-\cos\bar{r}$, so that its axial components are

$$\left.\begin{aligned} \varpi_1' &= \frac{2\sin i\cos i}{\sqrt{1-\mu^2}} G e^{\frac{2\pi}{\lambda'}\nu x} \,.\, e^{\iota\frac{2\pi}{\lambda'}(\omega' t + z + a/2)} \\ \varpi_3' &= \iota\frac{2\cos i\sqrt{\sin^2 i - \mu^2}}{\sqrt{1-\mu^2}} G e^{\frac{2\pi}{\lambda'}\nu x} \,.\, e^{\iota\frac{2\pi}{\lambda'}(\omega' t + z + a/2)} \end{aligned}\right\} \qquad \text{.........(32)},$$

and the extremity of the vector describes a small ellipse lying in the plane of incidence with its axes along the axes of x and z, the direction of revolution being the same as that in which the incident wave must revolve in order to decrease the angle of incidence.

109. A difficulty here arises in connection with these results for the refracted stream, as they apparently contradict those previously obtained for the reflected light, according to which the whole of the intensity of the incident stream is to be found in the reflected train of waves. What then is the source from which the energy of the refracted stream is derived?

Now if we multiply the second triplet of equations (1) by $\varpi_1 dT$, $\varpi_2 dT$, $\varpi_3 dT$ respectively, where dT is an element of volume, and integrate the sum of these products over a region T, we have

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2)\, dT$$
$$= \int\left\{\varpi_1\left(\frac{\partial e_3}{\partial y} - \frac{\partial e_2}{\partial z}\right) + \varpi_2\left(\frac{\partial e_1}{\partial z} - \frac{\partial e_3}{\partial x}\right) + \varpi_3\left(\frac{\partial e_2}{\partial x} - \frac{\partial e_1}{\partial y}\right)\right\} dT,$$

whence integrating the terms of the right-hand side by parts, we find by the aid of the first triplet of equations (1)

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\{\omega^2(u^2+v^2+w^2)+\varpi_1{}^2+\varpi_2{}^2+\varpi_3{}^2\}\,dT$$
$$=\omega^2\int\{(\varpi_2 w-\varpi_3 v)\cos nx+(\varpi_3 u-\varpi_1 w)\cos ny$$
$$+(\varpi_1 v-\varpi_2 u)\cos nz\}\,dS\,\ldots\ldots(33),$$

where dS is an element of the bounding surface of T and n is the normal to dS directed outwards. If now we extend the region of integration so far that the polarisation-vector vanishes on its bounding surface, this formula expresses that the integral on the left side does not alter its value with the time, and we may regard it as expressing to a factor independent of the time the whole of the energy of the luminous disturbance in the region in question. The right-hand side then expresses the energy that crosses the boundary of T, when the polarisation-vector does not vanish on its surface.

Let us then determine the energy that enters per period into the second medium through the interface of the media. By (33) since $\cos ny=\cos nz=0$, this is represented by

$$\omega'^2\int_0^\tau dt\int(\varpi_2' w'-\varpi_3' u')\,dS.$$

Now the actual values of ϖ_1', ϖ_2', ϖ_3' being the real parts of their symbolical expressions, we see that in the case of total reflection, each term of the integral has the form

$$\int dS\int_0^\tau 2A\sin\left(\frac{2\pi t}{\tau}+\delta\right)\cos\left(\frac{2\pi t}{\tau}+\delta\right)dt$$
$$=\int dS\int_0^\tau A\sin\left(\frac{4\pi t}{\tau}+2\delta\right)dt=0.$$

Thus on the whole no energy passes across the interface into the second medium, the flow of energy changing its direction four times during each period.

110. It has been assumed in the above investigation that the second medium extends so far from the surface at which reflection occurs, that the light-vector becomes insensibly small at its second limiting surface. When however this medium is an extremely thin plate, the superficial undulation within it gives rise to an homogeneous refracted wave at its second surface and the reflection ceases to be total.

To investigate this case, let us take the faces of the plate as the planes $x=0$ and $x=-d$ and suppose it to be bounded by media having different optical properties. Then assuming for the sake of generality that the incident waves are unhomogeneous, the complete specification of the systems of waves will be as follows:—

In the first medium,

incident wave

$$(\varpi_1,\ \varpi_2,\ \varpi_3) = (\bar{n},\ \bar{k},\ -\bar{l})\,\bar{D}\exp\{\iota(\bar{l}x + \bar{n}z + st)\} \quad\text{.............(34)},$$

reflected wave

$$(\varpi_1,\ \varpi_2,\ \varpi_3) = (\bar{n},\ \bar{k}_1,\ \bar{l})\,\bar{D}_1\exp\{\iota(-\bar{l}x + \bar{n}z + st)\} \quad\text{...........(35)}.$$

In the second medium,

wave incident on the second surface of the plate

$$(\varpi_1,\ \varpi_2,\ \varpi_3) = (\bar{n},\ \bar{k}',\ -\bar{l}')\,\bar{D}'\exp\{\iota(\bar{l}'x + \bar{n}z + st)\} \quad\text{..........(36)},$$

wave reflected at the second surface of the plate

$$(\varpi_1,\ \varpi_2,\ \varpi_3) = (\bar{n},\ \bar{k}_1',\ \bar{l}')\,\bar{D}_1'\exp\{\iota(-\bar{l}'x + \bar{n}z + st)\} \quad\text{..........(37)}.$$

In the third medium,

emergent wave

$$(\varpi_1,\ \varpi_2,\ \varpi_3) = (\bar{n},\ \bar{k}'',\ -\bar{l}'')\,\bar{D}''\exp\{\iota(\bar{l}''x + \bar{n}z + st)\} \quad\text{........(38)}.$$

The boundary conditions are the continuity of

$$\varpi_2\ (\text{or of } u),\quad \omega^2 w,\quad \varpi_3,\quad \varpi_1\ (\text{or of } \omega^2 v) \quad\text{...............(39)},$$

when $x = 0$ and $x = -d$. Hence introducing the components of the light-vectors parallel and perpendicular to the plane of incidence, we have as in § 101,

$$\left.\begin{aligned} &\bar{F} + \bar{F}_1 = \bar{F}' + \bar{F}_1', && (\bar{F} - \bar{F}_1)\sin\bar{\imath}\cos\bar{\imath} = (\bar{F}' - \bar{F}_1')\sin\bar{r}\cos\bar{r} \\ &(G - G_1)\cos\bar{\imath} = (\bar{G}' - \bar{G}_1')\cos\bar{r}, && (\bar{G} + \bar{G}_1)\sin\bar{\imath} = (\bar{G}' + \bar{G}_1')\sin\bar{r} \end{aligned}\right\} \quad\text{.........(40)},$$

and

$$\left.\begin{aligned} &q'\bar{F}' + q'^{-1}\bar{F}_1' = q''\bar{F}'', && (q'\bar{F}' - q'^{-1}\bar{F}_1')\sin\bar{r}\cos\bar{r} = q''\bar{F}''\sin\bar{\imath}''\cos\bar{\imath}'' \\ &(q'\bar{G}' - q'^{-1}\bar{G}_1')\cos\bar{r} = q''\bar{G}''\cos\bar{\imath}'', && (q'\bar{G}' + q'^{-1}\bar{G}_1')\sin\bar{r} = q''\bar{G}''\sin\bar{\imath}'' \end{aligned}\right\} \quad\text{.........(41)},$$

where $\quad q' = \exp(-\iota\bar{l}'d), \qquad q'' = \exp(-\iota\bar{l}''d),$

and $\bar{\imath}$, $\bar{r}$, $\bar{\imath}''$ are the complex angles of incidence, refraction and emergence.

The last set of equations gives

$$\left.\begin{aligned} \frac{q'\bar{F}'}{\sin(\bar{r}+\bar{\imath}'')\cos(\bar{r}-\bar{\imath}'')} &= \frac{q'^{-1}F_1'}{\cos(\bar{r}+\bar{\imath}'')\sin(\bar{r}-\bar{\imath}'')} = \frac{q''\bar{F}''}{\sin 2\bar{r}} \\ \frac{q'\bar{G}'}{\sin(\bar{r}+\bar{\imath}'')} &= \frac{q'^{-1}\bar{G}_1'}{-\sin(\bar{r}-\bar{\imath}'')} = \frac{q'''G''}{\sin 2\bar{r}} \end{aligned}\right\} \quad\text{......(42)},$$

or

$$\left.\begin{aligned} q'\bar{F}' &= \frac{q'^{-1}\bar{F}_1'}{-\bar{f}''} = \frac{q''\bar{F}''}{1-\bar{f}''} \\ q'G' &= \frac{q'^{-1}\bar{G}_1'}{-\bar{g}''} = \frac{q''\bar{G}''}{\dfrac{\sin\bar{r}}{\sin\bar{\imath}''}(1-\bar{g}'')} \end{aligned}\right\} \quad\text{.........................(43)},$$

where $\bar{f}'' = \tan(\bar{i}'' - \bar{r})/\tan(\bar{i}'' + \bar{r})$, $\bar{g}'' = -\sin(\bar{i}'' - \bar{r})/\sin(\bar{i}'' + \bar{r})$ ...(43).

Substituting for $\bar{F}'$, $\bar{F}_1'$, $\bar{G}'$, $\bar{G}_1'$ from these equations in (40) we obtain

$$\bar{F} + \bar{F}_1 = (q'^{-1} - q'\bar{f}'')\frac{q''\bar{F}''}{1-\bar{f}''}, \qquad \bar{F} - \bar{F}_1 = \frac{\sin 2\bar{r}}{\sin 2\bar{i}}(q'^{-1} + q'\bar{f}'')\frac{q''\bar{F}''}{1-\bar{f}''},$$

$$\bar{G} + \bar{G}_1 = \frac{\sin\bar{r}}{\sin\bar{i}}(q'^{-1} - q'\bar{g}'')\frac{\sin\bar{i}''}{\sin\bar{r}}\frac{q''\bar{G}''}{1-\bar{g}''},$$

$$\bar{G} - \bar{G}_1 = \frac{\cos\bar{r}}{\cos\bar{i}}(q'^{-1} + q'\bar{g}'')\frac{\sin\bar{i}''}{\sin\bar{r}}\frac{q''\bar{G}''}{1-\bar{g}''},$$

whence
$$\frac{\bar{F}}{\sin(\bar{i}+\bar{r})\cos(\bar{i}-\bar{r})\,q'^{-1} - \cos(\bar{i}+\bar{r})\sin(\bar{i}-\bar{r})\,q'\bar{f}''}$$
$$= \frac{\bar{F}_1}{\cos(\bar{i}+\bar{r})\sin(\bar{i}-\bar{r})\,q'^{-1} - \sin(\bar{i}+\bar{r})\cos(\bar{i}-\bar{r})\,q'\bar{f}''} = \frac{q''\bar{F}''}{\sin 2\bar{i}\,(1-\bar{f}'')}$$

and
$$\frac{\bar{G}}{\sin(\bar{i}+\bar{r})\,q'^{-1} + \sin(\bar{i}-\bar{r})\,q'\bar{g}''} = \frac{\bar{G}_1}{-\sin(\bar{i}-\bar{r})\,q'^{-1} - \sin(\bar{i}+\bar{r})\,q'\bar{g}''}$$
$$= \frac{q''\bar{G}''}{\sin 2\bar{i}\sin\bar{r}\,(1-\bar{g}'')/\sin\bar{i}''},$$

or
$$\left.\begin{aligned} \frac{\bar{F}}{q'^{-1} - q'\bar{f}\bar{f}''} &= \frac{\bar{F}_1}{q'^{-1}\bar{f} - q'\bar{f}''} = \frac{q''\bar{F}''}{(1+\bar{f})(1-\bar{f}'')} \\ \frac{\bar{G}}{q'^{-1} - q'\bar{g}\bar{g}''} &= \frac{\bar{G}_1}{q'^{-1}\bar{g} - q'\bar{g}''} = \frac{q''\bar{G}''}{(1+\bar{g})(1-\bar{g}'')\sin\bar{i}/\sin\bar{i}''} \end{aligned}\right\} \text{......(44)},$$

where $\bar{f} = \tan(\bar{i} - \bar{r})/\tan(\bar{i} + \bar{r})$, $\bar{g} = -\sin(\bar{i} - \bar{r})/\sin(\bar{i} + \bar{r})$......(45).

111. Let us now apply these general formulæ to the case, in which, the first and third media being identical and more highly refracting than the plate, homogeneous waves are incident at an angle exceeding the critical angle*.

In this case q' is real and equal to $\exp\{-2\pi d\sqrt{\mu^2\sin^2 i - 1}/\lambda\}$ where λ is the wave-length of homogeneous waves of the same frequency in the plate and μ is the refractive index of the surrounding medium relatively to the plate: also $\bar{g} = \bar{g}'' = \exp(\iota a)$, $\bar{f} = \bar{f}'' = \exp(\iota b)$, where

$$\tan\frac{b}{2} = \mu^2\tan\frac{a}{2} = \frac{\mu\sqrt{\mu^2\sin^2 i - 1}}{\cos i}.$$

Hence
$$\frac{\bar{G}_1}{\bar{G}} = \frac{(1-q'^2)e^{\iota a}}{1-q'^2e^{\iota 2a}} = \frac{(1-q'^2)(e^{\iota a} - q'^2e^{-\iota a})}{1-2q'^2\cos 2a + q'^4} = \rho e^{\iota\psi} \text{.............(46)},$$

where $\rho^2 = \dfrac{(1-q'^2)^2}{(1-q'^2)^2 + 4q'^2\sin^2 a}$...(47), $\tan\psi = \dfrac{1+q'^2}{1-q'^2}\tan a$ ....(48).

* Stokes, *Trans. Camb. Phil. Soc.* VIII. 642 (1849); *Math. and Phys. Papers*, II. 56.

If ρ be taken positive, ψ must be chosen so that $\cos\psi$ and $\cos a$ have the same sign: therefore $\sin\psi$ must be positive, since $\sin a$ is positive because a lies between 0 and π. Thus of the two angles lying between $-\pi$ and $+\pi$ that satisfy (48), we require that which lies between 0 and π.

To obtain the value of $\bar{F}_1/\bar{F}$, we have merely to write b for a in the above equations.

Considering now the transmitted light, we have $q'' = \exp\{-\iota\, 2\pi d \cos i/\lambda'\}$, λ' being the wave-length of the given light in the media bounding the plate, and

$$\frac{\bar{G}''}{\bar{G}} = \frac{q'(1-e^{\iota 2a})}{1-q'^2 e^{\iota 2a}} e^{\iota \frac{2\pi}{\lambda'} d\cos i} = -\frac{2\iota q' \sin a\,(e^{\iota a} - q'^2 e^{-\iota a})}{(1-q'^2)^2 + 4q'^2 \sin^2 a} e^{\iota \frac{2\pi}{\lambda'} d \cos i}$$

$$= \rho' e^{\iota\left(\psi' + \frac{2\pi}{\lambda'} d\cos i\right)} \quad\text{.........................(49),}$$

where $\rho'^2 = \dfrac{4q'^2 \sin^2 a}{(1-q'^2)^2 + 4q'^2 \sin^2 a}$ ...(50), $\tan\psi' = -\dfrac{1-q'^2}{1+q'^2}\cot a$......(51).

If we take ρ' positive, ψ' must be chosen so that $\cos\psi'$ is positive, that is, of the two angles between $-\pi$ and π that satisfy (51), we must take that which lies between $-\pi/2$ and $\pi/2$. Now from (48) and (51), $\psi' = \psi + \pi/2 + n\pi$ and we therefore must take $\psi' = \psi - \pi/2$.

The value of $\bar{F}''/\bar{F}$ is obtained by writing b for a in the above equations.

Since $\rho^2 + \rho'^2 = 1$, it follows that the sum of the intensities of the reflected and the transmitted light is equal to that of the incident light and it is therefore necessary to discuss the expression for the reflected stream alone.

Let us suppose that the plate is a thin film of air contained between the flat face of a prism and the convex surface of a lens, upon which the prism rests, the curvature of the lens being so small that the defect of parallelism of the surfaces of the film may be neglected.

At the point of contact itself, $d = 0$ and therefore $q' = 1$, $\rho = 0$ or there is absolute blackness: as d increases, q' decreases, but this decrease is at first extremely slow, for $d \propto y^2$, where y is the distance from the point of contact, and in consequence the intensity varies ultimately as y^4. There is apparently then perfect blackness for some distance round the point of contact. Further on q' decreases rapidly and finally becomes insensible: hence the intensity at first increases rapidly and afterwards more slowly until it attains its final value equal to that of the incident light.

Next as regards change of intensity as dependent upon colour, we have that a and b depend upon λ, but their changes are so small that they may be left out of account; the quantity that has to be considered is q'. Now the smaller λ is, the more rapidly q' changes on leaving the point of contact, and the central spot must therefore be smaller for blue light than for red;

that is beyond the edge of its central portion there is a preponderance of the colours of the blue end of the spectrum.

Finally the effect of the polarisation on the size of the spot has to be considered. Let s_1, s_2 be the ratio of the intensity of the transmitted light to that of the reflected light according as the incident stream is polarised in a plane parallel or perpendicular to the plane of incidence: then

$$s_1/s_2 = \sin^2 a/\sin^2 b = \{(\mu^2 + 1)\sin^2 i - 1\}^2.$$

Now according as s_1 (or s_2) is greater or less, the spot is more or less conspicuous as regards extent and intensity at some distance from the point of contact. Very near the critical angle, we have $s_2 = \mu^4 s_1$ and therefore the distinctness of the spot is the greater for light polarised perpendicularly to the plane of incidence. As i increases, the spots seen in the two cases become more and more nearly equal in size, and they become exactly of the same magnitude when $\sin^2 i = 2/(1 + \mu^2)$, that is when the difference of phase between the oppositely polarised streams, arising from reflection at the surface of the film, attains its maximum value. When i exceeds this value, the order of magnitude is reversed, and the spots become more and more unequal as i increases. When $i = \pi/2$, $s_1 = \mu^4 s_2$ so that the inequality becomes again relatively large.

112. The above investigation of the problem of reflection and refraction has been based upon the hypothesis that the transition from the one medium into the other takes place so rapidly, that the region within which the optical properties are variable may be regarded as vanishingly small.

One of the consequences of this assumption is that a stream of light plane polarised in any azimuth with respect to the plane of incidence gives rise, in the case of ordinary reflection, to a reflected stream that is in all cases also plane polarised, and in particular that at an angle of incidence $\tan^{-1}\mu$, the plane of polarisation of the reflected light coincides with the plane of incidence, and consequently at this angle light polarised in the perpendicular plane ceases to be reflected.

This is however by no means always the case and it was found by Brewster* and by Biot† that with certain highly refracting substances there is no angle of complete polarisation, while Airy‡ confirmed this result from observations of the behaviour of Newton's rings in polarised light and made the further deduction that the phase of the component stream polarised perpendicularly to the plane of incidence undergoes a continuous variation as the angle of incidence passes through the polarising angle, instead of changing abruptly as the theory requires.

* *Phil. Trans.* CIV. 230 (1814); CV. 152 (1815).

† *Traité de Phys.* IV. 288 (1816).

‡ *Camb. Phil. Trans.* IV. 279 (1831).

This question has been carefully investigated by Jamin* by direct measurement of the difference of phase and of the ratio of the amplitudes of the vibrations in the component reflected streams polarised in the principal azimuths, and he found that with few exceptions a stream of light polarised in any azimuth with respect to the plane of incidence except 0° and 90° occasions by reflection an elliptically polarised stream, its elliptic character however being only strongly marked at angles of incidence near the polarising angle. At the principal incidence itself the axes of the elliptic vibration of the polarisation-vector are in the principal azimuths, so that the components of the reflected light polarised in these azimuths have a phase-difference of $\pi/2$.

Jamin further recognised that transparent bodies may be arranged in three classes with respect to their action upon the light reflected from them. In the case of some substances, the phase of the component polarised in the plane of incidence is by reflection at the principal incidence retarded by $\pi/2$ relatively to that of the component polarised in the other principal azimuth: with others, it is accelerated by this amount; while intermediate to these classes there is a third, characterised by the property that the reflected light remains plane polarised. Substances belonging to these three classes he termed media of positive, negative and neutral reflection respectively, and he stated as a general rule that they are included in the first or second class according as their refractive index is greater or less than 1·46. Later investigations have however considerably modified this result.

The elliptic polarisation produced by reflection at the surface of transparent media has also been investigated by Quincke†, Wernicke‡, Cornu§ and others||.

The ellipticity of the polarisation of the reflected light is found to be to a great extent dependent upon the means employed to polish the reflecting surface and upon the time that has elapsed since the surface was made, and it is scarcely perceptible in the case of clean freshly formed surfaces, such as a clean surface of water¶ or a crystalline surface newly made by cleavage**. This fact indicates that the defect in the former investigation of the problem of reflection arises from the neglect of the thickness of the transition-layer, and that we must regard two homogeneous media as separated by a region of small but sensible thickness, within which the optical properties vary.

113. As we are ignorant of the nature and properties of this surface-layer, we must content ourselves with an approximate solution of the problem

* *Ann. de Ch. et de Phys.* (3) xxix. 263 (1850); xxxi. 165 (1851).

† *Pogg. Ann.* cxxviii. 355 (1866). ‡ *Wied. Ann.* xxv. 203 (1885).

§ *C. R.* cviii. 917, 1211 (1889).

|| Cf. Winkelmann, *Handb. der Phys.* ii. 761—771.

¶ Lord Rayleigh, *Phil. Mag.* (5) xxx. 400 (1890); xxxiii. 1 (1892).

** Drude, *Wied. Ann.* xxxvi. 532 (1889); xxxviii. 265 (1889).

of reflection. This we shall base on the method employed above, assuming that the ratio of the thickness of the layer to the wave-length of light is so small that its square may be neglected and that for all quantities that occur with this ratio as a factor we may substitute the values that are obtained by neglecting the thickness of the layer*.

Let us suppose that the medium, in which the incident light travels, is homogeneous from ∞ to $x=0$, that the transition-layer occupies the space from $x=0$ to $x=-d$ and that from this lower plane to $x=-\infty$ the properties of the second medium are unvaried; and let us further assume that the characteristic equations within the surface-layer have the form (1) and (2) in which ω is regarded as a function of x.

Taking the plane of incidence as the plane xz, we obtain by multiplying the last two of each pair of triplets (1) by dx and integrating from 0 to $-d$

$$\int_0^{-d} \dot{v}\,dx = -\int_0^{-d} \frac{\partial \varpi_1}{\partial z}\,dx + \varpi_3' - \varpi_3, \qquad \int_0^{-d} \dot{w}\,dx = -\varpi_2' + \varpi_2.$$

$$\int_0^{-d} \dot{\varpi}_2\,dx = \int_0^{-d} \frac{\partial e_1}{\partial z}\,dx - e_3' + e_3, \qquad \int_0^{-d} \dot{\varpi}_3\,dx = e_2' - e_2,$$

where the accents denote the values of quantities at the plane $x=-d$.

Now we have seen in § 99 that if the thickness of the layer be insensibly small, the quantities ϖ_1, ϖ_2, ϖ_3, u, $e_2=\omega^2 v$, $e_3=\omega^2 w$ are continuous across the interface: we may then, in accordance with the assumption made above, place these quantities outside the sign of integration, assigning to them their values at the plane $x=-d$. Then writing, for shortness,

$$\int_0^{-d} \omega_x^2\,dx = -Q\omega'^2 d, \qquad \int_0^{-d} \omega_x^{-2}\,dx = -P\omega'^{-2} d \quad \text{..............(52)},$$

where P and Q are simple numerics, we obtain the system of equations

$$[\varpi_3]_{x=0} = \left[\varpi_3 + Pd\dot{v} + \frac{\partial \varpi_1}{\partial z}\,d\right]_{x=-d} \quad \text{........................(53)},$$

$$[\varpi_2]_{x=0} = [\varpi_2 - Pd\dot{w}]_{x=-d} \quad \text{..................................(54)},$$

$$[e_3]_{x=0} = \left[e_3 - \dot{\varpi}_2 d + \frac{\partial e_1}{\partial z}\,Qd\right]_{x=-d} \quad \text{........................(55)},$$

$$[e_2]_{x=0} = [e_2 + \dot{\varpi}_3 d]_{x=-d} \quad \text{..................................(56)}.$$

Let us take as the specification of the system of waves, the expressions given in § 101, omitting the bars over the letters, as we shall only apply the

* Drude, *Wied. Ann.* XXXVI. 532, 865 (1889); XLIII. 126 (1891): *Lehrbuch der Optik*, p. 266. Voigt, *Komp. der Theor. Phys.* II. 700. Cf. also Zech, *Pogg. Ann.* CIX. 60 (1860). Van Kyn van Alkemade, *Wied. Ann.* XX. 22 (1883). Von der Mühll, *Math. Ann.* V. 505 (1872).

results to the case of homogeneous waves. Then neglecting terms involving d^2 and remembering that $s^2 = \omega^2 (l^2 + n^2)$, equations (53)—(56) give

$$\begin{aligned}
(kD + k_1\bar{D}_1) &= e^{-\iota l'd} (k'\bar{D}' + \iota k'l'dP\bar{D}') \\
&= k'\bar{D}' \{1 + \iota l'd (P-1)\}, \\
\frac{l}{l^2+n^2}(kD - k_1\bar{D}_1) &= e^{-\iota l'd}\left(\frac{l'}{l'^2+n^2} k'\bar{D}' + \iota dk'\bar{D}' - \iota \frac{n^2}{l'^2+n^2} dQk'\bar{D}'\right) \\
&= k'\bar{D}' \left\{\frac{l'}{l'^2+n^2} - \iota \frac{n^2}{l'^2+n^2} d(Q-1)\right\}, \\
l(D - \bar{D}_1) &= e^{-\iota l'd} \{l'\bar{D}' + \iota (l'^2+n^2) dP\bar{D}' - \iota n^2 d\bar{D}'\} \\
&= \bar{D}' \{l' + \iota (l'^2+n^2) d (P-1)\}, \\
D + \bar{D}_1 &= e^{-\iota l'd} (\bar{D}' + \iota l'd\bar{D}') \\
&= \bar{D}';
\end{aligned}$$

whence introducing the components perpendicular and parallel to the plane of incidence and the angles of incidence and refraction

$$\left.\begin{aligned}
F + \bar{F}_1 &= \bar{F}' \left\{1 + \iota \frac{2\pi}{\lambda'} \cos r (P-1) d\right\} \\
(F - \bar{F}_1) \sin i \cos i &= \bar{F}' \left\{\sin r \cos r - \iota \frac{2\pi}{\lambda'} \sin^3 r (Q-1) d\right\} \\
(G - \bar{G}_1) \cos i &= \bar{G}' \left\{\cos r + \iota \frac{2\pi}{\lambda'} (P-1) d\right\} \\
(G + \bar{G}_1) \sin i &= \bar{G}' \sin r
\end{aligned}\right\} \quad \ldots(57).$$

Hence

$$\frac{F}{\sin(i+r)\cos(i-r) + \iota \frac{2\pi}{\lambda'} d\{(P-1)\sin i \cos i \cos r - (Q-1)\sin^3 r\}}$$

$$= \frac{\bar{F}_1}{\sin(i-r)\cos(i+r) + \iota \frac{2\pi}{\lambda'} d\{(P-1)\sin i \cos i \cos r + (Q-1)\sin^3 r\}}$$

$$= \frac{\bar{F}'}{\sin 2i} \quad \ldots\ldots(58),$$

$$\frac{G}{\sin(i+r) + \iota \frac{2\pi}{\lambda'} \sin i (P-1) d} = \frac{\bar{G}_1}{-\sin(i-r) - \iota \frac{2\pi}{\lambda'} \sin i (P-1) d}$$

$$= \frac{\bar{G}'}{\sin 2i} \quad \ldots\ldots(59),$$

and omitting as before terms involving d^2,

$$\frac{\bar{F}_1}{\bar{F}} = \frac{\tan(i-r)}{\tan(i+r)}\left\{1 + \iota\,\frac{2\pi}{\lambda'}\,d\,\frac{(P-1)\cos^2 r + (Q-1)\sin^2 r}{\sin(i+r)\cos(i+r)\sin(i-r)\cos(i-r)}\sin 2i\sin r\right\} \quad \text{.........(60)},$$

$$\frac{\bar{G}_1}{\bar{G}} = -\frac{\sin(i-r)}{\sin(i+r)}\left\{1 + \iota\,\frac{2\pi}{\lambda'}(P-1)\,d\,\frac{\sin 2i\sin r}{\sin(i+r)\sin(i-r)}\right\} \quad \text{...............(61)},$$

$$\frac{\bar{F}_1}{\bar{G}_1} = -\frac{\cos(i+r)}{\cos(i-r)}\left\{1 + \iota\,\frac{2\pi}{\lambda'}\,d\,\frac{(P-1)\sin^2 i + (Q-1)\sin^2 r}{\sin(i+r)\cos(i+r)\sin(i-r)\cos(i-r)}\sin 2i\sin r\right\}\frac{F}{G}$$

$$= -\frac{\cos(i+r)}{\cos(i-r)}\left\{1 + \iota\,\frac{4\pi}{\lambda}\,d\,\frac{(P-1) + (Q-1)\mu^{-2}}{\sin(i+r)\cos(i+r)\sin(i-r)\cos(i-r)}\sin^4 i\cos i\right\}\frac{F}{G} \quad \text{.........(62)}.$$

At the polarising angle $i + r = \pi/2$ and

$$\frac{\bar{F}_1}{\bar{G}_1} = \iota\,\frac{4\pi}{\lambda}\,d\,\{(P-1) + (Q-1)\,\mu^{-2}\}\,\frac{\sin^4 i\cos i}{\sin^2 2i\cos 2i}\,\frac{F}{G}$$

$$= \iota\,\frac{\pi}{\lambda}\,d\,\{(P-1) + (Q-1)\,\mu^{-2}\}\,\frac{\mu^2\sqrt{1+\mu^2}}{1-\mu^2}\,\frac{F}{G} \quad \text{...............(63)}.$$

Hence if the incident light be polarised at 45° to the plane of incidence, the ratio of the amplitudes of the vibrations in the component reflected streams polarised in planes perpendicular and parallel to the plane of incidence is

$$\epsilon = \frac{\pi}{\lambda}\,d\,\{(P-1) + (Q-1)\,\mu^{-2}\}\,\frac{\mu^2\sqrt{1+\mu^2}}{1-\mu^2},$$

and the difference of phase between these components is $\pi/2$. Thus the reflected light is elliptically polarised with its planes of maximum and minimum polarisation in the principal azimuths.

The quantity ϵ is called the coefficient of ellipticity, and referring to (52) we see that its value is

$$\epsilon = \frac{\pi}{\lambda}\,\frac{\sqrt{1+\mu^2}}{1-\mu^2}\int_0^{-d}\frac{(\mu^2 - \mu_x^2)(\mu_x^2 - 1)}{\mu_x^2}\,dx,$$

where μ_x denotes the refractive index within the layer relatively to the first medium at a distance x from the plane at which this medium ceases to be homogeneous.

If $\mu > 1$, it follows that ϵ will be positive so long as the refractive index of the transition-layer is at all points between the values 1 and μ: if $\mu < 1$ the reverse is the case. As it is natural to assume that the index of the layer is between that of air and the second medium, we should expect to find that in most cases the reflection in air is positive: negative reflection requires that the polished layer must be the more refracting, and this is most likely to occur when the index of the refracting substance is small.

So long as the plane of polarisation of the incident light is not in one of the principal azimuths, a plane polarised stream will at any incidence give rise to a reflected stream of elliptically polarised light, the difference of phase between the components polarised in the principal azimuths being given by

$$\tan \delta = 4\epsilon \frac{\mu^2}{\sqrt{1+\mu^2}} \frac{\sin i \tan i}{\tan^2 i - \mu^2}:$$

since however ϵ is small, the ellipticity is only marked at incidences near the principal incidence. The ratio of the amplitudes of the vibrations is except at principal incidence approximately the same as that given by the simple theory, viz.:—

$$-\frac{\cos(i+r)}{\cos(i-r)} \frac{F}{G}.$$

In the case of bodies of positive reflection it is possible to assign an inferior limit to the thickness of the surface-layer: for the value of ϵ being given d will be a minimum, when μ_x is constant and of such a value that $(\mu^2 - \mu_x^2)(\mu_x^2 - 1)/\mu_x^2$ is a maximum, that is when $\mu_x^2 = \mu$. This gives

$$\frac{d}{\lambda} = \frac{\epsilon}{\pi} \frac{\mu+1}{\mu-1} \frac{1}{\sqrt{\mu^2+1}}.$$

Thus for heavy flint glass $\mu = 1{\cdot}75$, $\epsilon = {\cdot}03$, whence $d/\lambda = 0{\cdot}0175$.

CHAPTER XI.

DOUBLE REFRACTION.

114. It was discovered by Erasmus Bartholinus that a stream of light on entering a crystal of Iceland spar is in general divided into two refracted streams. By a careful series of experiments he found that the direction of one of these streams was determined by the ordinary law of refraction given by Snell, while the other stream was bent according to a different law, that had not been previously recognised.

An account of these observations was published in Copenhagen in 1669, and their publication led Huygens to investigate whether the new refraction could be accounted for by the principles that he had already successfully applied to the explanation of ordinary refraction, and for this purpose he proceeded to determine with accuracy the experimental laws of this new phenomenon.

According to Huygens' principle the existence of two refracted streams shows that an elementary disturbance at a point on the surface of the crystal occasions two disturbances spreading out into the medium at different rates, so that the wave-surface that determines the direction of the refracted streams must be a double surface or a surface of two sheets. As one of the streams follows the ordinary law of refraction, the corresponding wave-surface must, as in the case of isotropic media, be a sphere, and since it appeared that the law determining the refraction of the other stream, though less simple, was not much more complicated, Huygens assumed that for it the form of the wave-surface was a spheroid. Now from the measures that he made, it appeared that the radius of the spherical wave-surface was practically equal to the polar semi-axis of the spheroid, whence he inferred that the two surfaces touch in the axis, and finally observing that a rhombohedral crystal of spar behaved in precisely the same way whichever pair of faces the light passed through, he concluded that the polar axis of the spheroid must be symmetrically placed with respect to each of the planes of the rhombohedron and must therefore coincide with the direction of the axis of the crystal.

Huygens also discovered that each of the two refracted streams had acquired new properties with respect to their transmission through a second rhomb of spar. To this phenomenon, which has already been described in § 12, Malus afterwards gave the name of polarisation, and he found that the properties of a stream of light completely polarised by reflection are the same with reference to the plane of reflection, as are those of the ordinary stream with respect to the principal plane of the crystal and those of the extraordinary stream with respect to the perpendicular plane. Thus according to Malus' definition the ordinary stream is polarised in the principal plane, the extraordinary stream in the plane perpendicular to the principal plane of the crystal.

The correctness of Huygens' measures and deductions remained unrecognised for over a century, and little, if any, progress was made in the quantitative determination of the laws of double refraction, until Wollaston* in 1802 undertook at the suggestion of Young an experimental investigation of the subject. Wollaston's measures confirmed the accuracy of Huygens' law for the case of Iceland spar, and the evidence in its favour was further strengthened in 1810 by the publication of a memoir by Malus†, that gained the prize offered by the French Academy for an essay on the question of double refraction.

115. It was at first assumed that Huygens' law applied to all crystals that exhibited the phenomenon of double refraction, but Brewster‡ in 1818, while examining the rings surrounding the optic axis of a crystal in polarised light, discovered a number of crystals possessing two optic axes. He determined moreover that though these directions must not be regarded as the fundamental axes of the medium, they are connected with them by simple relations, the fundamental axes in fact being the internal and external bisectors of the angle between the optic axes and a direction perpendicular to their plane.

Brewster also succeeded in establishing a connection between the optical properties of crystals and their crystallographic form. Crystals are referred to six systems based upon their grade or type of symmetry and these systems are further grouped into three classes that correspond to the arrangement of crystals into divisions determined by their optical characteristics.

In the isometric class, containing the cubic system alone, there are three principal planes of symmetry at right-angles to one another and six secondary planes of symmetry that bisect the angles between the principal planes. Optically, crystals of this class are isotropic and the wave-surface for them is

* *Phil. Trans.* XCII. 381 (1802).

† *Mém. des Sav. Étrang.* II. 303 (1810).

‡ *Phil. Trans.* CVIII. 199 (1818).

in general a sphere, but there are a few cubic crystals possessing hemihedral or tetartohedral merosymmetry that exhibit rotary properties and for these the wave-surface consists of two concentric spheres. Such crystals though still isotropic show weak double refraction.

The isodimetric class comprises all forms that have a single principal plane of symmetry. In this class there are two crystal systems:—

(*a*) the tetragonal system, having four secondary planes of symmetry all at right-angles to the principal plane and inclined to one another at angles of $n\pi/4$;

(*b*) the hexagonal system with six secondary planes of symmetry intersecting the principal plane at right-angles and each other at angles of $n\pi/6$.

Crystals of this class are optically uniaxal, the optic axis coinciding with the principal axis of symmetry for all wave-lengths and temperatures, and the wave-surface is Huygens' system of a sphere and a spheroid touching one another in the axis. There are however some crystals having merosymmetry, that show rotary properties, and in these cases the sheets of the wave-surface no longer have a common tangent plane.

In the anisometric class there is no principal plane of symmetry and this is characteristic of three crystal systems:—

(*a*) the prismatic system, that has three secondary planes of symmetry at right-angles to one another;

(*b*) the monoclinic system having one secondary plane of symmetry;

(*c*) the anorthic system with no plane of symmetry.

Such crystals are optically biaxal, and in their case the wave-surface is a surface of the fourth degree with a centre of symmetry and three rectangular planes of symmetry determining by their intersections three axes of optical symmetry.

In the prismatic system, the axes of optical symmetry coincide with the crystallographic axes for all wave-lengths and temperatures. In the monoclinic system, one of the axes of the wave-surface coincides with the crystallographic axis in all cases, while the positions of the other two change with the wave-length and temperature. In the anorthic system, the orientation of all the three axes of optical symmetry is dependent upon the wave-length and the temperature.

Uniaxal Crystals.

116. Referred to a rectangular system of axes, of which the z-axis coincides with the optic axis of the medium, Huygens' wave-surface consists of

$$\left.\begin{array}{ll}\text{the sphere} & x^2+y^2+z^2=a^2 \\ \text{and the spheroid} & \dfrac{x^2+y^2}{c^2}+\dfrac{z^2}{a^2}=1\end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots(1),$$

where a and c are the principal wave-velocities, or if σ be the ray-velocity in a direction making an angle ψ with the optic axis, the equations of the sheets of the wave-surface may be written

$$\sigma^2 = a^2, \quad \frac{1}{\sigma^2} = \frac{\sin^2\psi}{c^2} + \frac{\cos^2\psi}{a^2} \quad \text{.....................(2)},$$

whence we obtain

$$\frac{1}{\sigma^2} - \frac{1}{a^2} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin^2\psi \quad \text{.........................(3)},$$

or the difference of the squares of the reciprocals of the ray-velocities in any direction is proportional to the square of the sine of the angle that this direction makes with the optic axis.

117. Since the wave-surface is a surface of revolution, it follows from symmetry that the extraordinary ray and the normal to the corresponding plane wave lie in a plane through the optic axis. Taking this plane as that of xz, let (x', z') be the coordinates of the extremity of an extraordinary ray OS: the corresponding plane wave W is perpendicular to the plane xz and cuts it in the line

$$\frac{xx'}{c^2} + \frac{zz'}{a^2} = 1,$$

touching the ellipse

$$\frac{x^2}{c^2} + \frac{z^2}{a^2} = 1$$

at the point (x', z').

But if ω be the wave-velocity and χ be the angle between the normal to the wave and the optic axis

$$x\sin\chi + z\cos\chi = \omega,$$

whence

$$x'/c^2 = \sin\chi/\omega, \quad z'/a^2 = \cos\chi/\omega \quad \text{.....................(4)}.$$

Substituting these values of x', z' in the equation of the ellipse, we obtain

$$\omega^2 = c^2\sin^2\chi + a^2\cos^2\chi.$$

Thus in polar coordinates the surface of wave-quickness consists of

$$\left.\begin{array}{rl} \text{the sphere} & \omega = a \\ \text{and the ovaloid} & \omega^2 = c^2\sin^2\chi + a^2\cos^2\chi \end{array}\right\} \quad \text{.........................(5)},$$

and in Cartesian coordinates the equations of these surfaces are

$$\left.\begin{array}{l} x^2 + y^2 + z^2 = a^2 \\ (x^2 + y^2 + z^2)^2 = c^2(x^2 + y^2) + a^2z^2 \end{array}\right\} \quad \text{.....................(6)}.$$

118. A comparison of equations (2) and (5) shows that we pass from the one to the other by changing

$$a,\ c,\ \sigma,\ \psi \ \text{into}\ a^{-1},\ c^{-1},\ \omega^{-1},\ \chi$$

respectively, and accordingly to each proposition referring to rays there

corresponds a similar proposition relating to waves. Thus we have in correspondence with (3)

$$\omega^2 - a^2 = (c^2 - a^2)\sin^2\chi \quad\text{.........................(7)},$$

or the difference of the squares of the wave-velocities in any given direction is proportional to the square of the sine of the angle that the direction makes with the optic axis.

From equations (4) we have

$$\tan\chi = \frac{a^2}{c^2}\frac{x'}{z'} = \frac{a^2}{c^2}\tan\psi \quad\text{...............................(8)},$$

thus the tangents of the angles between the optic axis and the ordinary and the extraordinary rays, corresponding to waves that have the same direction, are in a constant ratio, and the angle between these rays is given by

$$\tan\theta = \tan(\chi - \psi) = \frac{(a^2 - c^2)\tan\chi}{a^2 + c^2\tan^2\chi} = \frac{(a^2 - c^2)\sin\chi\cos\chi}{\omega^2} \quad\text{.........(9)};$$

this angle is the greatest when

$$\tan\chi = \pm\, a/c, \text{ or } \tan\psi = \pm\, c/a,$$

that is when the sum of the angles χ and ψ is a right-angle; its value then is $\tan^{-1}\{\pm(a^2 - c^2)/(2ac)\}$.

119. We see from equation (8) that $\chi \gtrless \psi$ or the ordinary ray is farther from or nearer to the optic axis than the extraordinary ray of a wave in the same direction, according as $a \gtrless c$, that is according as the spherical sheet is without or within the spheroidal sheet of the wave-surface, the spheroid being in the first case prolate and in the second oblate.

There are then two classes of uniaxal crystals and these Biot*, to whom their discovery is due, denominated attractive and repulsive respectively, ascribing the existence of the extraordinary ray in the theory of emission to attractive or repulsive forces emanating from the optic axis. These classes of crystals are now called positive and negative.

120. The surface of wave-slowness in an uniaxal crystal, being the inverse of the surface of wave-quickness, consists of a sphere of radius a^{-1} and an ellipsoid of revolution about the optic axis, the polar and equatorial semi-axes of which are a^{-1} and c^{-1} respectively.

Let Q be any point on the spheroid, QM the perpendicular from Q on the equatorial plane, O the centre of the surface, then

$$c^2OM^2 + a^2QM^2 = 1 \text{ or } c^2OQ^2 + (a^2 - c^2)\,QM^2 = 1;$$

but if θ_1, θ_2, θ_3, be the angles that the optic axis makes with the axes of x, y and z respectively, the equation of the equatorial plane is

$$x\cos\theta_1 + y\cos\theta_2 + z\cos\theta_3 = 0,$$

* *Mém. de la prem. classe de l'Inst.* XIII. (2) 19 (1814).

and hence the surface of wave-slowness consists of

$$\left.\begin{array}{ll}\text{the sphere} & x^2+y^2+z^2=a^{-2}\\ \text{and the spheroid} & \\ \quad c^2(x^2+y^2+z^2)+(a^2-c^2)(x\cos\theta_1+y\cos\theta_2+z\cos\theta_3)^2=1 & \end{array}\right\}\quad\text{......(10).}$$

Suppose now that a plane wave is incident at an angle i on a plane surface bounding an uniaxal crystal, and let the plane of incidence be taken as the plane of xz and the surface of the crystal as that of xy, the positive quadrant xz containing the direction in which the light travels*.

Since the ordinary wave within the crystal follows the ordinary law of refraction, its normal makes an angle r_0 with the normal to the surface given by

$$\sin r_0 = a\sin i/\Omega \quad\text{..............................(11),}$$

Ω being the propagational speed in the outer medium, and the ordinary ray coincides with the wave-normal. The plane of polarisation is the plane containing the wave-normal and the optic axis, and hence if α_0, β_0, γ_0 be the direction cosines of its normal

$$\alpha_0\cos\theta_1+\beta_0\cos\theta_2+\gamma_0\cos\theta_3=0,$$

$$\alpha_0\sin r_0 \qquad\qquad +\gamma_0\cos r_0=0,$$

whence the equation of the plane of polarisation of the ordinary wave is

$$\cos r_0\cos\theta_2\,x+(\sin r_0\cos\theta_3-\cos r_0\cos\theta_1)\,y-\sin r_0\cos\theta_2\,z=0\text{...(12).}$$

As regards the extraordinary wave, if r_e be the angle that its normal makes with the axis of z, r_e is determined by writing

$$x=\sin i/\Omega,\quad y=0,\quad z=\sin i\cot r_e/\Omega$$

in the equation of the spheroidal sheet of the surface of wave-slowness. This gives

$$c^2(1+\cot^2 r_e)+(a^2-c^2)(\cos\theta_1+\cos\theta_3\cot r_e)^2=\Omega^2/\sin^2 i,$$

or

$$[\{c^2+(a^2-c^2)\cos^2\theta_1\}\sin^2 i-\Omega^2]\tan^2 r_e+2(a^2-c^2)\cos\theta_1\cos\theta_3\sin^2 i\tan r_e$$
$$+\{c^2+(a^2-c^2)\cos^2\theta_3\}\sin^2 i=0\text{...............(13).}$$

If the propagational speed in the outer medium be greater than the greatest principal velocity in the crystal, this equation gives two real roots of opposite sign for $\tan r_e$, of which the positive one is that required, as from the nature of the problem r_e is positive and less than $\pi/2$. Hence r_e is determined and the velocity of the wave is given by

$$\omega=\Omega\sin r_e/\sin i.$$

The corresponding ray is in the direction of the perpendicular from the

* Beer, *Einleitung in die höhere Optik*, 2nd ed. p. 273.

centre on the tangent plane to the surface of wave-slowness at the point in which the normal to the wave meets it and is therefore parallel to the normal to the surface at this point. Whence we obtain for its equations

$$\frac{x}{c^2 \sin r_e + (a^2 - c^2)\cos\phi_e \cos\theta_1} = \frac{y}{(a^2 - c^2)\cos\phi_e \cos\theta_2}$$

$$= \frac{z}{c^2 \cos r_e + (a^2 - c^2)\cos\phi_e \cos\theta_3} \;\ldots\ldots\ldots\ldots(14),$$

where ϕ_e is the angle between the wave-normal and the optic axis, so that

$$\cos\phi_e = \sin r_e \cos\theta_1 + \cos r_e \cos\theta_3.$$

The plane of polarisation of the extraordinary wave is perpendicular to the wave and to the plane containing the wave-normal and the optic axis: hence if α_e, β_e, γ_e be the direction-cosines of its normal

$$\alpha_e \cos\theta_2 \cos r_e + \beta_e(\cos\theta_3 \sin r_e - \cos\theta_1 \cos r_e) - \gamma_e \cos\theta_2 \sin r_e = 0,$$

$$\alpha_e \sin r_e + \gamma_e \cos r_e = 0,$$

and the equation of the plane of polarisation is

$$x - \frac{\cos\theta_2}{\cos r_e(\cos\theta_3 \sin r_e - \cos\theta_1 \cos r_e)}\, y - \tan r_e\, z = 0 \ldots\ldots\ldots(15).$$

The angle between the planes of polarisation of the two refracted waves is Θ, where

$$\cos\Theta = \alpha_0\alpha_e + \beta_0\beta_e + \gamma_0\gamma_e$$

$$= \cos\theta_2 \operatorname{cosec}\phi_0 \cot\phi_e \sin(r_e - r_0) \;\ldots\ldots\ldots\ldots\ldots(16),$$

where ϕ_0, ϕ_e are the angles between the refracted wave-normals and the optic axis.

The extraordinary ray is in the plane of incidence, only when

$$\cos\phi_e = 0 \quad\text{or}\quad \cos\theta_2 = 0,$$

that is when the optic axis is either parallel to the refracted wave, or in the plane of incidence: in the latter case the plane of polarisation becomes indeterminate when

$$\cos\theta_1/\sin r_e = \cos\theta_3/\cos r_e,$$

which expresses that the optic axis is in the direction of the wave-normal.

The planes of polarisation of the two refracted waves are at right-angles, only when

$$\cos\theta_2 = 0, \quad \cos\phi_e = 0, \quad \sin(r_e - r_0) = 0,$$

that is when the optic axis is either in the plane of incidence or parallel to the extraordinary wave, and when the two refracted waves have the same direction, that is in the case of normal incidence.

Biaxal Crystals.

121. The first attempt to extend Huygens' construction to biaxal crystals was made by Young*, who suggested a sphere combined with an ellipsoid having three unequal axes as the form of the wave-surface in such media. It is readily seen however that this form of wave-surface is inconsistent with the biaxal character of the crystals, and in addition Fresnel discovered that any form with a spherical sheet must be rejected, since in biaxal crystals there is no ordinary refraction in all cases, as was at first supposed to be the case.

Fresnel arrived at this conclusion by the following considerations. Starting from the idea that light consists in transverse vibrations of the particles of the ether, he was led by the symmetry of uniaxal crystals about their axis to assume that vibrations perpendicular to this direction are propagated with the same speed in all directions†, and he pointed out that this explains the existence of an ordinary wave and the relation between its velocity and that of the extraordinary wave, provided the vibrations in a stream of polarised light are perpendicular to the plane of polarisation: for in that case, light polarised in the principal plane will travel with the same speed in all directions, as the vibrations are in all cases perpendicular to the optic axis; on the other hand light polarised in a plane perpendicular to the principal plane will have a speed dependent upon the direction of propagation, as the vibrations are in general oblique to the optic axis. As however the direction of propagation approaches that of the axis, the vibrations will become more and more nearly at right-angles to it and the speed will approximate to that of the ordinary waves.

It soon became obvious to Fresnel that this explanation could not be applied to the case of biaxal crystals, and that there was no reason to expect an ordinary wave in such media, since the existence of two optic axes indicates that they possess no single direction round which their optical properties are symmetrical. In order to test this inference, Fresnel took two prisms of topaz, cut in different directions with respect to the crystallographic axes and carefully worked so as to have the same angle, and of these he formed a single prism by attaching them together with their edges in the same straight line. After partially achromatising the system by prisms of crown glass, he observed through the combination a luminous line parallel to the edge of the prism and at once perceived that the image, hitherto regarded as due to ordinary refraction, was discontinuous, proving that the deviations produced by the two halves of the prism did not follow the same law‡.

* *Miscellaneous Works*, I. 317, 322.

† *Œuvres complètes*, I. No. XXII. § 14, p. 636. For an account of the sequence of Fresnel's ideas on Double Refraction, see the introduction to Fresnel's work by Verdet, pp. lxv—lxxxv, reprinted in Verdet's works, Vol. I. pp. 360—376.

‡ *Œuvres complètes*, II. No. XXXVIII. § 12, p. 271.

If then the laws of double refraction in biaxal crystals were to be deduced by Huygens' method, it would become necessary to look for a surface, having two sheets and probably of the fourth degree, that would reduce in the case of uniaxal crystals to Huygens' system of a sphere and a spheroid, and recognising the difficulties inherent in this method of procedure, Fresnel was led to consider the possibility of representing the phenomena by the aid of a simpler surface. Now he perceived that in the case of uniaxal crystals it was possible for this purpose to replace Huygens' wave-surface by a single spheroid, of which the polar and equatorial semi-axes are respectively the equatorial and polar axes of Huygens' spheroid, as the velocities of the two rays in any direction are given by the semi-axes of the section of this spheroid by a diametral plane perpendicular to the ray, and the plane of polarisation of either ray is perpendicular to the semi-axis that gives the ray-velocity*. It therefore suggested itself to Fresnel that the properties of biaxal crystals could be expressed by similar relations with respect to an ellipsoid with three unequal axes, and the results thus deduced he found to be in accordance with all the facts known about such crystals†.

This surface is called, for reasons that will appear later, "the reciprocal ellipsoid," and the wave-surface is the locus of points obtained by taking on the radii-vectores through its centre lengths equal to the semi-axes of the diametral sections perpendicular to their directions.

122. Turning now to the consideration of waves, it is clear that the speeds and polarisations of waves in an uniaxal crystal may be determined by the aid of a spheroid, of which the semi-axes are the reciprocals of those of the reciprocal spheroid, the wave-velocities in any direction being the reciprocals of the semi-axes of the diametral section parallel to the plane of the waves, and the plane of polarisation of each wave being perpendicular to the axis determining its speed. Hence it is natural, as in the case of rays, to extend this construction to biaxal crystals by the employment of an ellipsoid with three unequal axes. This ellipsoid is called "the polarisation ellipsoid."

Let the equation of the ellipsoid of polarisation, referred to its principal axes, be

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad \text{.............................(17)};$$

then to determine the speeds and the polarisations of the waves propagated in the direction, of which the direction-cosines are l, m, n, we have to find the axes of the section of this ellipsoid by the plane

$$lx + my + nz = 0 \quad \text{.................................(18)}.$$

Let $1/\omega$ be the length, α, β, γ the direction-cosines of any radius-vector

* Fresnel supposed that the vibrations were perpendicular to the ray (*Œuvres*, II. No. XXXVIII. § 22, p. 281), an assumption that he rejected afterwards.

† *Œuvres complètes*, II. Nos. XXXVIII., XXXIX., XL.

ON of the section; then if x, y, z be the coordinates of its extremity N, we have

$$x = \alpha/\omega, \quad y = \beta/\omega, \quad z = \gamma/\omega,$$

and from the equation of the ellipsoid

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 = \omega^2 \quad\text{.........................(19)},$$

also α, β, γ are connected by the relations

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad\text{..................................(20)},$$

$$\alpha l + \beta m + \gamma n = 0 \quad\text{..................................(21)},$$

the latter equation expressing that the radius-vector is in the plane of section.

If now ON be one of the semi-axes of the section, ω^2 must be either a maximum or a minimum subject to the conditions (20), (21); whence differentiating with respect to α, β, γ,

$$\left.\begin{aligned} a^2\alpha d\alpha + b^2\beta d\beta + c^2\gamma d\gamma = 0 \\ \alpha d\alpha + \beta d\beta + \gamma d\gamma = 0 \\ l d\alpha + m d\beta + n d\gamma = 0 \end{aligned}\right\} \quad\text{......................(22)},$$

and using indeterminate multipliers, we have

$$(a^2 - E)\,\alpha = Fl, \quad (b^2 - E)\,\beta = Fm, \quad (c^2 - E)\,\gamma = Fn.$$

Multiplying these equations by α, β, γ respectively and adding, we find

$$E = \omega^2,$$

whence

$$(a^2 - \omega^2)\,\alpha = Fl, \quad (b^2 - \omega^2)\,\beta = Fm, \quad (c^2 - \omega^2)\,\gamma = Fn \quad\text{.........(23)},$$

from which by eliminating α, β, γ we obtain

$$\frac{l^2}{a^2 - \omega^2} + \frac{m^2}{b^2 - \omega^2} + \frac{n^2}{c^2 - \omega^2} = 0 \quad\text{.........................(24)},$$

the roots of which give the two propagational speeds in the direction (l, m, n).

Again multiplying (23) by l, m, n respectively and adding, we get

$$F = a^2\alpha l + b^2\beta m + c^2\gamma n \quad\text{..........................(25)},$$

and also from (23) since $\alpha^2 + \beta^2 + \gamma^2 = 1$

$$\frac{1}{F^2} = \left(\frac{l}{a^2 - \omega^2}\right)^2 + \left(\frac{m}{b^2 - \omega^2}\right)^2 + \left(\frac{n}{c^2 - \omega^2}\right)^2 \quad\text{...........(26)},$$

which gives the value of F corresponding to either of the two waves, and then the direction-cosines of the corresponding polarisation-vector are obtained from (23).

On the other hand, if it be the plane of polarisation or α, β, γ, that we

know, the propagational speed is determined by (19), and squaring and adding equations (23), we have

$$F^2 + \omega^4 = a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2 \qquad \text{(27)},$$

which gives F and the direction-cosines of the wave-normal are by (23) given by

$$l = (a^2 - \omega^2)\,\alpha/F, \quad m = (b^2 - \omega^2)\,\beta/F, \quad n = (c^2 - \omega^2)\,\gamma/F.$$

123. Let ω_1 be the speed of the quicker, ω_2 that of the slower wave propagated in the direction (l, m, n), then since ω_1, ω_2 are the roots of equation (24), the equation

$$(\zeta^2 - \omega_1^2)(\zeta^2 - \omega_2^2)$$
$$= l^2(\zeta^2 - b^2)(\zeta^2 - c^2) + m^2(\zeta^2 - c^2)(\zeta^2 - a^2) + n^2(\zeta^2 - a^2)(\zeta^2 - b^2)$$

is identically true for all the values of ζ. Hence writing in turn a, b, c for ζ, we obtain

$$\left.\begin{aligned} l^2 &= \frac{(a^2 - \omega_1^2)(a^2 - \omega_2^2)}{(a^2 - b^2)(a^2 - c^2)} \\ m^2 &= \frac{(b^2 - \omega_1^2)(b^2 - \omega_2^2)}{(b^2 - c^2)(b^2 - a^2)} \\ n^2 &= \frac{(c^2 - \omega_1^2)(c^2 - \omega_2^2)}{(c^2 - a^2)(c^2 - b^2)} \end{aligned}\right\} \qquad \text{(28)},$$

whence we find

$$F_1^2 = \frac{(a^2 - \omega_1^2)(b^2 - \omega_1^2)(c^2 - \omega_1^2)}{\omega_1^2 - \omega_2^2}, \quad F_2^2 = \frac{(a^2 - \omega_2^2)(b^2 - \omega_2^2)(c^2 - \omega_2^2)}{\omega_2^2 - \omega_1^2} \quad \text{...(29)},$$

relations that we shall require later.

Now assuming, as we shall do in what follows, that $a^2 > b^2 > c^2$, the second of equations (28) shows that $(\omega_1^2 - b^2)(b^2 - \omega_2^2)$ is always positive and then the first and last of these equations give that ω_1 and ω_2 are both less than a and greater than c, so that $a > \omega_1 > b > \omega_2 > c$.

We see then that if ω_1 and ω_2 become equal, this can only occur by their both being equal to b; but if one of the speeds be b, we must have $m = 0$, that is the wave-normal lies in the plane of xz and if the second speed be also b, we have

$$l^2 = \frac{a^2 - b^2}{a^2 - c^2}, \quad n^2 = \frac{b^2 - c^2}{a^2 - c^2}.$$

At the same time the expressions for the direction-cosines of the polarisation-vector become indeterminate and hence in the directions given by

$$l_0 = \pm\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad m_0 = 0, \quad n_0 = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \quad \text{(30)},$$

all waves are propagated with the same speed, whatever may be their polarisation. Since these directions have the same property as the optic

axis of an uniaxal crystal, they are called the optic axes of the medium, and they are obviously the normals to the circular sections of the ellipsoid of polarisation.

Calling OA the optic axis in the quadrant xz, OB that in the quadrant $-xz$, and 2Ω the angle AOB, then

$$\sin\Omega = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad \cos\Omega = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \quad \text{...............(31).}$$

124. The quantities l, m, n given by equations (28) may be regarded as the coordinates of a point on a sphere of unit radius with the origin as centre. If ω_2 be kept constant, while ω_1 varies, the point will describe a spherical ellipse, determined by the cone

$$\frac{l^2}{a^2 - \omega_2{}^2} + \frac{m^2}{b^2 - \omega_2{}^2} + \frac{n^2}{c^2 - \omega_2{}^2} = 0 \quad \text{..................(32),}$$

the centre and foci of the ellipse being the points in which OZ, OA, and OB meet the sphere*.

Similarly if ω_2 vary, while ω_1 is constant, the point will describe a spherical ellipse given by the cone

$$\frac{l^2}{a^2 - \omega_1{}^2} + \frac{m^2}{b^2 - \omega_1{}^2} + \frac{n^2}{c^2 - \omega_1{}^2} = 0 \quad \text{..................(33)}$$

with its centre and foci at the points in which OX, OA and OB' (the prolongation of BO) meet the sphere.

Taking the spherical ellipse $\omega_1 = \text{const.}$, we have

$$ldl = -\frac{a^2 - \omega_1{}^2}{(a^2 - b^2)(a^2 - c^2)}\omega_2 d\omega_2, \quad mdm = -\frac{b^2 - \omega_1{}^2}{(b^2 - c^2)(b^2 - a^2)}\omega_2 d\omega_2,$$

$$ndn = -\frac{c^2 - \omega_1{}^2}{(c^2 - a^2)(c^2 - b^2)}\omega_2 d\omega_2,$$

whence if α_1, β_1, γ_1 be the direction-cosines of the normal to the plane of polarisation of the wave ω_1, we have

$$\alpha_1 dl + \beta_1 dm + \gamma_1 dn = 0,$$

or the plane of polarisation of the wave ω_1 cuts the sphere in a tangent to the spherical ellipse $\omega_1 = \text{const.}$

Similarly the plane of polarisation of the wave ω_2 cuts the sphere along a tangent to the ellipse $\omega_2 = \text{const.}$

But the tangent to a sphero-conic makes equal angles with the radii-vectores from the foci to the point of contact: hence, the planes of polarisation of the two waves propagated in any given direction bisect the angles between the planes drawn through this direction and the optic axes.

* Clebsch, *Prinzipien der math. Optik*, Augsburg (1887), p. 38.

Again let N be a point on the spherical ellipse $\omega_2 = \text{const.}$, and let the angles NA and NB be χ and χ' respectively; then $(\chi + \chi')/2$ is the major semi-axis of the ellipse, that is the angle between the axis of z and the

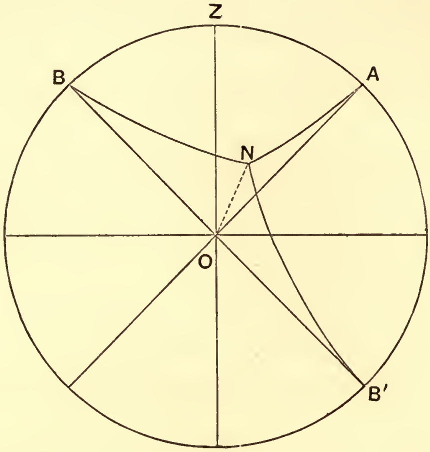


Fig. 27.

generating line of the cone (32) lying in the plane of xz. But writing $m = 0$, we have for this line

$$\frac{l^2}{a^2 - \omega_2^2} + \frac{n^2}{c^2 - \omega_2^2} = 0,$$

whence
$$\frac{a^2 - \omega_2^2}{\omega_2^2 - c^2} = \tan^2 \frac{\chi + \chi'}{2}$$

or
$$\omega_2^2 = \frac{a^2 + c^2}{2} + \frac{a^2 - c^2}{2} \cos(\chi + \chi') \quad \text{.................(34).}$$

Next regarding N as a point on the ellipse $\omega_1 = \text{const.}$ we obtain by interchanging a and c and writing $\pi - \chi'$ for χ',

$$\omega_1^2 = \frac{a^2 + c^2}{2} + \frac{a^2 - c^2}{2} \cos(\chi - \chi') \quad \text{..................(35),}$$

and from (34) and (35)

$$\omega_1^2 - \omega_2^2 = (a^2 - c^2) \sin \chi \sin \chi' \quad \text{.....................(36),}$$

or the difference of the squares of the speeds of two waves propagated in a given direction is proportional to the product of the sines of the angles between that direction and the optic axes.

We have further

$$a^2 - \omega_1^2 = (a^2 - c^2) \sin^2 \frac{\chi - \chi'}{2}, \quad c^2 - \omega_1^2 = -(a^2 - c^2) \cos^2 \frac{\chi - \chi'}{2},$$

$$b^2 - \omega_1^2 = -(a^2 - c^2)\left(\sin^2 \Omega - \sin^2 \frac{\chi - \chi'}{2}\right) = -\frac{a^2 - c^2}{2} \{\cos(\chi - \chi') - \cos 2\Omega\};$$

but if i be the angle ANB

$$\cos 2\Omega = \cos\chi \cos\chi' + \sin\chi \sin\chi' \cos i,$$

$$\therefore \quad b^2 - \omega_1^2 = -(a^2 - c^2)\sin\chi \sin\chi' \sin^2\frac{i}{2}.$$

Hence $$F_1 = \pm\tfrac{1}{2}(a^2 - c^2)\sin(\chi - \chi')\sin\frac{i}{2} \qquad (37),$$

and in the same way we find

$$F_2 = \pm\tfrac{1}{2}(a^2 - c^2)\sin(\chi + \chi')\cos\frac{i}{2} \qquad (38).$$

125. Considering now the transition from biaxal to uniaxal crystals, let us first suppose that the mean axis of the ellipsoid of polarisation gradually diminishes until it becomes equal to the least axis $2/a$. The medium then becomes ultimately an uniaxal crystal with its optic axis in the direction of the axis of z. Since in the limit χ and χ' become equal, we see that the quicker wave then has a constant speed and is polarised in the principal plane, while the slower wave has a speed dependent upon the direction of propagation and is polarised in a plane perpendicular to the principal plane. Hence when the acute angle between the optic axes is bisected by the greatest axis of the ellipsoid of polarisation, the biaxal crystal has a certain resemblance to a positive uniaxal crystal and the more acute the angle between the optic axes, the greater is the similarity.

Again by increasing the mean axis of the ellipsoid of polarisation until it becomes equal to the greatest axis $2/c$, we see that, when the acute angle between the optic axes of a biaxal crystal is bisected by the least axis of the ellipsoid of polarisation, the crystal to a certain extent resembles a negative uniaxal crystal with its axis in the direction of the axis of x.

The bisector of the acute angle between the optic axes is called "the first mean line," the bisector of the obtuse angle is termed "the second mean line." Thus a biaxal crystal is said to be positive or negative, according as the first or the second mean line coincides with the greatest axis of the ellipsoid of polarisation.

Calling 2Ω the angle between the optic axes that is bisected by the axis of z, we have

$$\cos 2\Omega = (2b^2 - a^2 - c^2)/(a^2 - c^2),$$

and from what precedes, the crystal is positive or negative according as

$$2\Omega \lesseqgtr \pi/2, \text{ that is according as } 2b^2 \gtrless a^2 + c^2.$$

126. The surface of wave-quickness is the locus of points obtained by taking on lines through the centre of the ellipsoid of polarisation lengths

representing the reciprocals of the semi-axes of the central sections of the ellipsoid perpendicular to the lines. Its equation in polar coordinates is

$$\frac{l^2}{a^2-\omega^2}+\frac{m^2}{b^2-\omega^2}+\frac{n^2}{c^2-\omega^2}=0 \quad\text{......................(39)},$$

and in Cartesian coordinates

$$\frac{x^2}{a^2-(x^2+y^2+z^2)}+\frac{y^2}{b^2-(x^2+y^2+z^2)}+\frac{z^2}{c^2-(x^2+y^2+z^2)}=0 \quad\text{.....(39')}.$$

The sections of this surface by the planes of symmetry are a circle and an oval, of which the equations are given in the following scheme:

Plane	Circle	Oval
yz	$y^2+z^2=a^2$	$(y^2+z^2)^2=c^2y^2+b^2z^2$,
zx	$z^2+x^2=b^2$	$(z^2+x^2)^2=a^2z^2+c^2x^2$,
xy	$x^2+y^2=c^2$	$(x^2+y^2)^2=b^2x^2+a^2y^2$.

Since the surface of wave-quickness is the pedal of the wave-surface, the circle in each plane of symmetry is common to these surfaces.

In the plane of xz the circle and oval intersect, and the radii-vectores to the points of intersection give the optic axes. We are thus afforded another method of determining these directions.

The equation of a plane wave propagated with speed ω in the direction given by the cosines l, m, n is

$$lx+my+nz=\omega,$$

l, m, n, ω being connected by (39). Taking as coordinates of the plane the negative reciprocals of the intercepts made by it on the axes, or writing

$$L=-l/\omega,\quad M=-m/\omega,\quad N=-n/\omega$$

the equation of the plane becomes

$$Lx+My+Nz+1=0,$$

where L, M, N are connected by the relations

$$\left.\begin{aligned}\frac{L^2}{a^2-\omega^2}+\frac{M^2}{b^2-\omega^2}+\frac{N^2}{c^2-\omega^2}&=0\\ \omega^2=(L^2+M^2+N^2)^{-1}&\end{aligned}\right\}.$$

Eliminating ω between these equations, we obtain

$$(L^2b^2c^2+M^2c^2a^2+N^2a^2b^2)(L^2+M^2+N^2)$$
$$-\{L^2(b^2+c^2)+M^2(c^2+a^2)+N^2(a^2+b^2)\}+1=0\text{...(40)},$$

which is the tangential equation of the envelope of the wave, that is of the wave-surface*.

* Plücker, *Crelle's J.* xix. 13 (1838).

127. The surface of wave-slowness is the reciprocal of the wave-surface with respect to a concentric sphere of unit radius: its Cartesian equation is thus obtained from the tangential equation of the wave-surface by changing tangential into Cartesian coordinates. Hence we obtain

$$(b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2)(x^2 + y^2 + z^2)$$
$$- \{(b^2 + c^2)\, x^2 + (c^2 + a^2)\, y^2 + (a^2 + b^2)\, z^2\} + 1 = 0 \quad\ldots\ldots(41).$$

It is also the inverse of the surface of wave-quickness, and hence its polar equation is

$$\frac{l^2}{a^2\omega^2 - 1} + \frac{m^2}{b^2\omega^2 - 1} + \frac{n^2}{c^2\omega^2 - 1} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(42).$$

Since it is the locus of points obtained by taking, on the normals to the diametral sections of the ellipsoid of polarisation drawn through its centre, lengths equal to the axes of the sections, the outer sheet of the surface of wave-slowness corresponds to the inner sheet of the surface of wave-quickness and *vice versâ*.

The section of the surface by each plane of symmetry consists of a circle and an ellipse, of which the equations are given in the following table:

Plane	Circle	Ellipse
yz	$y^2 + z^2 = a^{-2}$	$c^2y^2 + b^2z^2 = 1,$
zx	$z^2 + x^2 = b^{-2}$	$a^2z^2 + c^2x^2 = 1,$
xy	$x^2 + y^2 = c^{-2}$	$b^2x^2 + a^2y^2 = 1.$

In the plane of xz the circle and the ellipse intersect, the points of intersection being on the optic axes.

In order to determine the refracted waves corresponding to a plane wave incident in an isotropic medium on a plane surface of a biaxal crystal*, let us take new axes ξ, η, ζ such that the surface is the plane of $\xi\eta$ and the plane of incidence that of $\xi\zeta$, the positive direction of the new axes being so chosen that the positive quadrant $\xi\zeta$ contains the direction of propagation of the light, and that on regarding the plane of $\xi\zeta$ from the positive direction of the axis of η, the positive axis of ξ is made to coincide with that of ζ by a rotation in the direction of the hands of a watch. Let the new axes be given with reference to those of x, y, z by the scheme

	x	y	z
ξ	c_{11}	c_{12}	c_{13}
η	c_{21}	c_{22}	c_{23}
ζ	c_{31}	c_{32}	c_{33}.

* Liebisch, *N. Jahrb. für Min.* (1885) II. 181; *Phys. Kryst.* p. 353.

Then to obtain the equation of the surface of wave-slowness referred to the new system of coordinates, we must write

$$x = c_{11}\xi + c_{21}\eta + c_{31}\zeta, \qquad y = c_{12}\xi + c_{22}\eta + c_{32}\zeta,$$
$$z = c_{13}\xi + c_{23}\eta + c_{33}\zeta, \quad x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2,$$

and the directions of the refracted waves, and of those produced by them on reflection at a second surface parallel to the first, are found by writing in the equation thus obtained

$$\xi = \sin i/\Omega, \quad \eta = 0, \quad \zeta = \sin i/(\Omega \tan r).$$

Making these substitutions we obtain an equation

$$a_0 \tan^4 r + 4a_1 \tan^3 r + 6a_2 \tan^2 r + 4a_3 \tan r + a_4 = 0 \quad \text{........(43)},$$

where

$$a_0 = \frac{\sin^4 i}{\Omega^4}(b^2c^2c_{11}{}^2 + c^2a^2c_{12}{}^2 + a^2b^2c_{13}{}^2)$$
$$- \frac{\sin^2 i}{\Omega^2}\{(b^2 + c^2)\,c_{11}{}^2 + (c^2 + a^2)\,c_{12}{}^2 + (a^2 + b^2)\,c_{13}{}^2\} + 1,$$

$$4a_1 = 2\frac{\sin^4 i}{\Omega^4}(b^2c^2c_{11}c_{31} + c^2a^2c_{12}c_{32} + a^2b^2c_{13}c_{33})$$
$$- 2\frac{\sin^2 i}{\Omega^2}\{(b^2 + c^2)\,c_{11}c_{31} + (c^2 + a^2)\,c_{12}c_{32} + (a^2 + b^2)\,c_{13}c_{33}\},$$

$$6a_2 = \frac{\sin^4 i}{\Omega^4}\{b^2c^2(c_{11}{}^2 + c_{31}{}^2) + c^2a^2(c_{12}{}^2 + c_{32}{}^2) + a^2b^2(c_{13}{}^2 + c_{33}{}^2)\}$$
$$- \frac{\sin^2 i}{\Omega^2}\{(b^2 + c^2)\,c_{31}{}^2 + (c^2 + a^2)\,c_{32}{}^2 + (a^2 + b^2)\,c_{33}{}^2\},$$

$$4a_3 = 2\frac{\sin^4 i}{\Omega^4}(b^2c^2c_{11}c_{31} + c^2a^2c_{12}c_{32} + a^2b^2c_{13}c_{33}),$$

$$a_4 = \frac{\sin^4 i}{\Omega^4}(b^2c^2c_{31}{}^2 + c^2a^2c_{32}{}^2 + a^2b^2c_{33}{}^2).$$

In general this equation can only be solved by a method of approximation but in certain cases it assumes simple forms that give complete solutions of the problem. Thus, suppose that the surface of the crystal is parallel to one of the axes of symmetry, say the axis of z, and let the angle $(x\zeta)$ be μ and let the angle between the planes $\xi\zeta$ and xy be δ. Then

$$c_{11} = \sin\mu\cos\delta, \qquad c_{12} = \cos\mu\cos\delta, \qquad c_{13} = \sin\delta,$$
$$c_{31} = \cos\mu, \qquad c_{32} = -\sin\mu, \qquad c_{33} = 0,$$

and

$$a_0 = \frac{\sin^4 i}{\Omega^4}(b^2c^2\sin^2\mu\cos^2\delta + c^2a^2\cos^2\mu\cos^2\delta + a^2b^2\sin^2\delta)$$
$$- \frac{\sin^2 i}{\Omega^2}\{(b^2 + c^2)\sin^2\mu\cos^2\delta + (c^2 + a^2)\cos^2\mu\cos^2\delta + (a^2 + b^2)\sin^2\delta\} + 1,$$

$$4a_1 = 2\left(\frac{\sin^2 i}{\Omega^2}c^2 - 1\right)\frac{\sin^2 i}{\Omega^2}(b^2 - a^2)\sin\mu\cos\mu\cos\delta,$$

$$6a_2 = \frac{\sin^4 i}{\Omega^4}\{b^2c^2(1 - \sin^2\mu\sin^2\delta) + c^2a^2(1 - \cos^2\mu\sin^2\delta) + a^2b^2\sin^2\delta\}$$
$$- \frac{\sin^2 i}{\Omega^2}(c^2 + a^2\sin^2\mu + b^2\cos^2\mu),$$

$$4a_3 = 2\frac{\sin^4 i}{\Omega^4}c^2(b^2 - a^2)\sin\mu\cos\mu\cos\delta,$$

$$a_4 = \frac{\sin^4 i}{\Omega^4}c^2(a^2\sin^2\mu + b^2\cos^2\mu).$$

If the plane of incidence be parallel to the plane of xy, we have $\delta = 0$ and (43) becomes

$$f(r)\,\phi(r) = 0,$$

where
$$f(r) = \left(\frac{\sin^2 i}{\Omega^2}c^2 - 1\right)\tan^2 r + \frac{\sin^2 i}{\Omega^2}c^2,$$

$$\phi(r) = A_0\tan^2 r + 2A_1\tan r + A_2,$$

with
$$A_0 = \frac{\sin^2 i}{\Omega^2}(b^2\sin^2\mu + a^2\cos^2\mu) - 1,$$

$$A_1 = \frac{\sin^2 i}{\Omega^2}(b^2 - a^2)\sin\mu\cos\mu,$$

$$A_2 = \frac{\sin^2 i}{\Omega^2}(b^2\cos^2\mu + a^2\sin^2\mu).$$

If on the other hand the plane of incidence pass through the axis of z $\delta = \pi/2$ and (43) reduces to

$$a_0\tan^4 r + 6a_2\tan^2 r + a_4 = 0,$$

where
$$a_0 = \left(\frac{\sin^2 i}{\Omega^2}a^2 - 1\right)\left(\frac{\sin^2 i}{\Omega^2}b^2 - 1\right),$$

$$6a_2 = \frac{\sin^4 i}{\Omega^4}(b^2c^2\cos^2\mu + c^2a^2\sin^2\mu + a^2b^2)$$
$$- \frac{\sin^2 i}{\Omega^2}(c^2 + a^2\sin^2\mu + b^2\cos^2\mu),$$

$$a_4 = \frac{\sin^4 i}{\Omega^4}c^2(a^2\sin^2\mu + b^2\cos^2\mu).$$

Further if $\mu = \pi/2$ we have the case in which the surface of the crystal is parallel to the plane of symmetry xz and $\tan r$ is determined from

$$a_0\tan^4 r + 6a_2\tan^2 r + a_4 = 0,$$

where $$a_0=\left(\frac{\sin^2 i}{\Omega^2}b^2-1\right)\left\{\frac{\sin^2 i}{\Omega^2}(c^2\cos^2\delta+a^2\sin^2\delta)-1\right\},$$

$$6a_2=\frac{\sin^4 i}{\Omega^4}(c^2a^2+b^2c^2\cos^2\delta+a^2b^2\sin^2\delta)-\frac{\sin^2 i}{\Omega^2}(c^2+a^2),$$

$$a_4=\frac{\sin^4 i}{\Omega^4}c^2a^2,$$

and this breaks into two factors when $\delta=0$ or $\pi/2$. The equation also takes a simpler form when the plane of incidence passes through one of the optic axes, as then

$$\delta=\pm(\pi/2-\Omega)\quad\text{and}\quad c^2\cos^2\delta+a^2\sin^2\delta=c^2\sin^2\Omega+a^2\cos^2\Omega=b^2.$$

128. We have seen in § 121 that the wave-surface may be found at once from the reciprocal ellipsoid by a process similar to that by which the surface of wave-quickness is obtained from the ellipsoid of polarisation and it was thus in fact that Fresnel himself arrived at its equation*. It will however be convenient to proceed by a more direct method and to determine the wave-surface by its property of being the envelope of a system of plane waves, that have passed simultaneously through a given point and have travelled thence in different directions for unit time. This method was also given by Fresnel†, but he did not effect the elimination of the variable parameters: this was first done by Ampère‡ by a somewhat laborious process and afterwards in a far simpler fashion by Archibald Smith§.

The equation of a plane wave is

$$lx+my+nz=\omega \quad\text{.............................}(44),$$

wherein the parameters are connected by the relations

$$l^2+m^2+n^2=1 \quad\text{.............................}(45),$$

$$\frac{l^2}{a^2-\omega^2}+\frac{m^2}{b^2-\omega^2}+\frac{n^2}{c^2-\omega^2}=0 \quad\text{....................}(46).$$

In virtue of these relations only two of the parameters are entirely independent in their variations; but by multiplying equations (45) and (46) by the indeterminate quantities G and H respectively and adding them to (45), we obtain the equation

$$lx+my+nz+G(l^2+m^2+n^2)+H\left(\frac{l^2}{a^2-\omega^2}+\frac{m^2}{b^2-\omega^2}+\frac{n^2}{c^2-\omega^2}\right)=\omega+G,$$

* *Œuvres complètes,* II. No. XLVII. § 37, p. 561.

† *Ibid.* §§ 32—36, pp. 552—561.

‡ *Ann. de Ch. et de Phys.* (2) XXXIX. 113 (1828).

§ *Camb. Phil. Trans.* VI. 85 (1835); *Phil. Mag.* XII. 335 (1836).

in which all the parameters l, m, n, ω may be regarded as independent variables. Hence differentiating with respect to each in turn, we have

$$x + 2Gl + 2H\frac{l}{a^2 - \omega^2} = 0, \quad y + 2Gm + 2H\frac{m}{b^2 - \omega^2} = 0,$$

$$z + 2Gn + 2H\frac{n}{c^2 - \omega^2} = 0, \quad 2H\frac{\omega}{F^2} = 1,$$

where F is given by (26).

Multiplying the first three of these equations by l, m, n respectively and adding we find $2G = -\omega$, whence

$$\left.\begin{aligned} x &= l\omega - \frac{F^2}{\omega}\frac{l}{a^2 - \omega^2} \\ y &= m\omega - \frac{F^2}{\omega}\frac{m}{b^2 - \omega^2} \\ z &= n\omega - \frac{F^2}{\omega}\frac{n}{c^2 - \omega^2} \end{aligned}\right\} \quad\text{......................(47)},$$

which equations give the coordinates of the point of the wave-surface, at which it is touched by the wave (44).

Squaring and adding equations (47), we obtain

$$\begin{aligned} x^2 + y^2 + z^2 &= \omega^2 + \frac{F^4}{\omega^2}\left\{\left(\frac{l}{a^2 - \omega^2}\right)^2 + \left(\frac{m}{b^2 - \omega^2}\right)^2 + \left(\frac{n}{c^2 - \omega^2}\right)^2\right\} \\ &= \omega^2 + \frac{F^2}{\omega^2} \quad\text{..(48)}, \end{aligned}$$

whence, writing $x^2 + y^2 + z^2 = \sigma^2$, equations (47) become

$$\frac{x}{\sigma^2 - a^2} = -\frac{\omega l}{a^2 - \omega^2}, \quad \frac{y}{\sigma^2 - b^2} = -\frac{\omega m}{b^2 - \omega^2}, \quad \frac{z}{\sigma^2 - c^2} = -\frac{\omega n}{c^2 - \omega^2} \text{...(49)},$$

and multiplying these equations by equations (47) respectively and adding we have finally

$$\begin{aligned} \frac{x^2}{\sigma^2 - a^2} + \frac{y^2}{\sigma^2 - b^2} + \frac{z^2}{\sigma^2 - c^2} &= -\omega^2\left(\frac{l^2}{a^2 - \omega^2} + \frac{m^2}{b^2 - \omega^2} + \frac{n^2}{c^2 - \omega^2}\right) \\ &\quad + F^2\left\{\left(\frac{l}{a^2 - \omega^2}\right)^2 + \left(\frac{m}{b^2 - \omega^2}\right)^2 + \left(\frac{n}{c^2 - \omega^2}\right)^2\right\} \\ &= 1 \quad\text{..(50)}, \end{aligned}$$

the equation of the wave-surface.

129. Let λ, μ, ν be the direction-cosines of the ray σ, then

$$x = \lambda\sigma, \quad y = \mu\sigma, \quad z = \nu\sigma,$$

and introducing the direction-cosines of the polarisation-vector from (23), equations (47) become

$$\left.\begin{aligned}\lambda\sigma &= l\omega - F\alpha/\omega\\ \mu\sigma &= m\omega - F\beta/\omega\\ \nu\sigma &= n\omega - F\gamma/\omega\end{aligned}\right\} \quad \text{.........................(51).}$$

Eliminating σ, ω, F between these equations we have

$$\begin{vmatrix}\lambda, & l, & \alpha\\ \mu, & m, & \beta\\ \nu, & n, & \gamma\end{vmatrix} = 0,$$

which expresses that the ray, the wave-normal and the corresponding polarisation-vector are in one plane.

Consider now the normal to the ellipsoid of polarisation at the point in which the polarisation-vector meets it. This line is called "the reciprocal line" and if e, f, g be its direction-cosines

$$\frac{e}{a^2\alpha} = \frac{f}{b^2\beta} = \frac{g}{c^2\gamma} = \frac{1}{\sqrt{a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2}} = \frac{1}{\sqrt{F^2 + \omega^4}} = \frac{1}{\sigma\omega} \quad \text{......(52).}$$

Hence using the suffixes (1), (2) to distinguish between the two waves propagated in a given direction

$$e_1\alpha_2 + f_1\beta_2 + g_1\gamma_2 = 0, \qquad e_2\alpha_1 + f_2\beta_1 + g_2\gamma_1 = 0,$$

or the reciprocal line is in the same plane as the ray and the wave-normal.

Also from (51) and (52) we have

$$\begin{aligned}e\lambda + f\mu + g\nu &= (a^2\alpha l + b^2\beta m + c^2\gamma n)/\sigma^2 - (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)\,F/(\sigma^2\omega^2)\\ &= 0\end{aligned}$$

from (19) and (25); thus the reciprocal line is perpendicular to the ray.

We may therefore extend the proposition respecting the ellipsoid of polarisation as follows*:

The propagational speed of a plane wave in a crystal is given by the reciprocal of one of the semi-axes of the diametral section of the ellipsoid made by a plane parallel to that of the wave: the polarisation-vector of the wave is in the direction of that axis: the corresponding ray is parallel to the line of intersection of the tangent plane at the extremity of the axis and the plane containing the polarisation-vector and the wave-normal.

130. The angle between the ray and the wave-normal is given by

$$\tan(NS) = \sqrt{\sigma^2 - \omega^2}/\omega = F/\omega^2.$$

Hence we have from (37), (38) in the case of the quicker wave

$$\tan(NS_1) = \pm\frac{a^2 - c^2}{2\omega_1{}^2}\sin(\chi - \chi_1)\sin\frac{i}{2} \quad \text{..................(53),}$$

* Beer, *Höhere Optik*, 2nd ed. p. 319. Von Lang, *Wien. Ber.* XLIII. (2) 627 (1861).

and for the slower wave

$$\tan(NS_2) = \pm \frac{a^2 - c^2}{2\omega_2^2} \sin(\chi + \chi') \cos\frac{i}{2} \qquad \text{(54)},$$

where χ, χ' are the angles between the normal and the optic axes, and i is the angle between the planes through the normal and the optic axes.

In order to interpret these results, let a sphere be described round the origin as centre, and let the axes of symmetry of the crystal meet its surface in the points X, Y, Z and let the optic axes and the wave-normal intersect it in the points A, B and N. Then by § 124, the polarisation-vector of the quicker wave lies in the central plane bisecting the exterior angle between AN and BN and by § 129 the corresponding ray is in the same plane: the polarisation-vector and the ray of the slower wave are in the plane of the great circle bisecting the interior angle between these arcs.

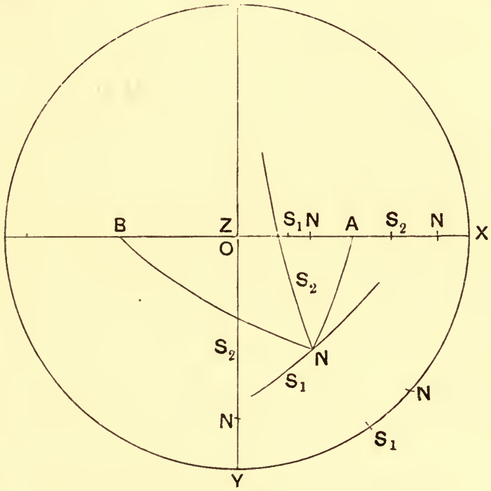


Fig. 28.

Considering now the cases in which the normal is in one of the planes of symmetry, we see that when it is in the plane of YZ, S_1 coincides with N and S_2 is between N and Z; when it is in the plane of XY, S_1 is between N and Y and S_2 coincides with N; finally in the case of the plane of XZ, when the normal is within the angle AOB, S_1 is between N and Z and S_2 and N are coincident, but when the normal is without this angle, S_1 and N coincide, and S_2 is between N and Z.

Collecting these results on the surface of the sphere, we see from continuity that S_2 must be within the angle ANB and S_1 must lie without the angle ANB', B' being the point on the sphere diametrically opposite to B*.

* Neumann, *Vorl. über theor. Optik*, p. 193.

131. The above method of determining the ray that corresponds to a given wave fails, when the wave-normal is in the direction of one of the optic axes, for the angle ANB loses its meaning when N coincides with A or B.

Suppose now that the point N, starting from some position other than A or B, moves along the great circle NA, till it comes to A; then in the limit when it reaches A, we have $\omega_1 = \omega_2 = b$, $\chi = 0$, $\chi' = 2\Omega$ and i is the angle BAN', so that the formulæ (53) and (54) become

$$\tan AS_1 = \frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{b^2} \sin \frac{i}{2},$$

$$\tan AS_2 = \frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{b^2} \cos \frac{i}{2},$$

S_1 being on the great circle bisecting the angle $N'AX$ and outside this angle and S_2 being on the great circle bisecting the angle $N'AZ$ and within the

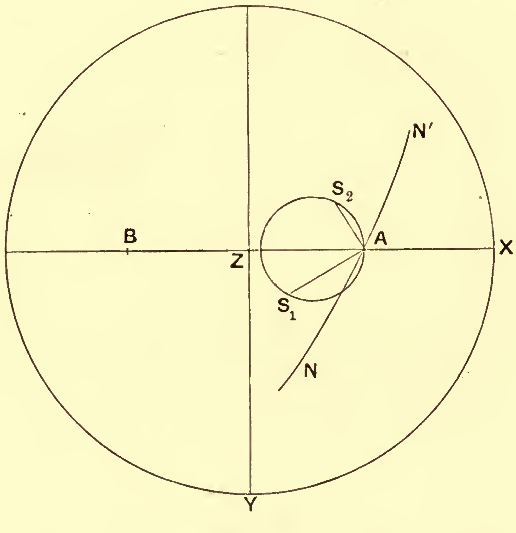


Fig. 29.

angle. Hence calling κ the angle ZAS, we may include these formulæ in the single expression

$$\tan AS = \frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{b^2} \cos \kappa \quad \text{.....................(55).}$$

But this result is independent of the particular path along which we have supposed N to travel and the same reasoning applies to all great circles through A and it hence follows that to the single wave-normal OA there correspond an infinite number of rays forming the generating lines of a cone.

Now in § 128 we have found that the coordinates of the extremity of a

ray are connected with the speed of the corresponding wave and with the direction-cosines of its normal by the relations

$$\frac{x}{\sigma^2 - a^2} = \frac{l\omega}{\omega^2 - a^2}, \quad \frac{y}{\sigma^2 - b^2} = \frac{m\omega}{\omega^2 - b^2}, \quad \frac{z}{\sigma^2 - c^2} = \frac{n\omega}{\omega^2 - c^2}.$$

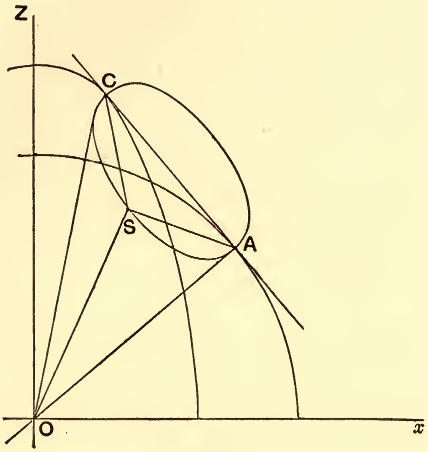


Fig. 30.

Writing the second of these relations in the form

$$\frac{y}{\sigma^2 - b^2} = \frac{\omega}{m}\left(\frac{l^2}{a^2 - \omega^2} + \frac{n^2}{c^2 - \omega^2}\right),$$

we see at once that the expression becomes indeterminate when the wave-normal coincides with the optic axis, since m and $l^2/(a^2 - \omega^2) + n^2/(c^2 - \omega^2)$ then vanish independently of one another. Consequently in this case the coordinates of the extremity of the ray have only to satisfy the two conditions

$$\frac{x}{\sigma^2 - a^2} = \frac{l_0 b}{b^2 - a^2} = -\frac{b}{\sqrt{(a^2 - b^2)(a^2 - c^2)}},$$

$$\frac{z}{\sigma^2 - c^2} = \frac{n_0 b}{b^2 - c^2} = \frac{b}{\sqrt{(b^2 - c^2)(a^2 - c^2)}},$$

and thus the extremities of the rays corresponding to the plane wave perpendicular to the optic axis lie on the intersection of the spheres

$$x^2 + y^2 + z^2 + \frac{\sqrt{(a^2 - b^2)(a^2 - c^2)}}{b} x - a^2 = 0 \quad \text{..............(56)},$$

and

$$x^2 + y^2 + z^2 - \frac{\sqrt{(b^2 - c^2)(a^2 - c^2)}}{b} z - c^2 = 0 \quad \text{..............(57)},$$

and the wave touches the wave-surface along a circle in the plane

$$\frac{1}{b}\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\, x + \frac{1}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\, z = 1 \quad \text{..................(58)}.$$

The equation of the cone of rays is found by combining (58) with (56) or (57) so as to form an homogeneous equation of the second degree, and we thus obtain

$$x^2 + y^2 + z^2 + \frac{\sqrt{a^2 - b^2}}{b^2}\left(\sqrt{a^2 - b^2}\, x + \sqrt{b^2 - c^2}\, z\right) x$$
$$- \frac{a^2}{b^2}\left(\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\, x + \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\, z\right)^2 = 0,$$

or $$a^2 (b^2 - c^2)\, x^2 + b^2 (a^2 - c^2)\, y^2 + c^2 (a^2 - b^2)\, z^2$$
$$= (a^2 + c^2) \sqrt{(a^2 - b^2)(b^2 - c^2)}\, xz \quad \text{............(59).}$$

The cone is symmetrical with respect to the plane of xz and its angular opening δ is found by writing $\kappa = 0$ in (55), which gives

$$\tan \delta = \sqrt{(a^2 - b^2)(b^2 - c^2)}/b^2 \quad \text{.....................(60).}$$

132. The fact that a plane perpendicular to the optic axis at its extremity touches the wave-surface in an infinite number of points constituting a circle, so that corresponding to a single wave-direction there are an infinite number of rays lying on a cone, occasions what is known as internal conical refraction.

In order to investigate the characteristics of this phenomenon, let us consider the simple case, in which a cylindrical pencil of rays, of small radius r, is incident normally on a plate of a biaxal crystal cut perpendicularly to one of the optic axes of the crystal.

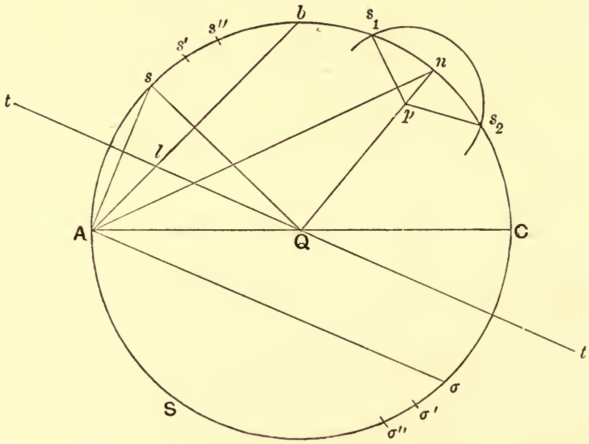


Fig. 31.

Round the point O in which the axis of the pencil meets the plate, describe the wave-surface within the plate and draw a tangent plane to it parallel to the faces of the plate: this will touch the wave-surface along a

circle and the lines joining O with the points of this circle will be the axes of refracted cylinders of light, that are determined by the illuminated portion of the first surface of the plate. The incident pencil is thus divided into an infinite number of diverging streams, the axes of which meet the second surface of the plate in a circle ASC, passing through the point in which the normal OA meets the surface and having its centre Q in the plane of the optic axes. On emergence these streams resume their primitive direction, and apart from loss of light due to refraction, the appearance on a screen parallel to the plate will be the same as on the second surface of the plate itself.

If AC be the diameter of the circle ASC, the angle AOC is given by

$$\tan AOC = \sqrt{(a^2 - b^2)(b^2 - c^2)}/b^2,$$

and if D be the thickness of the plate, the radius of the circle is

$$R = (D/2) \tan AOC = D\sqrt{(a^2 - b^2)(b^2 - c^2)}/(2b^2).$$

If $R > r$ we have on the second face of the plate a ring of light bounded by concentric circles of radii $R + r$ and $R - r$; if $R = r$ the central dark patch just vanishes; if $R < r$, there is a luminous circle, the inner portion of which of radius $r - R$ is due to the overlapping of the refracted streams.

Suppose that the incident light is plane polarised, and let us determine the intensity and the polarisation at a point on the second face of the plate.

Let tt be the direction of the polarisation-vector of the incident stream and let a be the amplitude of its vibrations. Draw As perpendicular to tt, meeting the circle ASC in the point s and dx being an element of an arc of unit radius, divide the circumference of the circle into elementary arcs ss', $s's''$, ..., corresponding to dx.

Now the incident stream may be regarded as the superposition of π/dx identical elementary streams, and these may be replaced by $2\pi/dx$ streams with their polarisation-vectors in the directions As, As' ... and the perpendicular directions $A\sigma$, $A\sigma'$ .... On entry into the plate each of these components will assume a direction corresponding to that of its polarisation-vector: thus the stream with its polarisation-vector parallel to Ab will meet the second face in a circle with its centre at b, and the amplitude of the vibrations in this stream is

$$(adx/\pi) \cos blt = (adx/\pi) \sin (\delta/2) \quad \text{.................(61)},$$

where δ is the angle sQb.

Consider a point p on the illuminated portion of the face: the light at p is due to a ray of each of the streams, the axes of which intersect the arc $s_1 s_2$, where s_1 and s_2 are the points in which ASC is cut by a circle described round p as centre with radius r.

Let Qp cut ASC in the point n and let $pQs_1 = pQs_2 = \phi$, $sQn = \delta$. Divide the angle ϕ into elements of magnitude dx, then the azimuths of the corresponding points of the arc $s_2 s_1$ measured from sQ are

$$\delta + \phi,\quad \delta + \phi - dx,\ \ldots,\quad \delta - \phi + dx,\quad \delta - \phi,$$

and hence from (61) the amplitudes of the vibrations in the streams that contribute to the illumination of p are

$$\frac{adx}{\pi} \sin \frac{\delta + \phi}{2},\quad \frac{adx}{\pi} \sin \frac{\delta + \phi - dx}{2},\ \ldots,\quad \frac{adx}{\pi} \sin \frac{\delta - \phi + dx}{2},\quad \frac{adx}{\pi} \sin \frac{\delta - \phi}{2},$$

and the azimuths of the corresponding polarisation-vectors measured from As are

$$\frac{\delta + \phi}{2},\quad \frac{\delta + \phi - dx}{2},\ \ldots,\quad \frac{\delta - \phi + dx}{2},\quad \frac{\delta - \phi}{2}.$$

Hence regarding the illumination at p as the effect of two streams with their polarisation-vectors parallel to As and $A\sigma$ respectively, we have for the amplitude of the vibrations of the first

$$Y = \frac{a}{\pi} \int_{-\phi}^{\phi} \sin \frac{\delta + x}{2} \cos \frac{\delta + x}{2}\, dx = \frac{a}{\pi} \sin \phi \sin \delta \quad \text{...........(62)},$$

and for the amplitude of the vibrations of the second

$$X = \frac{a}{\pi} \int_{-\phi}^{\phi} \sin \frac{\delta + x}{2} \sin \frac{\delta + x}{2}\, dx = \frac{a}{\pi} (\phi - \sin \phi \cos \delta) \quad \text{...(63)}.$$

Whence the intensity at the point p is

$$I = X^2 + Y^2 = \frac{a^2}{\pi^2} (\phi^2 - 2\phi \sin \phi \cos \delta + \sin^2 \phi) \quad \text{...........(64)},$$

and the polarisation-vector at the point is inclined at an angle ψ to that of the initial stream of light, where

$$\tan \psi = \frac{\sin \phi \sin \delta}{\phi - \sin \phi \cos \delta} \quad \text{........................(65)}.$$

Calling ρ the distance Qp, the angle ϕ is given by

$$\cos \phi = (R^2 - r^2 + \rho^2)/(2R\rho) \quad \text{....................(66)}.$$

In interpreting these results, it is necessary to consider separately the three cases mentioned above.

(1) If $R > r$, so that there is a ring of light with a dark centre, the value of ϕ, which remains constant over each circle concentric with the ring, is equal to zero at the edges of the ring, to which correspond the radii $R \pm r$, and attains its maximum value on the circle of radius $\sqrt{R^2 - r^2}$.

Along one and the same radius δ is constant, and the intensity changes in the same manner as ϕ: the polarisation-vector is at first inclined at an

angle $\pi/2 - \delta/2$ to that of the incident light; this angle increases to a maximum as we pass outwards and then decreases to its former value at the outer edge of the ring.

On one and the same circle ϕ is constant and the intensity increases from the minimum value $a^2(\phi - \sin\phi)^2/\pi^2$ at points on the radius Qs to the maximum value $a^2(\phi + \sin\phi)^2/\pi^2$ at the diametrically opposite points: the polarisation-vector has the same direction as that of the incident stream at points on the diameter $sQ\sigma$, and as we move round a given circle from one of these points, its deviation from this direction increases to a maximum and then decreases again, the deviations being equal and opposite at two points on the circle equidistant from the point in which Qs cuts it.

(2) When $R = r$, the central dark patch disappears and the centre is the intersection of the edges of all the elementary circles of light. At this point $\phi = \pi$, and on passing along a radius ϕ suddenly changes to $\pi/2$ and then decreases gradually to the value zero at the limit of the spot. Hence at the centre the intensity and the polarisation are the same as those of the incident light: they then change suddenly and thence alter gradually until at the outer edge the intensity becomes zero and the plane of polarisation is inclined at an angle $\pi/2 - \delta/2$ to its primitive position.

(3) If $R < r$, the elementary circles overlap on a circle of radius $r - R$. For all points within this circle $\phi = \pi$, and on passing outwards ϕ decreases gradually to the value zero at the limit of the spot. Thus the changes in the intensity and the polarisation are the same as in the former case, except that on crossing the edge of the central circle there is no sudden variation.

Taking now the case in which the primitive stream is unpolarised, we may regard the incident light as resulting from the superposition of two independent streams of equal intensity with their polarisation-vectors parallel and perpendicular respectively to AC. If I be the intensity of the unpolarised stream, then $I/2$ is the intensity of each of the polarised streams, and that with its polarisation-vector parallel to AC will give at the point p a stream of intensity $I\sin^2\phi\sin^2\delta/(2\pi^2)$, where δ is the angle AQp, with its polarisation-vector perpendicular to AC, together with a stream of intensity $I(\phi - \sin\phi\cos\delta)^2/(2\pi^2)$ with its polarisation-vector parallel to AC. On the other hand the second component of the incident light gives at the same point streams of intensities $I(\phi + \sin\phi\cos\delta)^2/(2\pi^2)$ and $I\sin^2\phi\sin^2\delta/(2\pi^2)$ with their polarisation-vectors perpendicular and parallel to AC respectively.

Hence the combined effect at p is a stream of intensity

$$\frac{I}{2\pi^2}(\phi^2 + \sin^2\phi + 2\phi\sin\phi\cos\delta) = \frac{I}{2\pi^2}\left\{(\phi - \sin\phi)^2 + 4\phi\sin\phi\cos^2\frac{\delta}{2}\right\},$$

with its polarisation-vector perpendicular to AC, and a stream of intensity

$$\frac{I}{2\pi^2}(\phi^2 + \sin^2\phi - 2\phi\sin\phi\cos\delta) = \frac{I}{2\pi^2}\left\{(\phi - \sin\phi)^2 + 4\phi\sin\phi\sin^2\frac{\delta}{2}\right\},$$

with its polarisation-vector parallel to AC. These are equivalent to a stream of common light of intensity $I(\phi - \sin\phi)^2/\pi^2$ together with a stream of polarised light of intensity $2I\phi\sin\phi/\pi^2$ with its polarisation-vector inclined at an angle of $\pi/2 - \delta/2$ to the direction AC. The total intensity is

$$I(\phi^2 + \sin^2\phi)/\pi^2,$$

and the measure of the polarisation is

$$\frac{2\phi\sin\phi}{\phi^2 + \sin^2\phi} = 2\left(\frac{\phi}{\sin\phi} + \frac{\sin\phi}{\phi}\right)^{-1}.$$

Applying these results to the three cases already considered, we have

(1) When $R > r$, the same intensity at all points on a circle concentric with the ring: on the edges the intensity is zero and it attains its maximum on a circle lying within the circle ASC. The light is partially plane polarised; the polarisation is the same for all points on a given radius, becoming more complete as the edges of the ring are approached, and the polarisation-vector is parallel to the line joining A to the point in which the radius cuts the circle ASC.

(2) When $R = r$, the inner limit of the ring contracts to a point, at which the light is unpolarised and the intensity that of the incident light. Passing outwards from the centre along a radius, there is a sudden change in intensity and the light becomes partially polarised and thence the intensity decreases and the measure of the polarisation increases as the edge of the spot is approached.

(3) When $R < r$, there is a circle of common light of radius $r - R$, over which the intensity is that of the incident light, and on moving thence to the edge of the spot the intensity gradually decreases and the measure of the polarisation increases*.

133. Returning to equations (49), we have

$$\frac{\lambda\sigma}{\sigma^2 - a^2} = -\frac{l\omega}{a^2 - \omega^2}, \quad \frac{\mu\sigma}{\sigma^2 - b^2} = -\frac{m\omega}{b^2 - \omega^2}, \quad \frac{\nu\sigma}{\sigma^2 - c^2} = -\frac{n\omega}{c^2 - \omega^2},$$

and from (23) and (52)

$$\frac{l}{a^2 - \omega^2} = \frac{\alpha}{F} = \frac{\sigma\omega}{a^2 F}e, \quad \frac{m}{b^2 - \omega^2} = \frac{\sigma\omega}{b^2 F}f, \quad \frac{n}{c^2 - \omega^2} = \frac{\sigma\omega}{c^2 F}g \quad \text{......(67)},$$

e, f, g being the direction-cosines of the reciprocal line, and we thus obtain

$$\left(\frac{1}{a^2} - \frac{1}{\sigma^2}\right)e = \frac{\lambda}{H}, \quad \left(\frac{1}{b^2} - \frac{1}{\sigma^2}\right)f = \frac{\mu}{H}, \quad \left(\frac{1}{c^2} - \frac{1}{\sigma^2}\right)g = \frac{\nu}{H} \quad \text{........(68)},$$

H being written for $-\omega^2\sigma^2/F$.

* Beer, *Pogg. Ann.* LXXXV. 67 (1852); *Höhere Optik,* 2nd ed. p. 346.

These equations connecting the ray-slowness and the directions of the ray and of the reciprocal line are of precisely the same form as equations (23) connecting the wave-velocity with the directions of the wave-normal and of the polarisation-vector, and we pass from the one set of equations to the other by writing

$$a^{-1},\ b^{-1},\ c^{-1};\ \lambda,\ \mu,\ \nu;\ H^{-1};\ \sigma^{-1};\ e,\ f,\ g,$$

for

$$a,\ b,\ c;\ l,\ m,\ n;\ F;\ \omega;\ \alpha,\ \beta,\ \gamma,$$

and (e, f, g) stand in exactly the same relation to (λ, μ, ν) as do (α, β, γ) to (l, m, n) § 129; whence it follows that all the propositions deduced from (23) may be extended by this change of letters. In this extension

the direction of the wave-normal becomes the direction of the ray,
the wave-velocity becomes the ray-slowness,
the polarisation-vector becomes the reciprocal line,
the polarisation ellipsoid becomes the reciprocal ellipsoid,

of which the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with the following properties:

The propagational speed along a ray in a given direction is equal to one of the semi-axes of the diametral section of the ellipsoid perpendicular to the ray; the plane through this axis and the ray is the corresponding plane of the polarisation-vector; the line of intersection of this plane and the tangent plane to the ellipsoid at the end of the said axis is parallel to the corresponding wave-normal*.

134. Corresponding to the optic axes or directions of single wave-velocity, we have two directions of single ray-velocity or ray-axes, the direction-cosines of which are

$$\lambda_0=\pm\frac{c}{b}\sqrt{\frac{a^2-b^2}{a^2-c^2}};\ 0;\ \nu_0=\pm\frac{a}{b}\sqrt{\frac{b^2-c^2}{a^2-c^2}}\ \dots\dots\dots\dots(69).$$

The ray-axes are thus in the plane of optical symmetry xz and in the directions of the radii-vectores to the points of intersection of the ellipse and circle, that are the section of the wave-surface made by this plane.

Also we obtain at once that the planes of polarisation of the waves corresponding to the two rays in a given direction bisect the angles between the planes through this direction and the ray-axes, and that, if ψ, ψ' be the angles that the direction makes with the ray-axes, the two ray-velocities are given by

$$\left.\begin{aligned}\sigma_1^{-2}&=\tfrac{1}{2}(a^{-2}+c^{-2})+\tfrac{1}{2}(a^{-2}-c^{-2})\cos(\psi-\psi')\\ \sigma_2^{-2}&=\tfrac{1}{2}(a^{-2}+c^{-2})+\tfrac{1}{2}(a^{-2}-c^{-2})\cos(\psi+\psi')\end{aligned}\right\}\ \dots\dots\dots(70),$$

* Beer, *Höhere Optik*, 2nd ed. p. 319. Von Lang, *loc. cit.*

whence $$\sigma_1^{-2} - \sigma_2^{-2} = (a^{-2} - c^{-2}) \sin \psi \sin \psi' \quad \text{.................. (71).}$$

Further

$$\left.\begin{aligned} H_1^{-1} &= \pm \tfrac{1}{2} (a^{-2} - c^{-2}) \sin (\psi - \psi') \sin \frac{j}{2} \\ H_2^{-1} &= \pm \tfrac{1}{2} (a^{-2} - c^{-2}) \sin (\psi + \psi') \cos \frac{j}{2} \end{aligned}\right\} \quad \text{............... (72),}$$

where j is the angle between the planes through the ray and the ray-axes in which the axis of z lies.

135. The angle between the ray and the wave-normal is given by

$$\tan (SN) = F/\omega^2 = -\sigma^2/H,$$

whence $$\left.\begin{aligned} \tan (SN_1) &= \pm \frac{a^{-2} - c^{-2}}{2\sigma_1^{-2}} \sin (\psi - \psi') \sin \frac{j}{2} \\ \tan (SN_2) &= \pm \frac{a^{-2} - c^{-2}}{2\sigma_2^{-2}} \sin (\psi + \psi') \cos \frac{j}{2} \end{aligned}\right\} \quad \text{............ (73),}$$

where, if $\alpha O \beta$ be the angle between the ray-axes $O\alpha$, $O\beta$ in which the axis of z lies, N_2 is the plane bisecting the angle $\alpha S \beta$ and without this angle, while N_1 is the plane bisecting the angle $\alpha S \beta'$ ($O\beta'$ being the prolongation of βO) and within the angle.

Thus the wave-normals corresponding to a given direction of a ray are completely determined.

136. This method, as in the case of the converse proposition of § 130, becomes indeterminate, when the ray coincides in direction with either of the ray-axes, so that the angle $\alpha S \beta$, on which it depends, is without meaning.

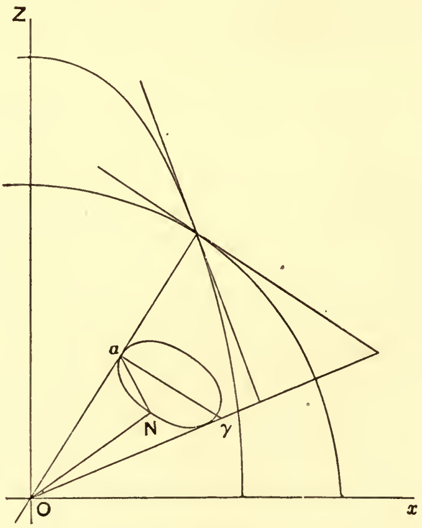


Fig. 32.

Proceeding as in the analogous case of § 131, it is seen at once that in this case to the single ray in the direction of the ray-axis there correspond an infinite number of wave-normals forming a cone, the equation of which may be written

$$\tan(\alpha N) = \frac{c^{-2} - a^{-2}}{2b^{-2}} \sin 2\Sigma \,.\, \cos\kappa,$$

where 2Σ is the angle $\alpha O\beta$ and κ is the angle that the plane αON makes with the plane of xz.

The section of this cone by a plane perpendicular to the ray-axis is a circle, for if $\alpha N\gamma$ be its section by the said plane, γ lying in the plane of xz, we have

$$\alpha\gamma/b = \tan(\alpha\gamma) = \frac{c^{-2} - a^{-2}}{2b^{-2}} \sin 2\Sigma,$$

and

$$\alpha N/b = \tan(\alpha N) = \frac{c^{-2} - a^{-2}}{2b^{-2}} \sin 2\Sigma \,.\, \cos\kappa\,;$$

$$\therefore\quad \alpha N/\alpha\gamma = \cos\kappa,$$

whence the angle $\alpha N\gamma$ is a right-angle, and the locus of N is a circle.

Now the coordinates ξ, η, ζ of the foot of the perpendicular from the centre on the tangent plane to the wave-surface at the point in which it is met by the ray σ with the direction-cosines λ, μ, ν satisfy the conditions

$$\frac{\xi}{\omega^2 - a^2} = \frac{\lambda\sigma}{\sigma^2 - a^2}, \quad \frac{\eta}{\omega^2 - b^2} = \frac{\mu\sigma}{\sigma^2 - b^2}, \quad \frac{\zeta}{\omega^2 - c^2} = \frac{\nu\sigma}{\sigma^2 - c^2},$$

and writing the second of these relations in the form

$$\frac{\eta}{\omega^2 - b^2} = -\frac{\sigma}{\mu b^2}\left(\frac{a^2\lambda^2}{\sigma^2 - a^2} + \frac{c^2\nu^2}{\sigma^2 - c^2}\right),$$

we see that it becomes indeterminate when the given ray coincides with the ray-axis, as μ and $a^2\lambda^2/(\sigma^2 - a^2) + c^2\nu^2/(\sigma^2 - c^2)$ then vanish independently. Hence in this case ξ, η, ζ have only to satisfy the two equations

$$\frac{\xi}{\omega^2 - a^2} = \frac{\lambda_0 b}{b^2 - a^2} \text{ and } \frac{\zeta}{\omega^2 - c^2} = \frac{\nu_0 b}{b^2 - c^2},$$

and thus the feet of the perpendiculars from the centre on the tangent planes to the wave-surface at the extremity of the ray-axis lie on the spheres

$$\left.\begin{aligned} \xi^2 + \eta^2 + \zeta^2 + \frac{\sqrt{(a^2 - b^2)(a^2 - c^2)}}{c}\,\xi - a^2 = 0 \\ \xi^2 + \eta^2 + \zeta^2 - \frac{\sqrt{(b^2 - c^2)(a^2 - c^2)}}{a}\,\zeta - c^2 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(75),$$

and thus their locus is a circle in the plane

$$\frac{1}{c}\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\,\xi + \frac{1}{a}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\,\zeta = 1 \ldots\ldots\ldots\ldots\ldots\ldots(76),$$

and this is the plane perpendicular to the plane of the ray-axes through the tangent at the end of the ray-axis to the elliptic section of the wave-surface made by the plane of xz.

Combining equation (76) with one of the equations (75) so as to form an homogeneous equation of the second degree, we obtain as the equation of the cone of wave-normals

$$F(\xi, \eta, \zeta) \equiv (b^2 - c^2)\,\xi^2 + (a^2 - c^2)\,\eta^2 + (a^2 - b^2)\,\zeta^2$$
$$- \frac{a^2 + c^2}{ac} \sqrt{(a^2 - b^2)(b^2 - c^2)}\,\xi\zeta = 0 \quad \text{......(77)}.$$

We may now obtain the equation of the tangent cone at the singular point of the wave-surface, for its generating lines pass through the extremity of the ray-axis, the coordinates of which are

$$x_0 = c\sqrt{(a^2 - b^2)/(a^2 - c^2)}, \quad y_0 = 0, \quad z_0 = a\sqrt{(b^2 - c^2)/(a^2 - c^2)},$$

and are perpendicular to the tangent planes of the cone of wave-normals.

Thus the equations of any one of them are

$$\frac{x - x_0}{\partial F/\partial \xi} = \frac{y}{\partial F/\partial \eta} = \frac{z - z_0}{\partial F/\partial \zeta} = \frac{1}{\rho} \text{ (say)}$$

whence

$$\left.\begin{aligned} 2(b^2 - c^2)\,\xi - \frac{a^2 + c^2}{ac}\sqrt{(a^2 - b^2)(b^2 - c^2)}\,\zeta &= \rho(x - x_0) \\ 2(a^2 - c^2)\,\eta &= \rho y \\ -\frac{a^2 + c^2}{ac}\sqrt{(a^2 - b^2)(b^2 - c^2)}\,\xi + 2(a^2 - b^2)\,\zeta &= \rho(z - z_0) \end{aligned}\right\},$$

which give

$$\frac{\xi}{ac\sqrt{a^2 - b^2}\,\{2ac\sqrt{a^2 - b^2}\,(x - x_0) + (a^2 + c^2)\sqrt{b^2 - c^2}\,(z - z_0)\}}$$
$$= \frac{\eta}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)\,y/2}$$
$$= \frac{\zeta}{ac\sqrt{b^2 - c^2}\,\{(a^2 + c^2)\sqrt{a^2 - b^2}\,(x - x_0) + 2ac\sqrt{b^2 - c^2}\,(z - z_0)\}}.$$

Substituting these values of ξ, η, ζ in (77) we obtain on reduction

$$a^2c^2(a^2 - b^2)(x - x_0)^2 + \tfrac{1}{4}(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)\,y^2 + a^2c^2(b^2 - c^2)(z - z_0)^2$$
$$+ ac(a^2 + c^2)\sqrt{(a^2 - b^2)(b^2 - c^2)}\,(x - x_0)(z - z_0) = 0 \quad \text{......(78)},$$

which is the equation of the tangent cone.

137. The existence of a conoidal cusp on the wave-surface at each of the four points, in which it is cut by the ray-axes, occasions the phenomena known as external conical refraction; for since at these points there are an

infinite number of tangent planes, forming a tangent cone of the second degree, it follows at once from Huygens' construction that a single ray within a biaxal crystal in the direction of one of the ray-axes will, on emergence into an isotropic medium, be divided into an infinite number of rays lying on the surface of a cone.

Let us take the case in which the surface of the crystal is perpendicular to the ray-axis and determine the equation of the cone of external rays. Let $O\alpha$ represent the normal to the plate, ON any wave-normal within the crystal corresponding to the ray-axis, ON' the corresponding emergent wave-normal, and suppose α, N, N' to lie in a plane perpendicular to $O\alpha$. Then the locus of N is, as we have seen, a circle with its diameter αP in the plane of the ray-axes.

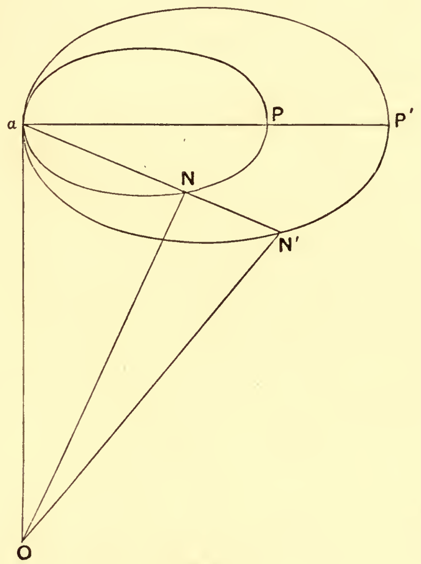


Fig. 33.

Now by Huygens' principle ON, ON' and $O\alpha$ are in one plane and

$$\sin \alpha ON : \sin \alpha ON' :: \omega : \Omega \quad \text{.....................(79)},$$

Ω being the propagational speed of light in the isotropic medium and ω the wave-velocity in the crystal in the direction ON. Also if κ be the angle that the plane αON makes with the plane of the ray-axes

$$\frac{\sqrt{b^2-\omega^2}}{\omega} = \tan \alpha ON = \frac{a^{-2}-c^{-2}}{2b^{-2}} \sin 2\Sigma \,.\, \cos \kappa,$$

whence $$\omega = \frac{abc}{\sqrt{a^2c^2 + (a^2-b^2)(b^2-c^2)\cos^2 \kappa}}.$$

Take the surface of the plate as the plane of xy, the plane of the ray-axes as

that of xz, the origin being at O: then the coordinates of N being x, y, z and ρ being written for $\sqrt{x^2+y^2}$, we have

$$\rho = \alpha P \cos\kappa = z \tan \alpha OP \,.\, \cos\kappa = z\sqrt{(a^2-b^2)(b^2-c^2)\cos^2\kappa}/ac \ldots(80).$$

But $\rho/z = \tan \alpha ON$, hence x', y', z' being the coordinates of N'

$$\frac{\rho^2}{\rho^2+z^2} = \sin^2 \alpha ON = \frac{\omega^2}{\Omega^2}\sin^2 \alpha ON' = \frac{a^2b^2c^2\Omega^{-2}}{a^2c^2+(a^2-b^2)(b^2-c^2)\cos^2\kappa}\cdot\frac{x'^2+y'^2}{x'^2+y'^2+z'^2},$$

and from (80)

$$\Omega^2(a^2-b^2)(b^2-c^2)\cos^2\kappa = a^2b^2c^2(x'^2+y'^2)/(x'^2+y'^2+z'^2),$$

and since $\cos\kappa = x'/\sqrt{x'^2+y'^2}$, we obtain

$$a^2b^2c^2(x'^2+y'^2)^2 = \Omega^2(a^2-b^2)(b^2-c^2)x'^2(x'^2+y'^2+z'^2) \ldots\ldots(81).$$

On account of the weak double refraction of all biaxal crystals, this equation takes approximately the form*

$$abc(x'^2+y'^2) = \Omega\sqrt{(a^2-b^2)(b^2-c^2)}\,x'z' \ldots\ldots\ldots\ldots\ldots(82),$$

whence it follows that the locus of N' is approximately a circle passing through α with the diameter $\alpha P'$ in the plane of the ray-axes of the crystal.

If now the stream of light within the crystal be limited by a circular cylinder having its axis in the direction of a ray-axis, the axes of the emergent streams will form the generating lines of the cone just obtained, and the section of any one of these streams by a plane parallel to the face of the crystal will be a circle equal to the section of the incident stream. The emergent light will therefore give a bright ring on a screen parallel to the face of the crystal and at a sufficient distance from it; but as the screen is moved towards the crystal, the ring will contract retaining the same width, until the central dark spot vanishes, and on a further approach of the screen the bright spot contracts, until at the surface it becomes equal to the section of the stream within the crystal.

The intensity and the polarisation at any point of the bright ring or spot may be calculated in the same manner as in the case of internal conical refraction, since the polarisation-vector for any part of one of the emergent streams is in the plane containing the axis of the stream and the normal $O\alpha$ to the face of the crystal. That this is so, is clear at once from the fact that for any one of the waves corresponding to the ray-axis, the polarisation-vector is in the plane through the ray-axis perpendicular to the wave-front, that is the plane of incidence of the wave†.

* This equation is obtained at once as follows: equation (79) gives approximately

$$\alpha N : \alpha N' :: \omega : \Omega :: b : \Omega \quad \text{or} \quad \rho/\rho' = b/\Omega,$$

whence substituting for ρ from (80), equation (82) is at once obtained.

† Beer, *Höhere Optik*, 2nd ed. p. 362.

138. It is to Sir William Hamilton that we owe the discovery of the conical refractions in biaxal crystals, as it was in the course of his researches on Fresnel's laws of double refraction by means of the surface of wave-slowness, that he found singularities of the wave-surface, that led him to anticipate the existence of these phenomena*. At his instigation Lloyd† undertook an experimental investigation of these cases of refraction, and entirely established the accuracy of the conclusions.

This confirmation of Hamilton's deductions was naturally regarded as a decisive proof of the general correctness of Fresnel's form of the wave-surface, but Stokes‡ has pointed out that the phenomena of conical refraction are not of themselves competent to decide between different theories that lead to Fresnel's surface as a near approximation.

Internal conical refraction depends upon the existence of a tangent plane touching the wave-surface along a plane curve. Let us then consider the result of supposing that the nearest approach to a plane curve of contact is a twisted curve. Let a plane be drawn touching the part where the surface bends over, at two points on opposite sides of the rim, and let this plane be moved parallel to itself towards the centre, after having been slightly tilted about one of the points of contact. The section of the wave-surface made by this plane will be of the general form represented in the following figures, from which we see that in four positions, as shown in *a*, *b*, *d*, *e*, the

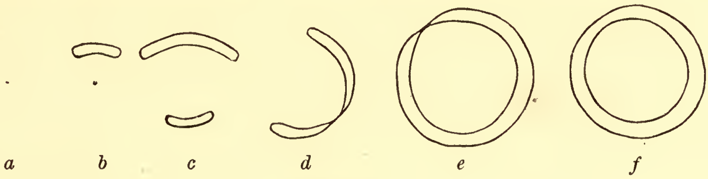


Fig. 34.

plane will touch the surface in one point, so that in the direction considered there will be four possible wave-velocities. On the other hand whatever theory of double refraction may be adopted, we are led to assign to a wave in a given direction three possible directions of the polarisation-vector, to each of which corresponds a different wave-velocity: but if these three parallel waves can be propagated, a fourth in the same direction is impossible, for replacing its polarisation-vector by its components in the three given directions, we should have three pairs of parallel waves and in each pair the waves would travel with different speeds though their polarisation-vectors were coincident. Thus the number of possible waves in a given direction is limited to three, or excluding waves with longitudinal vectors, to at most two. It follows then that a tangent plane with a plane curve of contact is a necessary property of the wave-surface and not a distinctive feature of Fresnel's form.

* *Trans. Roy. Irish Acad.* XVII. 134 (1832).

† *Ibid.* XVII. 145 (1833); *Papers on Physical Science*, p. 1.

‡ *B. A. Report*, 1862, p. 270.

This is not the case with the conoidal cusp on Fresnel's surface, on which the phenomenon of external conical refraction depends: but since the wave-surface must have approximately the same form as Fresnel's surface, and since it also has plane curves of contact with a tangent plane, the outer sheet will pass into the inner by what is very nearly a conoidal cusp, and hence taking into account the impossibility of reducing indefinitely the pencil of rays with which observations are made, we see that we should obtain a phenomenon that would be practically indistinguishable from true conical refraction.

CHAPTER XII.

DETERMINATION OF THE PRINCIPAL WAVE-VELOCITIES.

139. FROM an optical point of view, one of the most important practical questions connected with crystals is the measurement of the principal wave-velocities, as on these quantities the doubly refracting properties of the medium depend, and we will now pass in review the methods by which these determinations may be made, taking the three cases in which we have at our disposal (1) a plate, (2) a prism and (3) a single reflecting surface of the crystal.

Foci of lines seen through a crystalline plate.

140. In 1767 De Chaulnes* proposed the well-known method of determining the refractive index of a plate from measurements of its real and apparent thickness by means of a microscope. When the plate is isotropic, it may be easily shown that the refractive index is the ratio of these quantities: with crystalline plates, however, the case is not so simple, but when the object viewed through the plate consists of systems of lines at right-angles to one another, certain characteristic phenomena are observed, that serve to differentiate between singly refracting media, uniaxal and biaxal crystals, and that in general afford a method of deducing the principal indices of the plate†.

Consider a small pencil of rays emanating from a point O on the lower surface of the plate, in such a direction that its axis on emergence is perpendicular to the plate. Round O as centre describe a half wave-surface and considering only a single sheet of this surface, let its dimensions be such that the upper face of the plate touches this sheet at the point E: then by Huygens' construction OE will be the axis of the pencil considered.

Let an adjacent ray OPQ cut the wave-surface in P and the face of the plate in Q, then the form of the wave on emergence will by Huygens'

* *Mém. de l'Acad. Roy. des Sci. Paris*, 1767, p. 431.

† Stokes, *Proc. R. S.* xxvi. 386 (1877).

principle depend upon the time that the light requires to traverse PQ regarded as a function of the coordinates that determine the position of the point Q on the surface of the plate.

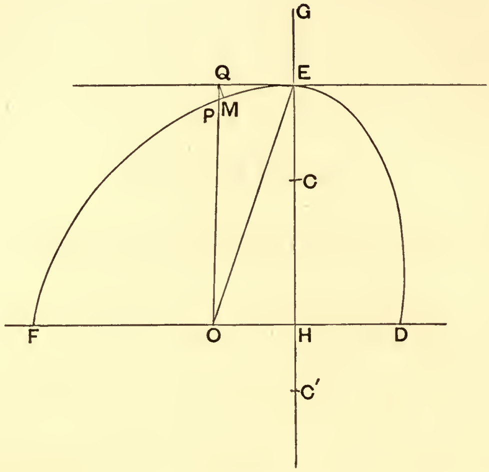


Fig. 35.

If QE be a small quantity of the first order, the retardation will be a small quantity of the second order, and it is only to this order that we require the retardation in determining the foci of the emergent pencil. We may then substitute for the retardation any quantity that bears to it a ratio that is ultimately one of equality. Now if QM be the normal to the sheet of the wave-surface drawn from the point Q, the wave-velocity in the direction QM will differ from that in a direction perpendicular to the plate by a small quantity of the first order, and the distance QM is a small quantity of the second order: hence we may neglect the variation of the wave-velocity and may regard the medium as if it were a singly refracting one, in which a wave is travelling that has already by some means acquired the form DEF.

Through the normal HE draw the two rectangular planes of principal curvature of the surface and let C, C' be the centres of curvature, ρ, ρ' the radii of curvature on the same scale as that in which EH represents the wave-velocity ω in the direction EH. Then by what precedes, we may regard the rays in that principal plane of curvature, the normals in which intersect in C, as diverging from C in an isotropic medium with refractive index Ω/ω and these will on emergence diverge from a focus distant $CE(\omega/\Omega)$ below the surface of the plate. But if τ be the thickness of the plate, $CE = \rho\tau/\omega$: hence the distance of the focus is $\rho\tau/\Omega$ and the apparent refractive index will be Ω/ρ. In the same way Ω/ρ' will be the apparent index in the perpendicular plane.

In each case the image of the point will be seen as a short line perpendicular to the plane of principal curvature, and hence in order that one or other of two rectangular systems of lines may be seen distinctly through the plate at a certain focal adjustment of the microscope, the lines must be perpendicular respectively to the two principal planes of curvature.

Hence taking into account the second sheet of the wave-surface, it follows that with biaxal plates there will be in general four focal distances at which lines properly orientated will be seen distinctly, and for each of the polarised streams the two necessary directions of the lines will be at right-angles to each other.

In the case of uniaxal plates, the focal distances are reduced to three, since one sheet of the wave-surface is spherical, and the image corresponding to the ordinary stream is free from astigmatism.

141. It may be shown* that, if ω_1 and ω_2 be the wave-velocities in a direction normal to the plate, ρ_1, ρ_1' and ρ_2, ρ_2' the principal radii of curvature of the corresponding sheets of the wave-surface at the points where they are touched by tangent planes parallel to the faces of the plate, then

$$\left.\begin{aligned}(\rho_1+\rho_1')\,\omega_1^3(\omega_1^2-\omega_2^2) &= \omega_1^4(a^2+b^2+c^2-2\omega_2^2)-a^2b^2c^2\\ (\rho_1-\rho_1')^2\,\omega_1^6(\omega_1^2-\omega_2^2)^4 &= [\{a^2b^2c^2-\omega_1^4(a^2+b^2+c^2-2\omega_1^2)\}(\omega_1^2-\omega_2^2)\\ &\quad+2p_1\omega_1^2]^2-4\omega_1^4p_1p_2\end{aligned}\right\}\ldots(1),$$

and

$$\left.\begin{aligned}(\rho_2+\rho_2')\,\omega_2^3(\omega_2^2-\omega_1^2) &= \omega_2^4(a^2+b^2+c^2-2\omega_1^2)-a^2b^2c^2\\ (\rho_2-\rho_2')^2\,\omega_2^6(\omega_2^2-\omega_1^2)^4 &= [\{a^2b^2c^2-\omega_2^4(a^2+b^2+c^2-2\omega_2^2)\}(\omega_2^2-\omega_1^2)\\ &\quad+2p_2\,\omega_2^2]^2-4\omega_2^4p_1p_2\end{aligned}\right\}\ldots(2),$$

where

$$\left.\begin{aligned}p_1&=(a^2-\omega_1^2)(b^2-\omega_1^2)(c^2-\omega_1^2)\\ p_2&=(a^2-\omega_2^2)(b^2-\omega_2^2)(c^2-\omega_2^2)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

142. In the case of an uniaxal plate, $\omega_1=a$ and $b=a$; hence

$$\rho_1+\rho_1'=2a,\quad \rho_1-\rho_1'=0,\quad \therefore\ \rho_1=\rho_1'=a,$$

and

$$(\rho_2+\rho_2')\,\omega_2^3=c^2(\omega_2^2+a^2),\quad (\rho_2-\rho_2')\,\omega_2^3=c^2(\omega_2^2-a^2).$$

$$\therefore\ \rho_2=c^2/\omega_2,\quad \rho_2'=a^2c^2/\omega_2^3.$$

(a) If the plate be perpendicular to the optic axis, $\omega_2=a$ and

$$\rho_2=\rho_2'=c^2/a.$$

In this case then there will be only two images, which are free from astigmatism and indistinguishable directly by their polarisation, and the apparent indices of the plate are μ_0 and μ_e^2/μ_0.

* See Appendix III. § 8. Hecht, *N. Jahrb. für Min. Beil.-Bd.* VI. 265 (1889).

(*b*) If the plate be parallel to the optic axis, $\omega_2 = c$, and

$$\rho_2 = c, \quad \rho_2' = a^2/c.$$

In this case there are three focal adjustments of the microscope, at which one or other or both systems of lines are seen distinctly—the one for the ordinary pencil polarised in the axial plane, at which both sets of lines are in focus together, giving the index μ_0; the other two for the extraordinary pencil polarised in the equatorial plane, at the one lines in that plane are seen distinctly and this gives an apparent index μ_0^2/μ_e, at the other lines in the axial plane are brought into focus and the index obtained is μ_e.

(*c*) In the general case, calling θ the angle between the optic axis and the normal to the plate, we have

$$\omega_2^2 = a^2 \cos^2\theta + c^2 \sin^2\theta;$$

and the apparent index obtained by focussing a line in the principal plane is

$$\Omega c^{-2} (a^2 \cos^2\theta + c^2 \sin^2\theta)^{\frac{1}{2}},$$

and that given by observing a line perpendicular to the principal plane is

$$\Omega a^{-2} c^{-2} (a^2 \cos^2\theta + c^2 \sin^2\theta)^{\frac{3}{2}}.$$

As θ increases from 0° to 90°, this latter index changes from μ_e^2/μ_0 to μ_0^2/μ_e, and therefore for an intermediate value of θ the index obtained will be μ_0. This value is given by

$$\tan^2\theta = \mu_0^{-\frac{4}{3}} \mu_e^{\frac{2}{3}} (\mu_0^{\frac{2}{3}} + \mu_e^{\frac{2}{3}}).$$

When the plate has this orientation, there are only two focal adjustments at which a distinct image is seen. This case is however easily distinguished from that of a plate perpendicular to the optic axis, because at one of the focal distances only lines parallel to the principal plane are brought into focus and this image is polarised in a plane perpendicular to the principal plane.

143. Let us now suppose that we have a biaxal plate, the faces of which are parallel to one of the axes of symmetry of the crystal and therefore perpendicular to one of the planes of symmetry.

Then either $p_1 = 0$ or $p_2 = 0$, and taking $p_1 = 0$, the equations (1) become

$$\left.\begin{aligned}(\rho_1 + \rho_1')\,\omega_1^3(\omega_1^2 - \omega_2^2) &= \omega_1^4(a^2 + b^2 + c^2 - 2\omega_2^2) - a^2b^2c^2\\ (\rho_1 - \rho_1')\,\omega_1^3(\omega_1^2 - \omega_2^2) &= a^2b^2c^2 - \omega_1^4(a^2 + b^2 + c^2 - 2\omega_1^2)\end{aligned}\right\} \quad \ldots\ldots(4),$$

which give

$$\rho_1 = \omega_1 \text{ and } \rho_1 = \frac{\omega_1^4(a^2 + b^2 + c^2 - \omega_1^2 - \omega_2^2) - a^2b^2c^2}{\omega_1^3(\omega_1^2 - \omega_2^2)} \quad \ldots\ldots\ldots(5),$$

and from (2) we obtain

$$(\rho_2 + \rho_2')\,\omega_1^2\omega_2^3(\omega_2^2 - \omega_1^2) = \omega_1^2\{\omega_2^4(a^2 + b^2 + c^2 - 2\omega_1^2) - a^2b^2c^2\} \quad \ldots\ldots(6),$$

$$\begin{aligned}(\rho_2 - \rho_2')\,\omega_1^2\omega_2^3(\omega_2^2 - \omega_1^2)^2 = \omega_1^2\{a^2b^2c^2 - \omega_2^4(a^2 + b^2 + c^2 - 2\omega_2^2)\}&(\omega_2^2 - \omega_1^2)\\ + 2\omega_2^2(\omega_1^2 p_2 - \omega_2^2 p_1)&\ldots\ldots\ldots\ldots(7).\end{aligned}$$

But

$$\omega_1^2 p_2 - \omega_2^2 p_1 = a^2b^2c^2(\omega_1^2 - \omega_2^2) + \omega_1^2\omega_2^2(\omega_2^2 - \omega_1^2)(a^2 + b^2 + c^2) + \omega_1^2\omega_2^2(\omega_1^4 - \omega_2^4)$$
$$= (\omega_2^2 - \omega_1^2)\{(a^2 + b^2 + c^2 - \omega_1^2 - \omega_2^2)\omega_1^2\omega_2^2 - a^2b^2c^2\} \quad \text{.........(8)},$$

and therefore (7) becomes

$$(\rho_2 - \rho_2')\,\omega_1^2\omega_2^3(\omega_2^2 - \omega_1^2) = a^2b^2c^2(\omega_1^2 - 2\omega_2^2) + \omega_1^2\omega_2^4(a^2 + b^2 + c^2 - 2\omega_1^2) \text{...(9)}.$$

Hence from (6) and (9)

$$\left.\begin{aligned} \rho_2 &= \frac{\omega_1^2\omega_2^2(a^2 + b^2 + c^2 - 2\omega_1^2) - a^2b^2c^2}{\omega_1^2\omega_2(\omega_2^2 - \omega_1^2)} \\ \rho_2' &= \frac{a^2b^2c^2}{\omega_1^2\omega_2^3} \end{aligned}\right\} \text{.....................(10)}.$$

(a) If the plate be parallel to the mean axis of the ellipsoid of polarisation, then

$$\omega_1 = b, \quad \omega_2^2 = a^2\cos^2\theta + c^2\sin^2\theta,$$

where θ is the angle between the normal to the plate and the greatest axis.

In this case therefore

$$\rho_1 = b, \quad \rho_1' = \frac{c^2(b^2 - a^2)\cos^2\theta + a^2(b^2 - c^2)\sin^2\theta}{b\{(b^2 - a^2)\cos^2\theta + (b^2 - c^2)\sin^2\theta\}},$$

$$\rho_2 = \frac{a^2(b^2 - a^2)\cos^2\theta + c^2(b^2 - c^2)\sin^2\theta}{(a^2\cos^2\theta + c^2\sin^2\theta)^{\frac{1}{2}}\{(b^2 - a^2)\cos^2\theta + (b^2 - c^2)\sin^2\theta\}},$$

$$\rho_2' = \frac{a^2c^2}{(a^2\cos^2\theta + c^2\sin^2\theta)^{\frac{3}{2}}}.$$

For light polarised in the principal plane, the apparent index is μ_b when a line perpendicular to that plane is brought into focus and Ω/ρ_1' in the case of focal adjustment for a line in that plane: for light polarised perpendicularly to the principal plane the apparent index is Ω/ρ_2 or Ω/ρ_2' according as a line in or perpendicular to the principal plane is the object of observation.

(b) The cases in which the plate is parallel to the greatest axis z and to the least axis x of the ellipsoid of polarisation are obtained from the above by changing a, b, c and x, y, z in cyclical order.

In each of the planes of symmetry xy and yz, the radii of curvature ρ_2 and ρ_2' at a point on the elliptic section of the wave-surface can become equal, the position of these umbilics being determined in the first plane by

$$\tan^4(NX) = b^2(c^2 - b^2)/\{a^2(c^2 - a^2)\},$$

and in the second plane by

$$\tan^4(NY) = c^2(a^2 - c^2)/\{b^2(a^2 - b^2)\},$$

N denoting the perpendicular from the centre on the tangent plane at the point.

If the plate be perpendicular to the normal at one of the umbilics, one of the polarised streams that it transmits will give images of both systems of cross-lines distinct together, and in this respect it acts as a plate of an uniaxal crystal cut in an arbitrary direction. The two cases may however be readily distinguished, when the double refraction is sufficiently strong to give a lateral separation of the two oppositely polarised images; for with an uniaxal plate the plane of polarisation of the image free from astigmatism is parallel to the plane of separation of the images, while with the biaxal plate it is perpendicular to that direction: further on rotating the plate about its normal, the image free from astigmatism will remain fixed in the case of the uniaxal plate, while with the biaxal crystal any point of this image will describe a small circle round its mean position.

144. When the plate is perpendicular to an axis of optical symmetry, $p_1 = p_2 = 0$ and we have from (8)

$$\omega_1^2 \omega_2^2 (a^2 + b^2 + c^2 - \omega_1^2 - \omega_2^2) = a^2 b^2 c^2.$$

Hence equations (5) and (10) give

$$\rho_1 = \omega_1, \quad \rho_1' = \frac{a^2 b^2 c^2}{\omega_1^3 \omega_2^2},$$

$$\rho_2 = \omega_2, \quad \rho_2' = \frac{a^2 b^2 c^2}{\omega_1^2 \omega_2^3}.$$

Thus if the plate be perpendicular to the least axis of the ellipsoid of polarisation, x, $\omega_1 = b$, $\omega_2 = c$, and we have the following results: when the line brought into focus is parallel to the mean axis y, the apparent indices are μ_b and μ_a^2/μ_c, the planes of polarisation being zx and xy respectively; the indices obtained by focussing on a line parallel to the greatest axis z are μ_c and μ_a^2/μ_b, the planes of polarisation being in these cases xy and zx.

The other cases are obtained from this by changing x, y, z and a, b, c in cyclical order.

*Prisms**.

145. We have already in Chapter I. considered the question of the passage of a stream of light through a prism from a general point of view without any assumption respecting the form of the wave-surface within the prism, and it is now only necessary to apply the results to the special cases, in which the wave-surface has one of the forms discussed in the last chapter.

Let us first take the case of a prism made of an uniaxal crystal, of which the crystallographic orientation is arbitrary but supposed to be known.

* Stokes, *Camb. and Dublin Math. J.* I. 183 (1846): *Math. and Phys. Papers*, I. 148. Senarmont, *Nouv. Ann. de Math.* XVI. 273 (1857). Von Lang, *Wien. Ber.* XXXIII. 155, 577 (1858). Liebisch, *N. Jahrb. für Min.* (1886) I. 14; (1900) I. 57: *Gött. Nachr.* (1888) 197: *Phys. Kryst.* pp. 376—403. Born, *N. Jahrb. für Min. Beil.-Bd.* V. 1 (1887). Viola, *Zeitschrift für Kryst. und Min.* XXXII. 66, 545 (1900): *Rend. Lincei* (5) IX. [1] 196 (1900).

Referring the prism to rectangular axes, such that the axis of ζ is the edge of the prism and the plane of $\xi\zeta$ bisects its angle, we may regard as given the angle μ that the optic axis makes with the normal section and the angle ϕ that its projection on this section makes with the axis of ξ, this angle being measured towards the direction in which the light is travelling.

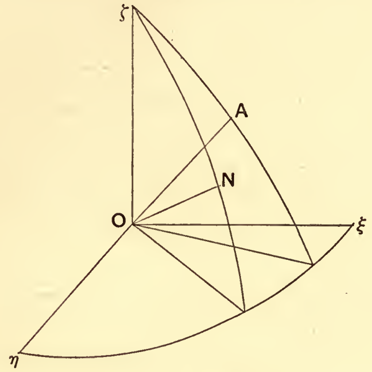


Fig. 36.

Suppose lines and planes, parallel to those that we have to consider, to be drawn through the centre of a sphere of unit radius, and let the coordinate axes meet the surface of the sphere in the points ξ, η, ζ and let the optic axis and the normal to the wave within the prism cut it in the points A and N respectively.

The surface of wave-quickness for the prism consists of a sphere of radius a and an ovaloid with the equation

$$\omega^2 = a^2 \cos^2 \theta + c^2 \sin^2 \theta \quad \text{......(11)},$$

where θ is the angle between the wave-normal and the optic axis; and if χ' be the angle that the wave-normal makes with the plane of $\xi\eta$ and ψ be the angle that its projection on this plane makes with the axis of ξ, the spherical triangle $A\zeta N$ gives

$$\cos \theta = \sin \mu \sin \chi' + \cos \mu \cos \chi' \cos (\psi - \phi) \quad \text{......(12)}.$$

If now we measure the angle of the prism A, the deviation D produced by it, the angle of incidence i and the angle χ that the incident wave makes with the edge of the prism, we have, § 7,

$$\sin (D_0/2) = \sec \chi \sin (D/2), \quad \cos i_0 = \sec \chi \cos i$$

and ψ and χ' are given by

$$\tan \psi = -\cot \frac{A}{2} \cot \left(i_0 - \frac{A + D_0}{2}\right) \tan \frac{A + D_0}{2},$$

$$\tan^2 \chi' = \tan^2 \chi \,(C_0^{-2} \cos^2 \psi + S_0^{-2} \sin^2 \psi),$$

where $$C_0 = \cos \frac{A + D_0}{2} \Big/ \cos \frac{A}{2}, \quad S_0 = \sin \frac{A + D_0}{2} \Big/ \sin \frac{A}{2},$$

and finally to determine ω we have

$$\omega^2 = \Omega^2 \sin^2 \chi' / \sin^2 \chi.$$

In the case of the ordinary wave, the one principal wave-velocity is given at once by $a = \omega$; while from measurements with the extraordinary wave, calculating θ from equation (12), the second principal velocity is determined from (11), or writing

$$\cos \Theta = a \cos \theta / \omega \quad \text{.............................(13)},$$

by

$$c = \omega \sin \Theta / \sin \theta \quad \text{.............................(14)}.$$

146. Passing now to the case in which the deviation is a minimum, we have in the first place that the wave-velocity within the prism is expressed in terms of the angles A, D_0, ψ, χ' by the relation

$$\frac{\omega^2}{\Omega^2} = \frac{\sin^2 \chi'}{\sin^2 \chi} = \sin^2 \chi' + \cos^2 \chi' \frac{\tan^2 \chi'}{\tan^2 \chi}$$

$$= \sin^2 \chi' + \cos^2 \chi' (M + N \cos 2\psi) \quad \text{..............(15)},$$

where

$$2M = C_0^{-2} + S_0^{-2}, \quad 2N = C_0^{-2} - S_0^{-2},$$

and on the other hand ω must satisfy the equation of the surface of wave-quickness, which may be written

$$f(\omega, \psi, \chi') = 0 \quad \text{.............................(16)},$$

whence eliminating ω between (15) and (16), we obtain an equation

$$F(D_0, \psi, \chi') = 0 \quad \text{.............................(17)},$$

and the minimum deviation being characterised by $dD_0/d\psi = 0$, we have in this case $\partial F/\partial \psi = 0$.

Thus in the case of the ordinary wave with an uniaxal prism, we have

$$\Omega^2 \{\sin^2 \chi' + \cos^2 \chi' (M + N \cos 2\psi)\} = a^2,$$

and in the case of minimum deviation

$$\Omega^2 \cos^2 \chi' \, . \, N \sin 2\psi = 0,$$

whence $\psi = \pi/2$ and

$$\tan \chi' = \tan \chi \sqrt{M - N} = \tan \chi \sin \frac{A}{2} \Big/ \sin \frac{A + \Delta_0}{2},$$

and

$$a^2/\Omega^2 = \sin^2 \chi' + \cos^2 \chi' (M - N) = \sin^2 \chi' + \cos^2 \chi' \sin^2 \frac{A}{2} \Big/ \sin^2 \frac{A + \Delta_0}{2},$$

Δ_0 being the minimum value of D_0.

In the case of the extraordinary wave, (17) becomes

$$\Omega^2 \{\sin^2 \chi' + \cos^2 \chi' (M + N \cos 2\psi)\}$$
$$- c^2 - (a^2 - c^2) \{\sin \mu \sin \chi' + \cos \mu \cos \chi' \cos (\psi - \phi)\}^2 = 0 \quad \text{......(18)},$$

and in the case of minimum deviation we have

$$\Omega^2 \cos^2 \chi' . N \sin 2\psi - (a^2 - c^2) \{\sin \mu \sin \chi' + \cos \mu \cos \chi' \cos (\psi - \phi)\}$$
$$\cos \mu \cos \chi' \sin (\psi - \phi) = 0 \quad \text{......(19)}.$$

It is clear however that with a prism of arbitrary orientation and in the general case of oblique refraction, no result of practical utility will be obtained, and we will therefore proceed to consider some special cases.

In the first place we will suppose that the incident waves are parallel to the edge of the prism; then we have $\chi' = 0$ and

$$\Omega^2 (M + N \cos 2\psi) - c^2 - (a^2 - c^2) \cos^2 \mu \cos^2 (\psi - \phi) = 0 \quad \text{......(20)},$$
$$\Omega^2 N \sin 2\psi - (a^2 - c^2) \cos^2 \mu \sin (\psi - \phi) \cos (\psi - \phi) = 0 \quad \text{......(21)},$$

whence

$$\Omega^2 M \sin 2\psi - c^2 \sin 2\psi - (a^2 - c^2) \cos^2 \mu \cos (\psi - \phi) \sin (\psi + \phi) = 0 \text{...(22)},$$

and from (21) and (22)

$$\{\Omega^2 (M + N) - c^2\} \sin 2\psi = (a^2 - c^2) \cos^2 \mu \cos (\psi - \phi) \, 2 \cos \phi \sin \psi,$$
$$\{\Omega^2 (M - N) - c^2\} \sin 2\psi = (a^2 - c^2) \cos^2 \mu \cos (\psi - \phi) \, 2 \sin \phi \cos \psi.$$

These equations give

$$(\Omega^2 C_0^{-2} - c^2)(\Omega^2 S_0^{-2} - c^2) \sin 2\psi = (a^2 - c^2)^2 \cos^4 \mu \cos^2 (\psi - \phi) \sin 2\phi$$

and

$$\{\Omega^2 (C_0^{-2} \sin^2 \phi + S_0^{-2} \cos^2 \phi) - c^2\} \sin 2\psi = (a^2 - c^2) \cos^2 \mu \cos^2 (\psi - \phi) \sin 2\phi,$$

whence

$$(\Omega^2 C_0^{-2} - c^2)(\Omega^2 S_0^{-2} - c^2) = (a^2 - c^2) \cos^2 \mu \{\Omega^2 (C_0^{-2} \sin^2 \phi + S_0^{-2} \cos^2 \phi) - c^2\}$$
$$\text{.........(23)},$$

which determines the principal wave-velocity c from measurements of the angle of the prism and the angle of minimum deviation of the extraordinary wave, the other principal wave-velocity a having been determined. Equation (23) is a quadratic in c^2, but since the double refraction of all known crystals is weak, that root must be taken for which ψ is very nearly $\pi/2$.

Returning to the general equations (18) and (19), let us next suppose that at minimum deviation the wave within the prism is parallel to the axis of ξ; then $\psi = \pi/2$, a case that is characterised by the vanishing of the lateral deviation, and we have

$$\Omega^2 \{\sin^2 \chi' + \cos^2 \chi' (M - N)\} - c^2 - (a^2 - c^2)(\sin \mu \sin \chi' + \cos \mu \cos \chi' \sin \phi)^2 = 0$$
$$\text{.........(24)},$$

$$(\sin \mu \sin \chi' + \cos \mu \cos \chi' \sin \phi) \cos \mu \cos \chi' \cos \phi = 0 \text{.........(25)},$$

and
$$\tan \chi' = \tan \chi \sqrt{M - N} = \tan \chi \sin \frac{A}{2} \Big/ \sin \frac{A + \Delta_0}{2}.$$

From (25) we have

$$\sin\mu\sin\chi' + \cos\mu\cos\chi'\sin\phi = 0, \text{ or } \cos\mu = 0, \text{ or } \cos\phi = 0.$$

In the first of these cases, the optic axis is perpendicular to the wave-normal and from (24)

$$\sin^2\chi' + \cos^2\chi'\sin^2\frac{A}{2}\Big/\sin^2\frac{A+\Delta_0}{2} = c^2/\Omega^2;$$

in the second case $\mu = \pi/2$ or the optic axis is parallel to the edge of the prism, and

$$\Omega^2\left\{\sin^2\chi' + \cos^2\chi'\sin^2\frac{A}{2}\Big/\sin^2\frac{A+\Delta_0}{2}\right\} = c^2 + (a^2 - c^2)\sin^2\chi';$$

and in the third case, $\phi = \pi/2$, or the plane through the edge of the prism and the optic axis is perpendicular to the plane bisecting the angle of the prism and we have

$$\Omega^2\left\{\sin^2\chi' + \cos^2\chi'\sin^2\frac{A}{2}\Big/\sin^2\frac{A+\Delta_0}{2}\right\} = c^2 + (a^2 - c^2)\cos^2(\mu - \chi').$$

147. Turning now to the case of a prism made from a biaxal crystal and referring it to the same axes (ξ, η, ζ) as before, let the direction-cosines of these axes with respect to the axes of optical symmetry (x, y, z) be given by

	x	y	z
ξ	α_1	β_1	γ_1
η	α_2	β_2	γ_2
ζ	α_3	β_3	γ_3.

Then (x, y, z) being the coordinates of a point referred to the crystallographic axes, and (ξ, η, ζ) the coordinates of the same point with respect to the axes of the prism, we have

$$\left.\begin{aligned} x &= \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta \\ y &= \beta_1\xi + \beta_2\eta + \beta_3\zeta \\ z &= \gamma_1\xi + \gamma_2\eta + \gamma_3\zeta \end{aligned}\right\} \quad \text{......................... (26),}$$

with the ordinary relations of orthogonal transformations.

Hence the equation of the surface of wave-quickness becomes

$$\frac{(\alpha_1\xi + \alpha_2\eta + \alpha_3\zeta)^2}{a^2 - \omega^2} + \frac{(\beta_1\xi + \beta_2\eta + \beta_3\zeta)^2}{b^2 - \omega^2} + \frac{(\gamma_1\xi + \gamma_2\eta + \gamma_3\zeta)^2}{c^2 - \omega^2} = 0 \text{ .. (27),}$$

or since $\xi = \omega\cos\psi\cos\chi'$, $\eta = \omega\sin\psi\cos\chi'$, $\zeta = \omega\sin\chi'$,

the equation in polar coordinates becomes

$$\begin{aligned} f(\omega, \psi, \chi') \equiv \omega^4 - \omega^2 (&L_{11}\cos^2\psi\cos^2\chi' + L_{22}\sin^2\psi\cos^2\chi' + L_{33}\sin^2\chi' \\ &+ L_{23}\sin\psi\sin 2\chi' + L_{31}\cos\psi\sin 2\chi' + L_{12}\sin 2\psi\cos^2\chi') \\ &+ M_{11}\cos^2\psi\cos^2\chi' + M_{22}\sin^2\psi\cos^2\chi' + M_{33}\sin^2\chi' \\ &+ M_{23}\sin\psi\sin 2\chi' + M_{31}\cos\psi\sin 2\chi' + M_{12}\sin 2\psi\cos^2\chi' = 0 \end{aligned}$$

.........(28),

$$\left.\begin{aligned} \text{where}\qquad L_{mn} &= (b^2+c^2)\,\alpha_m\alpha_n + (c^2+a^2)\,\beta_m\beta_n + (a^2+b^2)\,\gamma_m\gamma_n \\ M_{mn} &= b^2c^2\alpha_m\alpha_n + c^2a^2\beta_m\beta_n + a^2b^2\gamma_m\gamma_n \end{aligned}\right\} \quad\ldots\ldots(29).$$

148. Arranged according to a^2, b^2, c^2, the equation takes the form

$$f(\omega, \psi, \chi') \equiv Eb^2c^2 + Fc^2a^2 + Ga^2b^2$$
$$- \omega^2(F+G)\,a^2 - \omega^2(G+E)\,b^2 - \omega^2(E+F)\,c^2 + \omega^4 = 0 \ldots (30),$$

$$\left.\begin{aligned} \text{where}\qquad E &= (\alpha_1 \cos\psi \cos\chi' + \alpha_2 \sin\psi \cos\chi' + \alpha_3 \sin\chi')^2 \\ F &= (\beta_1 \cos\psi \cos\chi' + \beta_2 \sin\psi \cos\chi' + \beta_3 \sin\chi')^2 \\ G &= (\gamma_1 \cos\psi \cos\chi' + \gamma_2 \sin\psi \cos\chi' + \gamma_3 \sin\chi')^2 \end{aligned}\right\} \quad\ldots\ldots\ldots(31).$$

Suppose now that we have obtained three sets of corresponding values of ω, ψ, χ', so that we have three equations

$$E_nb^2c^2 + F_nc^2a^2 + G_na^2b^2 - \omega_n^2(F_n+G_n)\,a^2$$
$$- \omega_n^2(G_n+E_n)\,b^2 - \omega_n^2(E_n+F_n)\,c^2 + \omega_n^4 = 0, \quad (n = 1, 2, 3) \ldots\ldots(32),$$

then solving for b^2c^2, c^2a^2, a^2b^2, we shall obtain three equations of the form

$$b^2c^2 = A_1a^2 + B_1b^2 + C_1c^2 + D_1,$$
$$c^2a^2 = A_2a^2 + B_2b^2 + C_2c^2 + D_2,$$
$$a^2b^2 = A_3a^2 + B_3b^2 + C_3c^2 + D_3,$$

wherein A_1, $B_1 \ldots D_3$ depend only upon the coefficients E_1, $F_1 \ldots G_3$ and ω_1^2, ω_2^2, ω_3^2; the first two equations give

$$a^2 = \frac{A'c^4 + B'c^2 + C'}{A'''c^4 + B'''c^2 + C'''}, \qquad b^2 = \frac{A''c^4 + B''c^2 + C''}{A'''c^4 + B'''c^2 + C'''},$$

the coefficients being functions of E_1, $F_1 \ldots G_3$ and ω_1^2, ω_2^2, ω_3^2, and substituting these values in the third equation, we obtain an equation that involves only c^2 and coefficients deduced from measured quantities.

This equation is however of the fifth degree in c^2 and each root with the corresponding values of a^2 and b^2 gives a set of values that satisfy the three equations (32). The problem is thus indeterminate, unless we know approximate values a_0^2, b_0^2, c_0^2 of a^2, b^2, c^2.

One method of proceeding is as follows. If we obtain six sets of corresponding values of ω, ψ, χ', we shall have six equations of the form (32), from which we can deduce the values of b^2c^2, c^2a^2, a^2b^2, a^2, b^2, c^2. Calling these A, B, C, A_1, B_1, C_1 we have

$$b_0^2c_0^2 = A, \quad c_0^2a_0^2 = B, \quad a_0^2b_0^2 = C, \quad a_0^2 = A_1, \quad b_0^2 = B_1, \quad c_0^2 = C_1,$$

so that we have approximately

$$a_0^2 = \sqrt{BC/A}, \quad b_0^2 = \sqrt{CA/B}, \quad c_0^2 = \sqrt{AB/C}, \quad a_0^2 = A_1, \quad b_0^2 = B_1, \quad c_0^2 = C_1.$$

Now if the observations were absolutely exact, we should of course find

$$A_1^2 = BC/A, \quad B_1^2 = CA/B, \quad C_1^2 = AB/C,$$

and since $A \dots C_1$ are affected by errors, we can obtain more accurate values a_1^2, b_1^2, c_1^2, for a^2, b^2, c^2 by taking the geometric mean of the two values obtained for a_0^2, b_0^2 and c_0^2: thus

$$a_1^2 = \sqrt[4]{\frac{BCA_1^2}{A}}, \quad b_1^2 = \sqrt[4]{\frac{CAB_1^2}{B}}, \quad c_1^2 = \sqrt[4]{\frac{ABC_1^2}{C}}.$$

Suppose now that the three values of a^2, b^2 and c^2 are

$$a^2 = a_1^2 + x, \quad b^2 = b_1^2 + y, \quad c^2 = c_1^2 + z;$$

then substituting these values in the six equations of the form (32) and neglecting squares and products of the small quantities x, y, z we obtain, since $E + F + G = 1$,

$$\begin{aligned}
&x\,\{F_n(\omega_n^2 - c_1^2) + G_n(\omega_n^2 - b_1^2)\} + y\,\{G_n(\omega_n^2 - a_1^2) + E_n(\omega_n^2 - c_1^2)\} \\
&\quad + z\,\{E_n(\omega_n^2 - b_1^2) + F_n(\omega_n^2 - a_1^2)\} - E_n(\omega_n^2 - b_1^2)(\omega_n^2 - c_1^2) \\
&\quad - F_n(\omega_n^2 - c_1^2)(\omega_n^2 - a_1^2) - G_n(\omega_n^2 - a_1^2)(\omega_n^2 - b_1^2) = 0, \quad (n = 1, 2, \dots 6),
\end{aligned}$$

and from these six equations the values of x, y, z may be determined by the method of least squares*.

149. It is possible by means of prisms, as has been pointed out in Chapter I., to find any number of points on the surface of wave-quickness and hence to determine completely the form and orientation of this surface, and the question now arises whether in the case of crystalline media a determination of a plane central section suffices for this purpose†.

Taking this section as the plane of $\xi\eta$, the polar equation of the section may be written

$$\begin{aligned}
f(\omega, \psi) \equiv \omega^4 - \omega^2 (L_{11}\cos^2\psi + L_{22}\sin^2\psi + 2L_{12}\sin\psi\cos\psi) \\
+ M_{11}\cos^2\psi + M_{22}\sin^2\psi + 2M_{12}\sin\psi\cos\psi = 0 \quad \dots\dots\dots (33),
\end{aligned}$$

the values of $L_{11} \dots M_{12}$ being given by (29), where $\alpha_1 \dots \gamma_3$ are the direction-cosines of the principal axes of the surface referred to the axes of ξ, η, ζ.

This equation contains six coefficients, so that six pairs of corresponding values of ω and ψ suffice for their determination, and the problem is to deduce the values of a^2, b^2, c^2, $\alpha_1 \dots \gamma_3$ from the expressions for $L_{11} \dots M_{12}$ in terms of these quantities.

From the equations

$$\begin{aligned}
1 &= \alpha_1^2 + \beta_1^2 + \gamma_1^2, \\
L_{11} &= (b^2 + c^2)\,\alpha_1^2 + (c^2 + a^2)\,\beta_1^2 + (a^2 + b^2)\,\gamma_1^2, \\
M_{11} &= b^2c^2\alpha_1^2 + c^2a^2\beta_1^2 + a^2b^2\gamma_1^2,
\end{aligned}$$

* Born, *loc. cit.* p. 40.

† Bull, *Sitzungsb. Bayer. Acad.* (1883) 423; *Math. Ann.* XXXIV. 297 (1889). Liebisch, *N. Jahrb. für Min.* (1886) I. 31.

and
$$0 = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2,$$
$$L_{12} = (b^2+c^2)\,\alpha_1\alpha_2 + (c^2+a^2)\,\beta_1\beta_2 + (a^2+b^2)\,\gamma_1\gamma_2,$$
$$M_{12} = b^2c^2\alpha_1\alpha_2 + c^2a^2\beta_1\beta_2 + a^2b^2\gamma_1\gamma_2,$$
we have
$$\alpha_1^2 = \frac{a^4 - L_{11}a^2 + M_{11}}{(a^2-b^2)(a^2-c^2)}, \qquad \alpha_1\alpha_2 = \frac{-L_{12}a^2 + M_{12}}{(a^2-b^2)(a^2-c^2)},$$
$$\beta_1^2 = \frac{b^4 - L_{11}b^2 + M_{11}}{(b^2-c^2)(b^2-a^2)}, \qquad \beta_1\beta_2 = \frac{-L_{12}b^2 + M_{12}}{(b^2-c^2)(b^2-a^2)},$$
$$\gamma_1^2 = \frac{c^4 - L_{11}c^2 + M_{11}}{(c^2-a^2)(c^2-b^2)}, \qquad \gamma_1\gamma_2 = \frac{-L_{12}c^2 + M_{12}}{(c^2-a^2)(c^2-b^2)};$$
and in the same way α_2^2, β_2^2, γ_2^2 are obtained. Hence writing $a^2 = u$ and forming the expression $\alpha_1^2\alpha_2^2 = (\alpha_1\alpha_2)^2$, we find
$$(u^2 - L_{11}u + M_{11})(u^2 - L_{22}u + M_{22}) = (-L_{12}u + M_{12})^2 \quad\text{.........(34)},$$
and b^2, c^2 satisfy the same equation, as is easily seen.

Hence a^2, b^2, c^2 are three of the roots of (34) and calling the fourth root d^2, we have
$$a^2 + b^2 + c^2 + d^2 = L_{11} + L_{22} = a^2(1+\alpha_3^2) + b^2(1+\beta_3^2) + c^2(1+\gamma_3^2),$$
whence
$$d^2 = a^2\alpha_3^2 + b^2\beta_3^2 + c^2\gamma_3^2.$$

Thus d is the reciprocal of a semi-diameter of an ellipsoid, of which the principal semi-axes are $1/a$, $1/b$, $1/c$, and hence must lie between a and c, but may be either greater or less than b. Also if (33) represent a real section of the surface of wave-quickness, the four roots of (34) must be real.

Let us suppose that $a^2 > b^2 > d^2 > c^2$: if we assume that a, b, c are the principal wave-velocities required, we have
$$\alpha_3^2 = 1 - \alpha_1^2 - \alpha_2^2 = \frac{b^2c^2 + a^2d^2 - M_{11} - M_{22}}{(a^2-b^2)(a^2-c^2)},$$
$$\beta_3^2 = 1 - \beta_1^2 - \beta_2^2 = \frac{c^2a^2 + b^2d^2 - M_{11} - M_{22}}{(b^2-c^2)(b^2-a^2)},$$
$$\gamma_3^2 = 1 - \gamma_1^2 - \gamma_2^2 = \frac{a^2b^2 + c^2d^2 - M_{11} - M_{22}}{(c^2-a^2)(c^2-b^2)},$$
and if α_3, β_3, γ_3 be real, this assumption will give a surface of wave-quickness with its axes of symmetry inclined to the normal to the given central section at angles, of which the cosines are α_3, β_3, γ_3.

Again if we take a, d, c as the principal wave-velocities, we find that the principal axes of the surface are determined by
$$\alpha_3'^2 = \frac{c^2d^2 + a^2b^2 - M_{11} - M_{22}}{(a^2-d^2)(a^2-c^2)} = \gamma_3^2\,\frac{b^2-c^2}{a^2-d^2},$$

$$\beta_3'^2 = \frac{c^2a^2 + b^2d^2 - M_{11} - M_{22}}{(d^2 - c^2)(d^2 - a^2)} = \beta_3^2 \frac{(a^2 - b^2)(b^2 - c^2)}{(a^2 - d^2)(d^2 - c^2)},$$

$$\gamma_3'^2 = \frac{a^2d^2 + b^2c^2 - M_{11} - M_{22}}{(c^2 - a^2)(c^2 - d^2)} = \alpha_3^2 \frac{a^2 - b^2}{d^2 - c^2},$$

and α_3', β_3', γ_3' are real, if α_3, β_3, γ_3 be so. We have then a second real solution, if we have one.

On the other hand the assumptions, that a, b, d, or b, d, c are the principal wave-velocities, will give us positions of the principal axes that are imaginary, if α_3, β_3, γ_3 be real; and it therefore follows that there are two and only two real solutions of the problem of determining the surface of wave-quickness from its central section, if there be one such solution, and that the two surfaces determined will differ in respect of their mean axes while their greatest and least axes are the same.

The two surfaces will be identical only if $b = d$, which gives

$$\gamma_3/\alpha_3 = \pm \sqrt{a^2 - b^2}/\sqrt{b^2 - c^2},$$

that is if the given central section contain one of the optic axes.

150. Let us now take the case of minimum deviation with a biaxal prism. We have seen that the velocity of a wave within the prism satisfies the general relation (15), while on the other hand it is given in terms of the angles ψ and χ' by the equation of the surface of wave-quickness (28), and eliminating ω between these equations, we obtain

$$\begin{aligned} F(D, \psi, \chi') \equiv\ & \Omega^4 \{\sin^2\chi' + \cos^2\chi' (M + N\cos 2\psi)\}^2 \\ & - \Omega^2 \{\sin^2\chi' + \cos^2\chi' (M + N\cos 2\psi)\} \{L_{11}\cos^2\psi\cos^2\chi' \\ & + L_{22}\sin^2\psi\cos^2\chi' + L_{33}\sin^2\chi' + 2L_{23}\sin\psi\sin\chi'\cos\chi' \\ & + 2L_{31}\cos\psi\sin\chi'\cos\chi' + 2L_{12}\sin\psi\cos\psi\cos^2\chi'\} \\ & + M_{11}\cos^2\psi\cos^2\chi' + M_{22}\sin^2\psi\cos^2\chi' + M_{33}\sin^2\chi' \\ & + 2M_{23}\sin\psi\sin\chi'\cos\chi' + 2M_{31}\cos\psi\sin\chi'\cos\chi' \\ & + 2M_{12}\sin\psi\cos\psi\cos^2\chi' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(35). \end{aligned}$$

When the deviation is a minimum, we have $dD/d\psi = 0$, or $\partial F/\partial\psi = 0$, which gives

$$\begin{aligned} & \{\sin^2\chi' + \cos^2\chi' (M + N\cos 2\psi)\} \{4\Omega^4 N\cos^2\chi' \sin 2\psi \\ & \quad - \Omega^2 (L_{11}\sin 2\psi\cos^2\chi' - L_{22}\sin 2\psi\cos^2\chi' - 2L_{23}\cos\psi\sin\chi'\cos\chi' \\ & \quad + 2L_{31}\sin\psi\sin\chi'\cos\chi' - 2L_{12}\cos 2\psi\cos^2\chi')\} \\ & \quad - 2\Omega^2 N\sin 2\psi\cos^2\chi' (L_{11}\cos^2\psi\cos^2\chi' + L_{22}\sin^2\psi\cos^2\chi' \\ & \quad + L_{33}\sin^2\chi' + 2L_{23}\sin\psi\sin\chi'\cos\chi' + 2L_{31}\cos\psi\sin\chi'\cos\chi' \\ & \quad + 2L_{12}\sin\psi\cos\psi\cos^2\chi') + M_{11}\sin 2\psi\cos^2\chi' - M_{22}\sin 2\psi\cos^2\chi' \\ & \quad - 2M_{23}\cos\psi\sin\chi'\cos\chi' + 2M_{31}\sin\psi\sin\chi'\cos\chi' \\ & \quad - 2M_{12}\cos 2\psi\cos^2\chi' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(36). \end{aligned}$$

Equations (35) and (36) contain the solution of the problem, but in their general form they are too complicated to be of any practical use, and in fact even when the incident waves are parallel to the edge of the prism, so that $\chi' = 0$, no result of any value is obtained with a prism, of which the crystallographic orientation is quite arbitrary.

151. Confining then our attention to special cases of interest, let us first suppose that the refraction is direct and the normal section is the plane of symmetry xy. Then if the angle $\xi 0 x$ be μ, the direction-cosines $\alpha_1, \alpha_2 \ldots \gamma_3$ are given by the scheme

	x	y	z
ξ	$\cos\mu$	$-\sin\mu$	0
η	$\sin\mu$	$\cos\mu$	0
ζ	0	0	1,

and the equation of the section of the surface of wave-quickness by the normal section of the prism is

$$f(\omega, \psi) \equiv (\omega^2 - c^2)\{\omega^2 - b^2 - (a^2 - b^2)\sin^2(\psi - \mu)\} = 0.$$

The minimum deviation of the wave propagated with constant velocity, which is polarised in the principal section, gives at once the principal wave-velocity c by the ordinary formula: and in the case of minimum deviation of the other wave, we have

$$\Omega^2(M + N\cos 2\psi) - b^2 - (a^2 - b^2)\sin^2(\psi - \mu) = 0,$$

$$2\Omega^2 N \sin 2\psi + (a^2 - b^2)\sin 2(\psi - \mu) = 0,$$

and eliminating ψ between these equations, they give the relation

$$(\Omega^2 C_0^{-2} - a^2\sin^2\mu - b^2\cos^2\mu)(\Omega^2 S_0^{-2} - a^2\cos^2\mu - b^2\sin^2\mu) = (a^2 - b^2)^2\sin^2\mu\cos^2\mu.$$

152. Let us next consider the cases, in which the wave within the prism is parallel to the inner mean line, that is the line of intersection of the plane bisecting the angle of the prism and the normal section of the prism. The lateral deviation then vanishes, and since $\psi = \pi/2$, we have from equations (35) and (36)

$$\begin{aligned}\Omega^4\{\sin^2\chi' + (M - N)\cos^2\chi'\}^2 \\ - \Omega^2\{\sin^2\chi' + (M - N)\cos^2\chi'\}(L_{22}\cos^2\chi' + L_{33}\sin^2\chi' + 2L_{23}\sin\chi'\cos\chi') \\ + M_{22}\cos^2\chi' + M_{33}\sin^2\chi' + 2M_{23}\sin\chi'\cos\chi' = 0 \ldots\ldots\ldots\ldots(37),\end{aligned}$$

and

$$\begin{aligned}\Omega^2\{\sin^2\chi' + (M - N)\cos^2\chi'\}(L_{31}\sin\chi'\cos\chi' + L_{12}\cos^2\chi') \\ - (M_{31}\sin\chi'\cos\chi' + M_{12}\cos^2\chi') = 0 \ldots\ldots\ldots(38);\end{aligned}$$

or writing in the values of $L_{31}\ldots$

$$\begin{aligned}\Omega^4\{\sin^2\chi'+(M-N)\cos^2\chi'\}^2 &\\ -\Omega^2\{\sin^2\chi'+(M-N)\cos^2\chi'\}\{&(b^2+c^2)(\alpha_2\cos\chi'+\alpha_3\sin\chi')^2\\ &+(c^2+a^2)(\beta_2\cos\chi'+\beta_3\sin\chi')^2+(a^2+b^2)(\gamma_2\cos\chi'+\gamma_3\sin\chi')^2\}\\ &+b^2c^2(\alpha_2\cos\chi'+\alpha_3\sin\chi')^2+c^2a^2(\beta_2\cos\chi'+\beta_3\sin\chi')^2\\ &+a^2b^2(\gamma_2\cos\chi'+\gamma_3\sin\chi')^2=0 \qquad (39),\end{aligned}$$

and

$$\begin{aligned}\Omega^2\{\sin^2\chi'+(M-N)\cos^2\chi'\}\{&(b^2+c^2)\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')\\ &+(c^2+a^2)\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')+(a^2+b^2)\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')\}\\ &-b^2c^2\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')-c^2a^2\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')\\ &-a^2b^2\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')=0 \qquad (40).\end{aligned}$$

Now from the relations of orthogonal transformations, we have

$$\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')+\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')+\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')=0,$$

whence if

$$\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')=0,$$

equation (40) becomes

$$[\Omega^2\{\sin^2\chi'+(M-N)\cos^2\chi'\}-a^2](b^2-c^2)\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')=0 \quad\ldots(41)$$

and this is satisfied by

(A) $\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')=0$, whence also $\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')=0$,

or

(B) $\sin^2\chi'+(M-N)\cos^2\chi'=a^2/\Omega^2$.

(A) Let us first suppose that

$$\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')=\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')=\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')=0.$$

Then we have three cases to consider, of which the following are types:

(*a*) We may take

$$\begin{aligned}\alpha_1&=1, & \beta_1&=0, & \gamma_1&=0,\\ \alpha_2&=0, & \beta_2&=1, & \gamma_2&=0,\\ \alpha_3&=0, & \beta_3&=0, & \gamma_3&=1,\end{aligned}$$

which express that the coordinate axes (ξ, η, ζ) coincide with the axes of optical symmetry of the crystal. Then equation (39) gives

$$[\Omega^2\{\sin^2\chi'+(M-N)\cos^2\chi'\}-a^2][\Omega^2\{\sin^2\chi'+(M-N)\cos^2\chi'\} \\ -b^2\sin^2\chi'-c^2\cos^2\chi']=0.$$

Thus the minimum deviation gives in the one case the principal wave-velocity a, and in the other a linear relation between the other principal velocities b and c.

(*b*) The conditions are also satisfied by

$$\begin{array}{lll} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 0, \\ \alpha_2 = 0, & \beta_2 = \cos\mu, & \gamma_2 = -\sin\mu, \\ \alpha_3 = 0, & \beta_3 = \sin\mu, & \gamma_3 = \cos\mu; \end{array}$$

the axis of symmetry x then coincides with ξ, and the axis y is inclined at an angle μ to the axis η. In this case we have from (39)

$$[\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} - a^2]$$
$$\times[\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} - b^2\sin^2(\chi'-\mu) - c^2\cos^2(\chi'-\mu)] = 0.$$

(*c*) Finally we can have

$$\begin{array}{lll} \alpha_1 = 0, & \beta_1 = -\sin\mu, & \gamma_1 = \cos\mu, \\ \alpha_2 = \cos\chi', & \beta_2 = -\cos\mu\sin\chi', & \gamma_2 = -\sin\mu\sin\chi', \\ \alpha_3 = \sin\chi', & \beta_3 = \cos\mu\cos\chi', & \gamma_3 = \sin\mu\cos\chi'; \end{array}$$

the axis of symmetry x is then perpendicular to the axis ξ and the lateral deviation only vanishes when the wave within the prism is parallel to the plane of optical symmetry yz. In this case (39) gives

$$[\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} - b^2][\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} - c^2] = 0,$$

and the principal wave-velocities b and c are determined directly from the angles of minimum deviation of the two waves.

(B) Taking now the case in which

$$\alpha_1(\alpha_2\cos\chi' + \alpha_3\sin\chi') = 0, \text{ and } \Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} = a^2,$$

we have from (39), since

$$(\alpha_2\cos\chi' + \alpha_3\sin\chi')^2 + (\beta_2\cos\chi' + \beta_3\sin\chi')^2 + (\gamma_2\cos\chi' + \gamma_3\sin\chi')^2 = 1,$$

$$\Omega^4\{\sin^2\chi' + (M-N)\cos^2\chi'\}^2$$
$$-\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\}\{c^2 + a^2 - (a^2-b^2)(\alpha_2\cos\chi' + \alpha_3\sin\chi')^2$$
$$+ (b^2-c^2)(\gamma_2\cos\chi' + \gamma_3\sin\chi')^2\}$$
$$+ c^2a^2 - c^2(a^2-b^2)(\alpha_2\cos\chi' + \alpha_3\sin\chi')^2 + a^2(b^2-c^2)(\gamma_2\cos\chi' + \gamma_3\sin\chi')^2 = 0,$$

and this equation which is independent of α_1 is satisfied if

$$\alpha_2\cos\chi' + \alpha_3\sin\chi' = 0$$

and $$\Omega^2\{\sin^2\chi' + (M-N)\cos^2\chi'\} - a^2 = 0.$$

In this case then $\alpha_2\cos\chi' + \alpha_3\sin\chi' = 0$; that is the axis of symmetry x is in the plane of the wave, which is therefore perpendicular to the plane of symmetry yz, and the minimum deviation, characterised by the absence of lateral deviation, gives the principal wave-velocity a. Similar results are obtained when the wave within the prism is perpendicular to the planes of symmetry zx and xy.

(C) Returning to the general condition (40), we have

$$\omega^2 = \Omega^2 \{\sin^2\chi' + (M-N)\cos^2\chi'\}$$

$$= \frac{b^2c^2\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')+c^2a^2\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')+a^2b^2\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')}{(b^2+c^2)\alpha_1(\alpha_2\cos\chi'+\alpha_3\sin\chi')+(c^2+a^2)\beta_1(\beta_2\cos\chi'+\beta_3\sin\chi')+(a^2+b^2)\gamma_1(\gamma_2\cos\chi'+\gamma_3\sin\chi')},$$

and this is the expression for the square of the velocity of a wave, parallel to the inner mean line of the prism with its light-vector in the direction of this line. We have then the further case that the deviation is a minimum for symmetrical passage through the prism, when the wave has its plane of polarisation parallel to the line, in which the plane bisecting the angle of the prism cuts the normal section.

Total Reflection.

153. When a single reflecting surface of a crystal is all that can be obtained, recourse must be had to total reflection for the determination of the principal wave-velocities of the medium. This method of finding refractive indices was first employed by Wollaston* in 1802, but it is only in recent years that instruments for measuring the phenomenon have been brought to perfection or indeed that the theory in the case of crystalline substances has been worked out with any approach to completeness.

The crystal must be in contact with a more highly refracting medium, and this is effected, either by suspending it in a liquid†, or by placing its reflecting surface against the flat face of a solid substance, such as dense glass, in the form of a prism‡, a cylinder§ or a hemispherical lens‖, a drop of liquid being interposed between the solid and the crystal. If under these circumstances diffused monochromatic light be directed upon the surface and the reflected light be received in a telescope focussed on infinity, the field of view with a proper orientation of the crystal will be divided into parts of greater and less intensity by lines that mark the limits of total reflection corresponding to the two streams that the crystal is capable of transmitting. When the field is small, these lines are nearly straight.

Now each point of the focal plane of the telescope corresponds to a system of parallel rays reflected from the surface, the direction of which is given by the line joining the point to the optical centre C of the object-glass, and the lines separating the brighter and darker regions are the intersections of the focal plane with cones having their vertices at C and parallel to the limiting

* *Phil. Trans.* XCII. 381 (1802).

† F. Kohlrausch, *Wied. Ann.* IV. 1 (1878).

‡ Feussner, *Diss. Marburg.* (1882). F. Kohlrausch, *Wied. Ann.* XVI. 609 (1882). Liebisch, *Zeitschr. für Instrumentk.* IV. 185 (1884); V. 13 (1885).

§ Pulfrich, *Wied. Ann.* XXX. 193, 317, 487; XXXI. 724 (1887); XXXVI. 561 (1889).

‖ This is the form adopted in Abbe's refractometer: cf. Czapski, *Zeitschr. für Instrumentk.* X. 246, 269 (1890); *N. Jahrb. für Min. Beil-Bd.* VII. 175 (1891); (1892) I. 209. Pulfrich, *Zeitschr. für Kryst.* XXX. 568 (1899).

cones K, K' of total reflection at the point O, in which the optical axis of the telescope meets the surface of the crystal.

If then the telescope be so placed that its optical axis FO intersects one or other of these bounding lines, OF will be one of the generating lines of the cone K (or K') and the angle χ that the line makes with the plane of reflection of the ray OF is the angle between this plane and the tangent plane to the cone along OF. The ray OF is thus characterised by the angle χ and by the limiting angle of total reflection i_0 (or i_0').

The problem then is to express these angles in terms of the optical constants of the crystal and of the medium in contact with its face, and of the angles that define the crystallographic orientation of the surface and of the plane of reflection*.

We have seen in Chapter I, that the equations of the cones K and K' are obtained by equating to zero the discriminant of the equation

$$a_0 \tan^4 r + 4a_1 \tan^3 r + 6a_2 \tan^2 r + 4a_3 \tan r + a_4 = 0 \qquad (42),$$

that gives the directions of the refracted waves in terms of the angle of incidence, the angles defining the plane of incidence and the refracting surface and the constants of the contiguous media, and in Chapter XI we have found the form that this equation assumes in the case of uniaxal and biaxal crystals. We will now apply these results to certain special cases.

154. Let us take first the case of an uniaxal crystal, and suppose that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction-cosines of the optic axis referred to a system of rectangular coordinate axes, such that the reflecting surface is the plane of xy and the plane of incidence is that of xz.

Then the directions of the refracted waves are given by

$$\sin^2 r = a^2 \sin^2 i/\Omega^2$$

and

$$a_0 \tan^2 r + 2a_1 \tan r + a_2 = 0$$

where

$$a_0 = \{c^2 + (a^2 - c^2)\cos^2\alpha\}\sin^2 i - \Omega^2$$
$$a_1 = (a^2 - c^2)\cos\alpha\cos\gamma\sin^2 i$$
$$a_2 = \{c^2 + (a^2 - c^2)\cos^2\gamma\}\sin^2 i.$$

Hence the limiting angles of total reflection are given by

$$\sin^2 i_0 = \Omega^2/a^2 \qquad (43)$$

and

$$a_0 a_2 = a_1^2$$

or

$$\sin^2 i_0' = \frac{\Omega^2}{c^2}\,\frac{c^2 + (a^2 - c^2)\cos^2\gamma}{c^2 + (a^2 - c^2)(\cos^2\alpha + \cos^2\gamma)}$$
$$= \frac{\Omega^2}{c^2}\,\frac{c^2 + (a^2 - c^2)\cos^2\gamma}{a^2 - (a^2 - c^2)\cos^2\beta}$$
$$= \frac{\Omega^2}{c^2}\,\frac{c^2 + (a^2 - c^2)\cos^2\mu}{a^2 - (a^2 - c^2)\sin^2\mu\sin^2\theta} \qquad (44),$$

* Liebisch, *N. Jahrb. für Min.* (1885) I. 245; II. 181; (1886) II. 47: *Phys. Kryst.* p. 404.

if μ be the angle that the optic axis makes with the normal to the surface and θ be the angle between the plane of incidence E and the principal plane of the surface, H.

Let us now take rectangular axes (ξ, η, ζ), such that the plane of $\xi\eta$ is the reflecting surface, and that of $\xi\zeta$ is its principal plane; then if (ξ, η, ζ) be the coordinates with respect to these axes of a point distant ρ from the origin on the ray defined by the angles i_0 (or i_0') and θ, we have

$$\xi = \rho \sin i_0 \cos\theta, \qquad \eta = \rho \sin i_0 \sin\theta, \qquad \zeta = \rho\cos i_0,$$

and the equations of the cones K and K' become respectively

$$(a^2 - \Omega^2)(\xi^2 + \eta^2) - \Omega^2\zeta^2 = 0 \quad\ldots\ldots\ldots\ldots(45),$$

and

$$\left(\frac{a^2c^2}{a^2\cos^2\mu + c^2\sin^2\mu} - \Omega^2\right)\xi^2 + (c^2 - \Omega^2)\eta^2 - \Omega^2\zeta^2 = 0 \quad\ldots\ldots(46).$$

The difference of these equations gives

$$a^2\cos^2\mu\,\xi^2 + (a^2\cos^2\mu + c^2\sin^2\mu)\,\eta^2 = 0,$$

and therefore the cones in general have only their vertices in common, but if the optic axis is in the surface of the crystal $(\mu = \pi/2)$, the cones touch one another along the axis of ξ.

The cone, K, is a right circular cone with its axis perpendicular to the surface: hence for this cone the angle χ between the tangent plane along any ray and the plane of reflection for that ray is a right-angle.

In the case of the cone, K', the plane of incidence of the ray defined by the angles i_0' and θ is

$$\xi = \cot\theta \,.\, \eta$$

and the tangent plane to the cone along this ray is

$$\left(\frac{a^2c^2}{a^2\cos^2\mu + c^2\sin^2\mu} - \Omega^2\right)\cos\theta\,.\,\xi + (c^2 - \Omega^2)\sin\theta\,.\,\eta - \Omega^2\cot i_0'\,.\,\zeta = 0.$$

Hence

$$\cos\chi = c\,(a^2 - c^2)\sin^2\mu\sin\theta\cos\theta/D \quad\ldots\ldots\ldots\ldots(47)$$

where introducing the value of $\cot^2 i_0'$ from (44)

$$D^2 = a^2\{a^2c^2 - \Omega^2(a^2\cos^2\mu + c^2\sin^2\mu)\}\cos^2\theta$$
$$+ (c^2 - \Omega^2)(a^2\cos^2\mu + c^2\sin^2\mu)^2\sin^2\theta \quad\ldots\ldots(48)$$

and χ is only equal to $\pi/2$, if the optic axis of the crystal be perpendicular to the reflecting surface $(\mu = 0)$, or the optic axis having any direction, if the plane of incidence be parallel or perpendicular to the principal plane of the surface $(\theta = 0$ or $\pi/2)$.

Now when $\theta = 0$, we have

$$\sin^2 i_0 = \Omega^2/a^2, \qquad \sin^2 i_0' = \Omega^2\{a^{-2} + (c^{-2} - a^{-2})\cos^2\mu\} = \Omega^2/\sigma^2$$

where σ is the ray-velocity along the line of intersection of the surface of the crystal with its principal plane: and when $\theta = \pi/2$

$$\sin^2 i_0 = \Omega^2/a^2, \qquad \sin^2 i_0' = \Omega^2/c^2.$$

Hence from measurements of the limiting angles of total reflection, when the lines separating the regions of partially and totally reflected light are perpendicular to the plane of incidence, we can determine the principal wave-velocities and the angle that the optic axis of the crystal makes with the normal to the reflecting surface.

155. With biaxal crystals the most interesting cases are those in which equation (42) assumes the form

$$a_0 \tan^4 r + 6a_2 \tan^2 r + a_4 = 0,$$

as we then have two pairs of equal and opposite roots $\pm \tan r_1$ and $\pm \tan r_2$ and equality of the roots of either of these pairs can only occur when both are either zero or infinity. Hence since the value zero corresponds to normal incidence, we must have at the limit of total reflection infinity as the common value of the roots and the critical angles are given by

$$a_0 = 0.$$

These cases occur when either the reflecting surface is a plane of symmetry or its intersection with the plane of incidence is an axis of optical symmetry.

Let us suppose that the reflecting surface is parallel to the plane of the optic axes xz: then θ being the azimuth of the plane of reflection measured from yz,

$$a_0 = \left(\frac{\sin^2 i}{\Omega^2} b^2 - 1\right) \left\{\frac{\sin^2 i}{\Omega^2} (a^2 \cos^2 \theta + c^2 \sin^2 \theta) - 1\right\}.$$

Hence the limiting angles of total reflection are given by

$$\sin^2 i_0 = \Omega^2/b^2, \qquad \sin^2 i_0' = \Omega^2/(a^2 \cos^2 \theta + c^2 \sin^2 \theta)$$

and the two cones of limiting rays K, K' can be represented as distinct.

Taking new axes (ξ, η, ζ), such that the surface of the crystal is the plane of $\xi\eta$ and the plane of symmetry yz is that of $\xi\zeta$, the equations of the cones K, K' are respectively

$$(b^2 - \Omega^2)(\xi^2 + \eta^2) - \Omega^2\zeta^2 = 0$$

and
$$(a^2 - \Omega^2)\,\xi^2 + (c^2 - \Omega^2)\,\eta^2 - \Omega^2\zeta^2 = 0:$$

for the cone K, χ is always a right-angle, while for the cone K'

$$\cos \chi = \frac{(a^2 - c^2)\sin\theta\cos\theta}{\sqrt{c^2(c^2 - \Omega^2)\sin^2\theta + a^2(a^2 - \Omega^2)\cos^2\theta}}.$$

The other cases in which the reflecting surface is parallel to a plane of optical symmetry are obtained at once from this by changing a, b, c and x, y, z in cyclical order.

156. The present case is however of special interest both because the cones K and K' have four lines in common, namely those that lie in planes through the optic axes normal to the reflecting surface, and because the limiting rays of total reflection, in addition to being generating lines of these cones, form part of the surfaces of two other cones L and L' that correspond to refracted rays in the directions of the ray-axes, and that are determined by the singular tangent planes to the surface of wave-slowness. These tangent planes are perpendicular to the plane of the optic axes and pass through the common tangents to the ellipse and the circle, in which this plane cuts the surface.

Hence a tangent cylinder to the surface of wave-slowness of the crystal, perpendicular to the plane of the optic axes, touches the surface not only along this ellipse and circle, but also along the four circles of contact of these tangent planes, and the cones L, L' are determined by joining the centre O of the surface to the curves in which these singular tangent planes intersect the sphere with the same centre and of radius $1/\Omega$, that is the surface of wave-slowness for the outer medium.

Now since these tangent planes are perpendicular to the ray-axes, their equations are

$$a\sqrt{\frac{b^2-c^2}{a^2-c^2}}\cdot\xi \pm c\sqrt{\frac{a^2-b^2}{a^2-c^2}}\cdot\eta = 1,$$

and combining this with the equation

$$\xi^2+\eta^2+\zeta^2 = 1/\Omega^2,$$

so as to form an homogeneous equation of the second degree, we obtain as the equations of the cones L and L'

$$\Omega^2(\xi^2+\eta^2+\zeta^2) = \left\{a\sqrt{\frac{b^2-c^2}{a^2-c^2}}\,\xi \pm c\sqrt{\frac{a^2-b^2}{a^2-c^2}}\,\eta\right\}^2.$$

Each of the cones L and L' has a line in common with each of the cones K and K': that common to K and L or L' is in the plane through the corresponding ray-axis normal to the reflecting surface; that common to K' and L or L' is in the central plane perpendicular to the reflecting surface through the point of contact of the corresponding singular tangent plane with the ellipse, in which the plane of the optic axes cuts the surface of wave-slowness of the crystal. It is clear that only the part of the cones L and L' between these two lines give limiting rays of total reflection: all rays on these cones outside this portion are totally reflected, since the perpendiculars on the reflecting surface from the points, in which they meet the sphere of radius $1/\Omega$, neither touch nor intersect the surface of wave-slowness of the crystal.

It is now easy to determine the nature of the phenomenon of total reflection, when the plane of incidence is nearly parallel to one of the optic axes. Let N be the foot of the perpendicular on the reflecting surface from

the point in which the incident wave-normal meets the sphere of radius $1/\Omega$, and let us suppose first of all that the plane of incidence passes through the optic axis OA: then if this direction meets the common tangent to the ellipse and circle in the point B, it is clear that total reflection does not commence until N falls outside OB, for until this occurs the perpendicular cuts

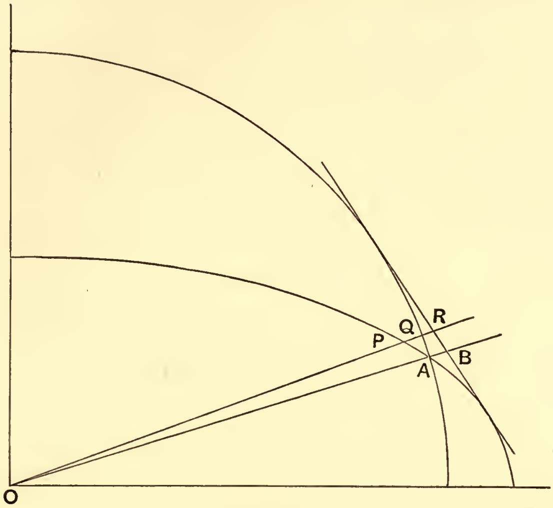


Fig. 37.

the surface of wave-slowness in two points within the crystal; only when N is at A, two of the infinite number of streams into which the light is divided are at the limit of total reflection, viz.: those polarised in planes parallel and perpendicular to the surface of the crystal.

Next let the trace of the plane of incidence on the surface be $OPQR$, P, Q, R being the points in which it meets the ellipse, the circle and the common tangent to these curves respectively. When N falls between O and P, the perpendicular meets the surface of wave-slowness at two points within the crystal, there are two refracted waves and no total reflection; when N is between P and Q, the perpendicular intersects the surface of wave-slowness at only one point within the crystal and one wave is totally reflected: when N is between Q and R, the surface of wave-slowness is again cut by the perpendicular in two points, where the surface bends over, so that there are again two refracted waves and no total reflection; and finally when N is beyond R, total reflection is complete, as the perpendicular is entirely outside the surface of wave-slowness. The appearance presented will consequently be that represented in fig. 38*.

157. When the reflecting surface is parallel to an axis of optical symmetry, the limiting angles of total reflection are determined by $a_0 = 0$ if

* Soret, *C. R.* cvii. 479 (1888); *Zeitschr. für Kryst.* xv. 45 (1889). Mallard, *J. de Phys.* (2) v. 389 (1886). W. Kohlrausch, *Wied. Ann.* vi. 113 (1879).

the plane of incidence contain this axis, and a_0 is then the product of two factors that are linear functions of $\sin^2 i$. On the other hand, if the plane of incidence be perpendicular to the axis of symmetry, equation (42) takes the form

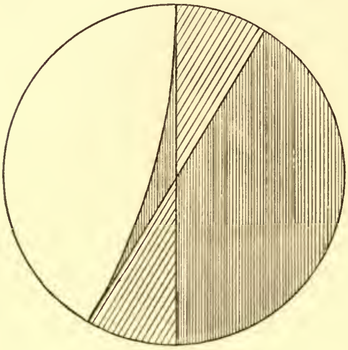


Fig. 38.

$$(a_0 \tan^2 r + a_2)(A_0 \tan^2 r + 2A_1 \tan r + A_2) = 0,$$

wherein a_0 is a linear function of $\sin^2 i$, and the critical angles are then given by

$$a_0 = 0 \quad \text{and} \quad A_0 A_2 = A_1^2.$$

Thus if the surface of the crystal be parallel to the axis of x and if μ be the angle between the axis of y and the normal to the face, we have

$$\sin i_0 = \Omega/b\,;\ \sin i_0' = \Omega/c$$

when the plane of incidence passes through x, and

$$\sin i_0 = \Omega/a, \quad \sin^2 i_0' = \Omega^2 \{c^{-2} + (b^{-2} - c^{-2}) \cos^2 \mu\}$$

when the plane of incidence is parallel to the plane of symmetry yz.

Hence from measurements in these planes, which are experimentally determined from the fact that in these cases the lines bounding the regions of total reflection are perpendicular to the planes of reflection, we can find the three principal wave-velocities and the orientation of the face of the crystal.

158. Let us now suppose that the reflecting surface is neither parallel nor perpendicular to a plane of symmetry of the crystal*, and that it cuts the three planes of symmetry in the lines ON_a, ON_b, ON_c, the points N_a, N_b, N_c being on circular sections of the surface of wave-slowness in these planes, the radii of which are $1/a$, $1/b$, $1/c$ respectively.

Then N_a, N_b, N_c are points on the tangent cylinder to the surface perpendicular to the face of the crystal, and the corresponding critical angles are given by

$$\sin i_{0a} = \Omega/a, \quad \sin i_{0b} = \Omega/b, \quad \sin i_{0c} = \Omega/c.$$

Now the greatest and least radii of the section of the surface of wave-slowness made by the face of the crystal are $1/c$ and $1/a$ respectively, and $1/b$ will be the greatest radius of the inner curve of the section or the least radius of the outer curve, according as the section cuts the plane of xz within or without the angle between the optic axes that is bisected by the axis of x.

* Soret, *C. R.* CVII. 176, 479 (1888); *Arch. de Genève* (3) XX. 263 (1888). Perrot, *C. R.* CVIII. 137 (1889); *Arch. de Genève* (3) XXI. 113 (1889). Hecht, *N. Jahrb. für Min. Beil-Bd.* VI. 241 (1889).

It follows then that measurements of the greatest and the least critical angles, that can be obtained by varying the azimuth of the plane of reflection, give the extreme wave-velocities of the crystal. But we cannot determine without having recourse to other considerations, whether the mean principal wave-velocity b corresponds to the greater critical angle of the inner cone or to the less critical angle of the outer cone of rays of total reflection.

This ambiguity may in general be removed, as Viola* has pointed out, by a determination of the polarisation of the rays: for the planes of optical symmetry of the crystal are the planes through the limiting rays, that determine the principal wave-velocities, perpendicular to their respective planes of polarisation, and if A, B, C be the corresponding planes of reflection, the angles α, β, γ between the planes of optical symmetry and the crystalline surface are given by

$$\cos^2\alpha = \cot \widehat{AB} \,.\, \cot \widehat{CA}, \quad \cos^2\beta = \cot \widehat{BC} \,.\, \cot \widehat{AB}, \quad \cos^2\gamma = \cot \widehat{CA} \,.\, \cot \widehat{BC} \qquad \text{.........(49).}$$

On the other hand these angles may easily be determined by an analyser placed in the eye-piece of the observing telescope, and the agreement of the measured with the calculated angles will indicate the plane of reflection of the limiting ray that gives the wave-velocity b.

Cornu† has suggested another method of procedure. If ω be the wave-velocity calculated from the fourth angle, we have the relation

$$\omega^{-2} = a^{-2}\cos^2\alpha + b^{-2}\cos^2\beta + c^{-2}\cos^2\gamma,$$

where $\cos^2\alpha$, $\cos^2\beta$, $\cos^2\gamma$ have the values (49), and the verification of this formula will decide whether the proper angles have been selected for the calculation of b, α, β and γ.

* *Zeitschr. für Kryst.* XXXI. 40 (1899); *Rend. Lincei* (5) VIII. [1] 276 (1899).

† *J. de Phys.* (4) I. 136 (1902).

CHAPTER XIII.

CRYSTALLINE REFLECTION AND REFRACTION.

159. BEFORE considering the question of the intensity of the light reflected and refracted at the surface of a crystal, it is necessary to obtain the differential equations of the polarisation-vector in crystalline media and to determine the surface conditions that must be satisfied at the confines of such substances. This may be done, as in the case of isotropic media, by the application of the principle of interference.

According to Fresnel's laws of double refraction, the polarisation-vector of any wave is in the direction of one of the axes of the central section of the ellipsoid of polarisation parallel to the plane of the wave, and the corresponding wave-velocity ω is given by the reciprocal of that axis. Hence if the equation of the ellipsoid be

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy = 1 \text{..........(1)},$$

we obtain as in § 122

$$\left.\begin{aligned}(a_{11} - \omega^2)\,\alpha + a_{12}\beta + a_{31}\gamma &= Fl \\ a_{12}\alpha + (a_{22} - \omega^2)\,\beta + a_{23}\gamma &= Fm \\ a_{31}\alpha + a_{23}\beta + (a_{33} - \omega^2)\,\gamma &= Fn\end{aligned}\right\} \text{.....................(2)},$$

and

$$F = (a_{11}\alpha + a_{12}\beta + a_{31}\gamma)\,l + (a_{12}\alpha + a_{22}\beta + a_{23}\gamma)m + (a_{31}\alpha + a_{23}\beta + a_{33}\gamma)n \text{...(3)},$$

α, β, γ being the direction-cosines of the polarisation-vector, and l, m, n those of the normal to the wave-front.

Now the principle of interference is expressed by

$$u = \Sigma\alpha D,\;\; v = \Sigma\beta D,\;\; w = \Sigma\gamma D,\;\; D = A\exp\{\iota\kappa\,(lx + my + nz - \omega t)\}\text{...(4)},$$

where $\kappa = 2\pi/\lambda$: whence

$$\ddot{u} = -\Sigma\kappa^2\alpha\omega^2 D,\;\; \ddot{v} = -\Sigma\kappa^2\beta\omega^2 D,\;\; \ddot{w} = -\Sigma\kappa^2\gamma\omega^2 D,$$

and from (2)

$$\alpha\omega^2 = (a_{11}\alpha + a_{12}\beta + a_{31}\gamma)(l^2 + m^2 + n^2) - Fl,$$

and two similar equations; whence substituting for $\alpha\omega^2$, $\beta\omega^2$ and $\gamma\omega^2$, and eliminating the direction-cosines, we find at once

$$(\ddot{u}, \ddot{v}, \ddot{w}) = \nabla^2 \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right) \Phi$$

$$- \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \left(\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial v} + \frac{\partial}{\partial z}\frac{\partial \Phi}{\partial w}\right) \qquad (5),$$

where

$$2\Phi = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + 2a_{23}vw + 2a_{31}wu + 2a_{12}uv \qquad (6).$$

These equations, as in the case of those relating to isotropic media, may be put into the more convenient form,

$$\dot{D} = - \operatorname{curl} \varpi, \quad \dot{\varpi} = \operatorname{curl} E \qquad (7),$$

where the components of E are given by

$$(E_1, E_2, E_3) = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right) \Phi \qquad (8).$$

As regards the vector E, it is clearly parallel to the normal to the ellipsoid of polarisation at the point in which the polarisation-vector meets it, and is therefore perpendicular to the ray corresponding to the given wave, and if χ be the angle between D and E

$$\cos\chi = (\alpha E_1 + \beta E_2 + \gamma E_3)/E = (a_{11}\alpha^2 + a_{22}\beta^2 + \ldots)\, D/E$$

$$= \omega^2 D/E.$$

Equations (7) having the same form as those that relate to isotropic bodies, the surface conditions will be the same as those that hold at the interface of such media. Thus the interface being the plane $x = 0$ we have that

$$\varpi_2, \ E_3, \ \varpi_3, \ E_2$$

are continuous across the surface, to which we may add the further conditions, clearly dependent upon the former, that u and ϖ_1 are also continuous.

160. We can now determine the significance of the ray in the theory of double refraction. Proceeding as in § 109 we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int (\varpi^2 + 2\Phi)\, dT = \int \{(\varpi_2 E_3 - \varpi_3 E_2) \cos nx$$

$$+ (\varpi_3 E_1 - \varpi_1 E_3) \cos ny + (\varpi_1 E_2 - \varpi_2 E_1) \cos nz\}\, dS \ldots (9),$$

the integration being extended over a certain region T, dS being an element of the surface of T, and n the normal to dS drawn outwards, and this equation may be regarded as representing the change of energy within the space T that results from a flow of energy across its surface. We see then that the direction of this flow is perpendicular to the vectors E and ϖ and is therefore along the ray.

161. Since the three vectors D, ϖ, and E are connected merely by geometrical relations, we may take which we please as characteristic of a stream of light, and we shall in the remainder of this chapter employ the light-vector ϖ, as by so doing the calculations are somewhat simplified.

Let the plane of incidence be taken as that of xz, the reflecting surface being the plane of yz and the medium in which the light is incident lying on the side of negative x: then since the vector ϖ is in the plane of the wave, we may write

$$(\varpi_1, \varpi_2, \varpi_3) = (n, k, -l) D \exp \{\iota (lx + nz + st)\} \quad \text{.........(10)},$$

where $$k = 2\pi \tan \phi/\lambda, \quad D = \lambda \cos \phi \,.\, A/(2\pi) \quad \text{..................(11)},$$

ϕ being the angle that the vector makes with the plane of incidence. Now the vector ϖ being independent of y, we have from equations (7)

$$\left.\begin{aligned} \ddot{\varpi}_1 &= -\frac{\partial}{\partial z}\left\{a_{12}\frac{\partial \varpi_2}{\partial z} - a_{22}\left(\frac{\partial \varpi_1}{\partial z} - \frac{\partial \varpi_3}{\partial x}\right) - a_{23}\frac{\partial \varpi_2}{\partial x}\right\} \\ \ddot{\varpi}_2 &= \frac{\partial}{\partial z}\left\{a_{11}\frac{\partial \varpi_2}{\partial z} - a_{12}\left(\frac{\partial \varpi_1}{\partial z} - \frac{\partial \varpi_3}{\partial x}\right) - a_{31}\frac{\partial \varpi_2}{\partial x}\right\} \\ &\quad -\frac{\partial}{\partial x}\left\{a_{31}\frac{\partial \varpi_2}{\partial z} - a_{23}\left(\frac{\partial \varpi_1}{\partial z} - \frac{\partial \varpi_3}{\partial x}\right) - a_{33}\frac{\partial \varpi_2}{\partial x}\right\} \\ \ddot{\varpi}_3 &= \frac{\partial}{\partial x}\left\{a_{12}\frac{\partial \varpi_2}{\partial z} - a_{22}\left(\frac{\partial \varpi_1}{\partial z} - \frac{\partial \varpi_3}{\partial x}\right) - a_{23}\frac{\partial \varpi_2}{\partial x}\right\} \end{aligned}\right\} \quad \text{.........(12)};$$

whence, substituting the values of ϖ_1, ϖ_2, ϖ_3, we have

$$\left.\begin{aligned} s^2 - a_{22}(l^2 + n^2) &= k(a_{23}l - a_{12}n) \\ k(s^2 - a_{33}l^2 - a_{11}n^2 + 2a_{31}ln) &= (l^2 + n^2)(a_{23}l - a_{12}n) \end{aligned}\right\} \quad \text{.........(13)},$$

which give

$$\{s^2 - a_{22}(l^2 + n^2)\}\{s^2 - a_{33}l^2 - a_{11}n^2 + 2a_{31}ln\} = (a_{23}l - a_{12}n)^2(l^2 + n^2) \quad \text{......(14)},$$

and $$k = \frac{(a_{23}l - a_{12}n)(l^2 + n^2)}{s^2 - a_{33}l^2 - a_{11}n^2 + 2a_{31}ln} = \frac{s^2 - a_{22}(l^2 + n^2)}{a_{23}l - a_{12}n} \quad \text{............(15)}.$$

Similar equations with b written for a apply to the second medium.

The first of these two equations determines the value of l when n and s are given: since it is of the fourth degree, it is evident that there are four possible values corresponding to waves of given period, the traces of which on the interface of the media travel at a given rate. Two of these waves approach the surface and two leave it, and the question which of the values of l correspond to the receding, that is the reflected or refracted waves, is decided by the position of the corresponding rays as determined by Huygens' construction.

The quantity l having been thus determined, equation (15) gives the corresponding value of k, and thus the directions and the polarisations of the reflected and the refracted waves are known.

162. Let the suffixes ($_1$), ($_2$), the accents ($'$), ($''$) and the suffixes ($_0$), ($_e$) refer respectively to the incident and to the reflected waves in the first medium and to the refracted waves in the second medium, and suppose that there are no incident waves in the latter medium.

Then introducing the boundary conditions, we have from the continuity of ϖ_3 when $x=0$

$$l_1D_1 + l_2D_2 + l'D' + l''D'' = l_0D_0 + l_eD_e \quad\text{..............(16)},$$

from the continuity of E_2 or of ϖ_1

$$D_1 + D_2 + D' + D'' = D_0 + D_e \quad\text{.....................(17)},$$

from the continuity of ϖ_2 or of u

$$k_1D_1 + k_2D_2 + k'D' + k''D'' = k_0D_0 + k_eD_e \quad\text{..............(18)},$$

while the continuity of E_3 gives

$$\Sigma\,\{a_{31}kn - a_{23}(l^2+n^2) - a_{33}kl\}\,D = \Sigma\,\{b_{31}kn - b_{23}(l^2+n^2) - a_{33}kl\}\,D \quad\text{...(19)},$$

the summations extending to the four waves in the first medium and to the two waves in the second medium respectively.

The last of these equations may be put into a different form, that is somewhat more convenient. The direction-cosines of the vector E are proportional to E_1, E_2, E_3 and from (13) we have

$$E_1 = kna_{11} - (l^2+n^2)\,a_{12} - kla_{31} = (ks^2 + lE_3)/n,$$

$$E_2 = kna_{12} - (l^2+n^2)\,a_{22} - kla_{23} = -s^2,$$

$$E_3 = kna_{31} - (l^2+n^2)\,a_{23} - kla_{33};$$

and since the ray is perpendicular to the vectors E and ϖ, its direction-cosines are

$$(kE_3 + lE_2)/R, \quad -(lE_1 + nE_3)/R, \quad (nE_2 - kE_1)/R,$$

where
$$R^2 = (kE_3 + lE_2)^2 + (lE_1 + nE_3)^2 + (nE_2 - kE_1)^2$$
$$= \{(l^2+n^2)\,E_3{}^2 + 2kls^2E_3 + (k^2+n^2)\,s^4\}\,(l^2+k^2+n^2)/n^2,$$

and the angle between the ray and the wave-normal is given by

$$\cos\chi = \frac{l\,(kE_3 + lE_2) + n\,(nE_2 - kE_1)}{R\sqrt{l^2+n^2}}$$

$$= -\frac{s^2n\sqrt{l^2+k^2+n^2}}{\sqrt{l^2+n^2}\,\sqrt{(l^2+n^2)\,E_3{}^2 + 2klE_3s^2 + (k^2+n^2)\,s^4}}.$$

We thus obtain

$$\tan\chi = -\frac{(l^2+n^2)\,E_3 - kls^2}{s^2n\sqrt{l^2+k^2+n^2}},$$

and
$$E_3 = -s^2\left\{\frac{kl}{l^2+n^2}+\frac{n\sqrt{l^2+k^2+n^2}}{l^2+n^2}\tan\chi\right\},$$

and (19) becomes

$$\Sigma\{kl+n\sqrt{l^2+k^2+n^2}\,.\tan\chi\}\frac{D}{l^2+n^2}$$
$$=\Sigma\{kl+n\sqrt{l^2+k^2+n^2}\,.\tan\chi\}\frac{D}{l^2+n^2}\;\ldots\ldots(19').$$

Introducing the angles of incidence, reflection and refraction, and the values of k and D from (11), the equations (16)...(19′) become

$$\Sigma\cos i\cos\phi A=\Sigma\cos r\cos\phi B\;\ldots\ldots(20),$$
$$\Sigma\sin i\cos\phi A=\Sigma\sin r\cos\phi B\;\ldots\ldots(21),$$
$$\Sigma\sin\phi A=\Sigma\sin\phi B\;\ldots\ldots(22),$$

and

$$\Sigma\{(a_{31}\sin i-a_{33}\cos i)\sin\phi-a_{23}\cos\phi\}\frac{A}{\sin i}$$
$$=\Sigma\{(b_{31}\sin r-b_{33}\cos r)\sin\phi-b_{23}\cos\phi\}\frac{B}{\sin r}\;\ldots\ldots(23),$$

or

$$\Sigma\sin i(\cos i\sin\phi+\sin i\tan\chi)A=\Sigma\sin r(\cos r\sin\phi+\sin r\tan\chi)B\ldots(23'),$$

B representing the amplitude of the vibrations in the second medium.

163. As a first application of these formulæ, let us take the case in which the first medium is isotropic. Then

$$a_{11}=a_{22}=a_{33}=\Omega^2,\quad a_{12}=a_{23}=a_{31}=0,$$

Ω being the wave-velocity in the medium.

The values of l for this medium are $\pm l$ and k becomes indeterminate, but introducing the components of the vector ϖ parallel and perpendicular to the plane of incidence, and calling the amplitude of these components G and F for the incident waves and G' and F' for the reflected wave, we have

$$(G-G')\cos i=\cos r_0\cos\phi_0B_0+\cos r_e\cos\phi_eB_e\;\ldots\ldots(24),$$
$$(G+G')\sin i=\sin r_0\cos\phi_0B_0+\sin r_e\cos\phi_eB_e\;\ldots\ldots(25),$$
$$F+F'=\sin\phi_0B_0+\sin\phi_eB_e\;\ldots\ldots(26),$$

$$\Omega^2(F-F')\frac{\cos i}{\sin i}=\{(b_{33}\cos r_0-b_{31}\sin r_0)\sin\phi_0+b_{23}\cos\phi_0\}\frac{B_0}{\sin r_0}$$
$$+\{(b_{33}\cos r_e-b_{31}\sin r_e)\sin\phi_e+b_{23}\cos\phi_e\}\frac{B_e}{\sin r_e}\;\ldots(27),$$

or

$$(F-F')\sin i\cos i=\sin r_0(\cos r_0\sin\phi_0+\sin r_0\tan\chi_0)B_0$$
$$+\sin r_e(\cos r_e\sin\phi_e+\sin r_e\tan\chi_e)B_e\;\ldots\ldots(27'),$$

where

$$\tan\phi_0 = \frac{\omega_0^2 - b_{22}}{b_{23}\cos r_0 - b_{12}\sin r_0} = \frac{b_{23}\cos r_0 - b_{12}\sin r_0}{\omega_0^2 - b_{33}\cos^2 r_0 - b_{11}\sin^2 r_0 + 2b_{31}\sin r_0\cos r_0},$$

with a similar expression for $\tan\phi_e$, ω_0 and ω_e being the propagational speeds of the two waves within the crystal.

For the further consideration of these equations it is convenient, in accordance with a plan due to MacCullagh*, to consider first of all the special cases in which only a single wave of amplitude unity exists in the second medium. Using the suffixes $(_0)$ and $(_e)$ to distinguish the cases in which $B_0 = 1$, $B_e = 0$ and $B_0 = 0$, $B_e = 1$, we have

$$(G_0 - G_0')\cos i = \cos r_0 \cos\phi_0,$$

$$(G_0 + G_0')\sin i = \sin r_0 \cos\phi_0,$$

$$F_0 + F_0' = \sin\phi_0,$$

$$(F_0 - F_0')\sin i \cos i = \sin r_0(\cos r_0 \sin\phi_0 + \sin r_0 \tan\chi_0),$$

with similar equations for the second case. We thus have eight equations, from which the eight quantities F_0, $G_0 \ldots F_e'$, G_e' may be determined, and then the ratios F_0/G_0, F_e/G_e give the azimuths of the vector ϖ with respect to the plane of incidence, for which the wave $(_e)$ and the wave $(_0)$ vanish within the crystal, while F_0'/G_0', F_e'/G_e' give the corresponding azimuths in the reflected stream of light.

If now B_0 and B_e, instead of being either 0 or 1, have any values, it follows that

$$F = F_0B_0 + F_eB_e, \qquad G = G_0B_0 + G_eB_e,$$

$$F' = F_0'B_0 + F_e'B_e, \qquad G' = G_0'B_0 + G_e'B_e,$$

whence if F and G be given, F', G', B_0, B_e may be determined; for

$$B_0 = \frac{FG_e - GF_e}{F_0G_e - G_0F_e}, \quad B_e = \frac{GF_0 - FG_0}{F_0G_e - G_0F_e},$$

and $$(F_0G_e - G_0F_e)F' = F(F_0'G_e - F_e'G_0) + G(F_0F_e' - F_eF_0'),$$

$$(F_0G_e - G_0F_e)G' = F(G_0'G_e - G_e'G_0) + G(F_0G_e' - F_eG_0').$$

Also if ϕ, ϕ' be the azimuths of the vector ϖ with respect to the plane of incidence in the case of the incident and reflected streams

$$\tan\phi' = \frac{F_0'B_0 + F_e'B_e}{G_0'B_0 + G_e'B_e},$$

wherein the ratio B_0/B_e is determined from

$$\tan\phi = \frac{F_0B_0 + F_eB_e}{G_0B_0 + G_eB_e}.$$

Now $\tan\phi'$ is independent of $\tan\phi$, if $F_0'/G_0' = F_e'/G_e'$, and this condition

* *Collected Works*, p. 98.

determines a special value of the angle of incidence, that is called the polarising angle of the crystalline surface with respect to the isotropic medium. A stream of common light incident at this angle is reflected as a plane polarised stream; for we may represent the incident light by two independent streams of the like intensities polarised in perpendicular planes, and since each of the streams incident at the polarising angle gives the same azimuth for the plane of polarisation of the reflected stream, it follows that the stream of common light will give a reflected stream plane polarised in this azimuth given by

$$\tan\phi = F_0'/G_0' = F_e'/G_e'.$$

164. Suppose now that the crystalline medium is uniaxal, and that the direction-cosines of its optic axis are p_1, p_2, p_3, then the equation of the ellipsoid of polarisation is

$$a^2(x^2+y^2+z^2)+(c^2-a^2)(p_1x+p_2y+p_3z)^2=1,$$

whence

$$b_{mm}=a^2+(c^2-a^2)p_m^2, \quad b_{mn}=(c^2-a^2)p_m p_n.$$

When the refracted wave is ordinary, the propagational speed is constant and equal to a and we have

$$\sin r_0=\frac{a}{\Omega}\sin i, \qquad \tan\phi_0=\frac{p_2}{p_1\sin r_0-p_3\cos r_0},$$

and $\chi_0=0$, since the ordinary ray coincides with its wave-normal. Thus the ordinary uniradial system is determined from

$$(G_0-G_0')\cos i=\cos r_0\cos\phi_0, \qquad F_0+F_0'=\sin\phi_0$$

$$(G_0+G_0')\sin i=\sin r_0\cos\phi_0, \qquad (F_0-F_0')\sin i\cos i=\sin r_0\cos r_0\sin\phi_0.$$

The extraordinary wave-velocity is given by

$$\omega_e^2=c^2+(a^2-c^2)(p_1\cos r_e+p_3\sin r_e)^2$$

and

$$\sin^2 r_e=\frac{\sin^2 i}{\Omega^2}\{c^2+(a^2-c^2)(p_1\cos r_e+p_3\sin r_e)\}^2$$

$$\tan\phi_e=\frac{p_3\cos r_e-p_1\sin r_e}{p_2},$$

whence the extraordinary uniradial system is determined from

$$(G_e-G_e')\cos i=\cos r_e\cos\phi_e, \qquad (G_e+G_e')\sin i=\sin r_e\cos\phi_e$$

$$F_e+F_e'=\sin\phi_e,$$

$$\Omega^2(F_e-F_e')\frac{\cos i}{\sin i}$$

$$=\frac{1}{\sin r_e}[a^2\cos r_e\sin\phi_e+(c^2-a^2)p_3\{(p_3\cos r_e-p_1\sin r_e)\sin\phi_e+p_2\cos\phi_e\}].$$

When the plane of incidence is the principal plane of the reflecting

surface, $p_2=0$, $p_1=\sin\mu$, $p_3=\cos\mu$, where μ is the angle that the optic axis makes with the surface. Then

$$\phi_0=0, \qquad \phi_e=\pi/2,$$

and for the ordinary uniradial system

$$(G_0-G_0')\cos i=\cos r_0, \qquad (G_0+G_0')\sin i=\sin r_0,$$

$$F_0+F_0'=F_0-F_0'=0,$$

while for the extraordinary uniradial system

$$G_e-G_e'=G_e+G_e'=0, \qquad F_e+F_e'=1$$

$$\Omega^2(F_e-F_e')\frac{\cos i}{\sin i}=\frac{1}{\sin r_e}\{a^2\cos r_e+(c^2-a^2)\cos\mu\cos(\mu+r_e)\}$$

where $$\Omega^2\sin^2 r_e=\sin^2 i\,\{c^2-(c^2-a^2)\sin^2(\mu+r_e)\}.$$

Thus the vector ϖ is parallel to the plane of incidence in the ordinary uniradial system, and perpendicular to it in the extraordinary system.

In this particular case, the polarising angle is determined by the condition

$$F_e'G_0'=0,$$

or since G_0' can only be zero, if $i=r_0$, by the condition

$$F_e'=0.$$

In this case, $F_e=1$, so that if I be the polarising angle, R_e the corresponding angle of refraction for the extraordinary stream

$$\Omega^2\frac{\cos I}{\sin I}=\frac{1}{\sin R_e}\{a^2\cos R_e+(c^2-a^2)\cos\mu\cos(\mu+R_e)\},$$

where

$$\Omega^2\sin^2 R_e=\sin^2 I\,\{c^2-(c^2-a^2)\sin^2(\mu+R_e)\};$$

whence eliminating R_e between these equations

$$\sin^2 I=\frac{\Omega^2(\Omega^2-a^2\sin^2\mu-c^2\cos^2\mu)}{\Omega^4-a^2c^2}.$$

165. Another interesting case is that of reflection at a twin surface of a crystal*. Taking the surface as the plane $x=0$, the only difference between the two media is then that which corresponds to a rotation through 180° about the axis of x perpendicular to the twin plane.

Let us assume that there is a plane perpendicular to the twin surface, with respect to which each medium is symmetrical, and let us consider only the two cases in which the plane of incidence is parallel and perpendicular respectively to this plane of symmetry.

* Lord Rayleigh, *Phil. Mag.* (5) xxvi. 246 (1888).

When the plane of incidence is coincident with the plane of symmetry the axis of y is a principal axis, and we have

$$a_{12} = a_{23} = 0, \qquad b_{12} = b_{23} = 0,$$

$$b_{11} = a_{11}, \quad b_{22} = a_{22}, \quad b_{33} = a_{33}, \quad b_{31} = -a_{31}.$$

Then the values of l are determined from

$$\{s^2 - a_{22}(l^2 + n^2)\}\{s^2 - a_{33}l^2 - a_{11}n^2 \pm 2a_{31}ln\} = 0,$$

the upper and lower sign corresponding to the upper and lower medium respectively and $k = 0$ or ∞ according as

$$s^2 = a_{22}(l^2 + n^2) \quad \text{or} \quad s^2 = a_{33}l^2 + a_{11}n^2 \mp 2a_{31}ln.$$

When the vector ϖ is in the plane of incidence, $s^2 = a_{22}(l^2 + n^2)$ and $l' = -l$, $l_0 = l$, and the conditions (16) and (17) give

$$D_1 - D' = D_0, \qquad D_1 + D' = D_0,$$

$\therefore$ $D' = 0$, and there is no reflected wave.

Again, when ϖ is perpendicular to the plane of incidence,

$$s^2 = a_{33}l^2 + a_{11}n^2 \mp 2a_{31}ln,$$

and the conditions (18) and (19) give

$$D_2 + D'' = D_e, \quad (a_{33}l_2 - a_{31}n)D_2 + (a_{33}l'' - a_{31}n)D'' = (a_{33}l_e + a_{31}n)D_e,$$

whence $$a_{33}(l'' - l_2)D'' = \{a_{33}(l_e - l_2) + 2a_{31}n\}D_e.$$

But $$s^2 = a_{33}l_e^2 + a_{11}n^2 + 2a_{31}l_e n = a_{33}l_2^2 + a_{11}n^2 - 2a_{31}l_2 n,$$

$$\therefore\ a_{33}(l_e^2 - l_2^2) + 2a_{31}n(l_e + l_2) = 0,$$

or $$a_{33}(l_e - l_2) + 2a_{31}n = 0.$$

Hence D'' vanishes and in this case again there is no reflected wave. It follows then that when light, whether common or polarised in any manner, is incident in the plane of symmetry there is no reflection at the surface of the twin plane.

Next let the plane of incidence be perpendicular to the plane of symmetry, then the axis of z is a principal axis, and we have

$$a_{31} = a_{23} = 0, \qquad b_{31} = b_{23} = 0, \qquad b_{12} = -a_{12}$$

$$b_{11} = a_{11}, \qquad b_{22} = a_{22}, \qquad b_{33} = a_{33},$$

and consequently for both media

$$\{s^2 - a_{22}(l^2 + n^2)\}\{s^2 - a_{33}l^2 - a_{11}n^2\} = (l^2 + n^2)n^2 a_{12}^2 \quad \text{.........(28)},$$

while $$k = \frac{a_{22}(l^2 + n^2) - s^2}{\pm a_{12}n} \quad \text{..........................(29)},$$

the upper and lower sign referring to the upper and lower medium respectively.

The values of l are thus $\pm l$ and $\pm l'$ for both media, and the corresponding values of k are k and k' for the upper, and $-k$, $-k'$ for the lower medium, and taking the positive values of l to refer to the incident waves, we must also take the positive values for the refracted waves.

The boundary conditions then give

$$lD_1 + l'D_2 - lD' - l'D'' = lD_0 + l'D_e$$
$$D_1 + D_2 + D' + D'' = D_0 + D_e$$
$$kD_1 + k'D_2 + kD' + k'D'' = -kD_0 - k'D_e$$
$$klD_1 + k'l'D_2 - klD' - k'l'D'' = -klD_0 - k'l'D_e,$$

or writing $K = k'/k$, $L = l'/l$,

$$\left.\begin{aligned} D_1 - D' + \quad L(D_2 - D'') &= D_0 + LD_e \\ D_1 + D' + \qquad D_2 + D'' &= D_0 + D_e \\ D_1 + D' + \quad K(D_2 + D'') &= -D_0 - KD_e \\ D_1 - D' + KL(D_2 - D'') &= -D_0 - KLD_e \end{aligned}\right\} \text{............(30)}.$$

Solving these equations we obtain

$$\left.\begin{aligned} D' &= \frac{K(L-1)}{(L-K)(1-KL)}\{(L+1)D_1 + L(K+1)D_2\} \\ D'' &= -\frac{L-1}{(L-K)(1-KL)}\{(K+1)D_1 + K(L+1)D_2\} \end{aligned}\right\} \text{...(31)}.$$

166. We will now introduce the simplification, that the doubly refracting energy is small. Then a_{11}, a_{22}, a_{33} are nearly equal and a_{12} is small, under which circumstances l and l' are nearly equal, so that $L \fallingdotseq 1$, and

$$\left.\begin{aligned} D' &\fallingdotseq \frac{k'(l'-l)}{l(k-k')^2}\{2kD_1 + (k+k')D_2\} \\ D'' &\fallingdotseq -\frac{k(l'-l)}{l(k-k')^2}\{(k+k')D_1 + 2k'D_2\} \end{aligned}\right\} \text{...............(32)}.$$

If now the waves D_1, D_2 be regarded as due to a stream of light from an isotropic medium passing into the crystal through a surface parallel to the twin plane, under the condition (such as gradual transition) that no light is lost by reflection, and if the optical power of the medium be so nearly the same as that of the crystal that the refraction may be neglected, then denoting the amplitude of the components of the vector ϖ perpendicular and parallel to the plane of incidence by F and G for the incident stream and by F' and G' for the reflected stream, we have

$$\left.\begin{aligned} F &= kD_1 + k'D_2, \quad G = \sqrt{l^2+n^2}\,(D_1 + D_2) \\ F' &= kD' + k'D'', \quad G' = \sqrt{l^2+n^2}\,(D' + D'') \end{aligned}\right\} \text{............(33)},$$

and in these equations we may identify $D_1 \ldots D''$ with the quantities in equations (32), provided the thickness of the plate is so small that its effect may be neglected.

We have then

$$F' = -\frac{kk'(l'-l)}{l(k'-k)}(D_1 + D_2) = -\frac{kk'(l'-l)}{l(k'-k)}\frac{G}{\sqrt{l^2+n^2}},$$

$$G' = \sqrt{l^2+n^2}\,\frac{l'-l}{l(k'-k)}(kD_1 + k'D_2) = \frac{l'-l}{l(k'-k)}\sqrt{l^2+n^2}\,.\,F.$$

Now eliminating s^2 between (28) and (29), we obtain

$$a_{12}\,nk^2 - \{(a_{22}-a_{33})\,l^2 + (a_{22}-a_{11})\,n^2\}\,k - a_{12}\,n\,(l^2+n^2) = 0,$$

and this equation may be regarded as a quadratic for determining the two values of k if the difference between l and l' be neglected: hence

$$kk' = -(l^2+n^2).$$

Also $$a_{12}nk = a_{22}(l^2+n^2) - s^2, \quad a_{12}nk' = a_{22}(l'^2+n^2) - s^2,$$

$$\therefore\; a_{12}n\,(k'-k) = a_{22}\,(l'^2 - l^2),$$

and $$\frac{l'-l}{k'-k} \doteq \frac{a_{12}n}{2a_{22}l}.$$

We thus have finally

$$F' = \frac{n\sqrt{l^2+n^2}}{2l^2}\cdot\frac{a_{12}}{a_{22}}\,G = \frac{\sin i}{2\cos^2 i}\cdot\frac{a_{12}}{a_{22}}\,G$$

$$G' = \frac{n\sqrt{l^2+n^2}}{2l^2}\cdot\frac{a_{12}}{a_{22}}\,F = \frac{\sin i}{2\cos^2 i}\cdot\frac{a_{12}}{a_{22}}\,F,$$

and if I, I' be the intensities of the incident and the reflected light

$$I' = \frac{\sin^2 i}{4\cos^4 i}\left(\frac{a_{12}}{a_{22}}\right)^2 I.$$

Thus the intensity of the reflected light is proportional to that of the incident light, whatever the state of the latter as regards polarisation: the reflected light is unpolarised, if the incident light be so; while, if the incident light be polarised in a plane parallel or perpendicular to the plane of incidence, the reflected light is polarised in the opposite manner.

If the thickness of the plate cannot be neglected, the retardations of the streams in their passage to and from the twin surface will generally modify the relations between the polarisations of the light before entering and after leaving the crystal. It is clear, however, that if the incident light be unpolarised, so is also the light reflected from the crystal; for there is nothing to alter this character in the passage of the light through the plate, neither is it lost, as has been shown, in the act of the reflection. On the other hand, if the plate be thick, the reversal of the polarisation of the reflected light, when the initial stream is polarised in one of the principal azimuths, will only occur in all probability for small angles of incidence*.

* Lord Rayleigh, *loc. cit.* p. 255.

167. The interest of this investigation lies in the explanation that it affords of the chief features of a remarkable phenomenon of crystalline reflection exhibited by iridescent crystals of chlorate of potash, that is ascribed by Stokes* to a thin layer, that he regards as twin stratum, situated within the crystal and about a thousandth of an inch in thickness.

The chief peculiarities of this internal coloured reflection, as described by Stokes, have been summarised by Lord Rayleigh† as follows:

(1) If one of the crystalline plates be turned round in its own plane, without alteration of the angle of incidence, the peculiar reflection vanishes twice in a revolution, viz.—when the plane of incidence coincides with the plane of symmetry of the crystal.

(2) As the incidence is increased, the reflected light becomes brighter and rises in refrangibility.

(3) The colours are not due to absorption, the transmitted light being strictly complementary to the reflected.

(4) The coloured light is not polarised. It is produced indifferently whether the incident light be common light or light polarised in any plane, and is seen whether the reflected light be viewed directly or through a Nicol's prism turned in any way.

(5) The spectrum of the reflected light is frequently found to consist entirely of a comparatively narrow band. When the angle of incidence is increased, the band moves in the direction of increasing refrangibility and at the same time increases rapidly in width. In many cases the reflection appears to be almost total.

To these Lord Rayleigh has added the further peculiarity, first predicted by his theoretical investigation, that when the light is incident in a plane perpendicular to the plane of symmetry, the polarisation is for small angles of incidence reversed in the reflected stream, if it be either parallel or perpendicular to the plane of incidence.

The theory of reflection at a twin plane is however incompetent to explain the copiousness and the highly selective character of the reflected light, and Lord Rayleigh‡ is inclined to attribute these characteristics to repeated alternations of structure due to a large number of twin planes, within the thin stratum that is the seat of the colour. He has in fact shown that the narrowness of the band in the spectrum of the reflected light at nearly normal incidence and its widening as the incidence increases is what would be expected in the case of reflection from such a laminated medium: while the movement of the band towards the blue end of the spectrum is accounted for by the increasing obliquity within the crystal, as in the ordinary theory of thin plates.

* *Proc. R. S.* XXXVIII. 174 (1885).

† *Phil. Mag.* (5) XXVI. 256 (1888).

‡ *loc. cit.* p. 257.

CHAPTER XIV.

THE INTERFERENCE OF POLARISED LIGHT.

168. THE first discovery of the interference that occurs when a stream of polarised light is transmitted through crystalline substances was made by Arago in 1811*. Malus had already observed that, when a plate of a doubly refracting crystal is interposed between a polariser† and an analyser regulated for extinction, the light is partially restored; and Arago found that in the case of white light and with a plate that is moderately thin, the light is no longer white but coloured, and that a variation of brilliancy but not of tint is produced by a rotation of the plate in its own plane, the polariser and analyser remaining fixed, while a rotation of the analyser, the plate and the polariser retaining their positions, causes a change of colour, which passes through white into the complementary tint.

On the publication of Arago's memoir this chromatic polarisation, as it is sometimes called, was subjected to a searching investigation by Biot‡, who during the years 1812 to 1814 succeeded in establishing the experimental laws of the phenomenon. Biot's earliest researches were limited to the case in which a stream of nearly parallel light fell upon the plate of crystal: the phenomena of rings and brushes, that are seen when the incident pencil is conical, were first discovered by Brewster§ in the case of uniaxal crystals in 1813, and in that of biaxal crystals in 1814.

169. The first to apply the principles of the wave-theory to the explanation of Chromatic Polarisation was Thomas Young||. From the

* *Mém. de la prem. Classe de l'Instit.* XII. 93 (1812): *Œuvres Complètes*, x. 36.

† A polariser is an instrument for obtaining a polarised beam of light: it is called an analyser, when it is used for investigating the character of a stream of light or for reducing it to a given plane of polarisation.

‡ *Mém. de la prem. Classe de l'Instit.* XII. 135; XIII. 1re partie 1, 2e partie 1, 31 (1812). *Mém. d'Arceuil*, III. 132 (1813). *Traité de Phys.* IV. (1816).

§ *Treatise on New Philosophical Instruments.* Edinburgh (1813), p. 336. *Phil. Trans.* CIV. 187 (1814).

|| *Quarterly Review*, XI. 42 (1814): *Misc. Works*, I. 269.

results of Biot's experiments he observed, that a plate of crystal in polarised light exhibits the same tint as a thin plate of air in transmitted light, when its thickness is such that the relative retardation of the ordinary and the extraordinary streams produced by the crystal is the same as that of the interfering streams in the case of the plate of air, and from this fact he drew the inference that the phenomenon is the result of the interference of these two streams.

This explanation is, as was recognised by its author, incomplete, for it makes the phenomenon of colour depend upon the plate alone and leaves out of account the action of the polariser and the analyser that are found to be necessary for the production of the interference. In order to remove this flaw in Young's explanation, Fresnel and Arago* devised a series of experiments to determine whether and in what manner polarisation of the light modifies the ordinary laws of interference. The results of these researches are summed up in the following five laws of the interference of polarised light.

(1) Two streams of light polarised in perpendicular planes do not interfere under the same circumstances as two streams of common light.

(2) Two streams polarised in parallel planes give the same phenomena of interference as two streams of common light.

(3) Two streams, polarised at right angles and coming originally from a stream of common light, can be brought to the same plane of polarisation without thereby acquiring the faculty of interfering.

(4) Two streams, polarised in perpendicular planes and coming originally from a beam of polarised light, interfere as common light when brought to the same plane of polarisation.

(5) When two streams, coming from a stream of polarised light, are first polarised at right-angles and then brought to the same plane of polarisation, it is necessary in calculating the conditions of the interference to add a half wave-length to the actual relative retardation measured in length, unless the initial and final planes of polarisation lie in the same angle between the two perpendicular planes.

These laws are a direct consequence of the transversality of the polarisation-vector, already deduced in Chapter II as a result of the first law. Thus the gain or loss of half an undulation required in accordance with the fifth law appears at once as due to the process of resolution of the vector; and this explains the necessity of the polarisation of the primitive light for the production of interference with two streams polarised at right-angles and subsequently analysed, for common light may be represented by two independent streams polarised at right-angles, and as the interference

* Fresnel's experiments were commenced in 1816 (*Œuvres Complètes*, I. 385); Fresnel and Arago published their memoir in 1819 (*Ann. de Ch. et de Phys.* (2) x. 288; *Œuvres Complètes de Fresnel*, I. 509; *Œuvres Complètes d'Arago*, x. 132).

phenomena due to these two streams are complementary, they will obliterate one another.

170. In the final series of experiments by which Fresnel and Arago established the laws of interference of polarised light, the arrangement was adopted that had already been employed by Young for producing interference fringes with common light.

A stream of light from a luminous point fell upon an opaque screen pierced with two parallel slits near to one another, and after passing through these apertures was received in an eye-lens. The light that traverses the slits gives rise to two systems of diffraction bands, with which we need not concern ourselves, and intermediate to these a set of interference fringes, that will be displaced to the right or the left, according as the stream from the slit on the right or the left side is retarded relatively to the other.

On placing a thin plate of selenite before the two slits it was found that no change in the phenomenon occurred, a single system of fringes being produced exactly as was the case before the plate was introduced. From the position of these fringes it is clear that they are due to streams that have not acquired any relative retardation in traversing the selenite, and they must consequently be ascribed to the superposition of two systems of bands, the one produced by the ordinary streams, the other by the extraordinary streams coming from the two slits. It follows then that two streams polarised in parallel planes interfere as common light.

If streams polarised in perpendicular planes also interfere, there could be two additional systems of fringes, situated on either side of the former and arising from the interference of the ordinary stream from the one slit and the extraordinary stream from the other slit. No trace of these fringes was however seen under any circumstances, nor did they become visible when the light after passing the eye-lens was analysed in a plane inclined to the principal section of the selenite.

In order to place this result beyond a doubt, the plate of selenite was then cut in half, and replaced in front of the slits, after the half covering one slit had been turned in its own plane through a right-angle. The central system of fringes then disappeared and was replaced by the two lateral systems, due to the ordinary stream from the one slit interfering with the extraordinary stream from the other, these now being polarised in parallel planes and retarded relatively to one another. The two ordinary streams, as also the two extraordinary streams, no longer interfered, as they were polarised in perpendicular planes.

Returning to the arrangement of the first experiment, the light incident on the selenite was next polarised in a plane at 45° to its principal section, and a rhomb of Iceland spar was placed before the eye-lens with its principal

section parallel to the primitive plane of polarisation: then in each image given by the spar the central system of fringes, together with the two lateral ones, was produced, and the lateral systems in the extraordinary image were seen to be displaced so as to become complementary to the lateral systems in the ordinary image. This experiment proves the fourth and the fifth laws; but in order to check this result, Fresnel substituted for the rhomb of spar a plate of selenite too thin to give sensible separation of the images, and then found that the six systems of fringes gave by their superposition only one, the lateral systems being blotted out, which proves that these systems in the case of one plane of analysation are obtained from those analysed in the perpendicular plane by the addition of a half-wave to the actual difference of path.

171. Fresnel's and Arago's experiments have been modified and extended by subsequent observers*, and we owe in particular to Mach† an experiment that may be described, as it possesses a special theoretical interest.

We have seen that, when a telescope is focussed on a narrow line of monochromatic light and the object-glass is limited to a slit parallel to the line, the geometrical image of the line is bordered by a system of diffraction fringes, and that on covering one half of the slit with a retarding plate the bands of an odd order are shifted towards the side of the retarded stream, while those of an even order retain their position. If the light that passes be white, the diffraction phenomenon may be analysed by a spectroscope with its slit in the plane of the pattern and perpendicular to the fringes, and a spectrum is then obtained with dark bands running along it, that approach one another as the blue end of the spectrum is neared.

This was the arrangement that was adopted by Mach, who covered the two halves of the slit with equal plates of quartz cut parallel to the optic axis and so placed that their principal sections were perpendicular to one another.

If we suppose that the slit is vertical and the plate on the left-hand side has its principal section vertical, the streams that we have to consider are L_v and L_h from the left-hand half polarised respectively in a vertical and horizontal plane, and the corresponding streams R_v and R_h from the right-hand half of the slit, and of these L_h and R_v are retarded relatively to L_v and R_h by an amount that increases from red to violet light.

Now L_v and R_v, being polarised in parallel planes, will give rise to a system of fringes, and as R_v is retarded relatively to L_v, the bands of an even order will retain their former positions, but those of an odd order will be displaced towards the right by an amount that increases considerably as the wave-length diminishes. A similar result is obtained from the streams

* Stefan, *Wien. Ber.* LIII. (2) 548 (1866); LXVI. (2) 425 (1872).

† Mach and Rosicky, *ibid.* LXXII. (2) 197 (1876).

L_h and R_h, the displacement being in this case to the left. Finally, as regards the stream resulting from L_v and R_v, as also that resulting from L_h and R_h, these are polarised in perpendicular planes and consequently give rise to no interference. Hence instead of the bands a and b seen with an uncovered slit, the spectrum will be traversed by three series of lines a, c, d.

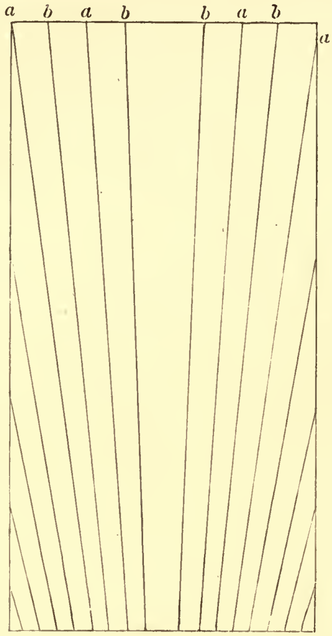


Fig. 39.

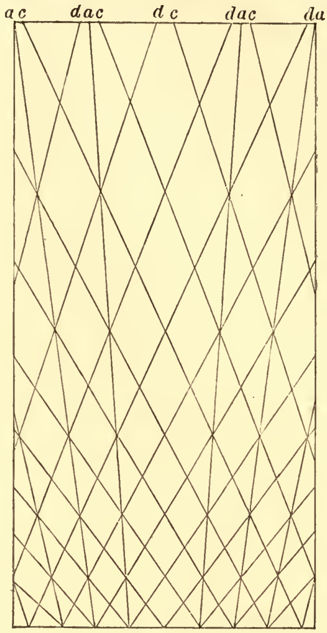


Fig. 40.

The insertion of a polarising prism either before the slit or in the eye-piece does not affect the phenomenon, unless the plane of polarisation or of analysation is either vertical or horizontal, but in these two cases the bands d and c respectively disappear.

Next let a polarising prism be introduced both before the slit and in the eye-piece of the telescope, the planes of polarisation and analysation being inclined at an angle of 45° to the vertical: then we have four streams L_v, L_h, R_v, R_h of equal intensity coming from the same polarised stream and finally brought to the same plane of polarisation. If then the planes of primitive and final polarisation be parallel, the streams L_v and R_h, as also the streams L_h and R_v will give the system of bands a and b, since they start from the slit without any relative retardation, and in addition, inasmuch as the stream resulting from L_h and R_v is retarded relatively to that resulting from L_v and R_h by an amount that is constant for any one wave-length, and increases from the red to the violet, there will be a set of horizontal bands e, exactly the same as would be obtained if one of the plates of quartz were placed between the polariser and the analyser and the light traversing the system were

analysed by the spectroscope. When the planes of polarisation and analysation are crossed, we have to add $\lambda/2$ to the actual retardation in length, and the system a will remain unchanged, while the system b will be displaced by an amount corresponding to $\lambda/2$, which will bring them into coincidence with the bands a, the central band becoming dark: similarly the horizontal system of bands e will be replaced by the complementary system e'.

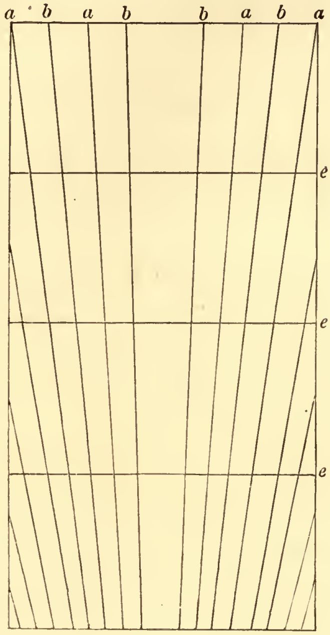


Fig. 41.

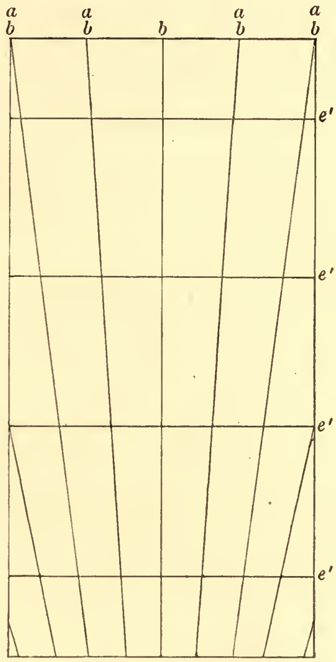


Fig. 42.

172. Returning now to the interference phenomena produced by crystalline plates, let us suppose that a stream of light is received on a screen after it has traversed an optical system containing a polariser, a plate of crystal and a second series of lenses containing an analyser. The interference at any point on the screen is the same as at its image formed by the second set of lenses, when the light emanates from the image of the actual source due to the first optical system, and we may consequently suppress the lenses and consider merely the passage of light from a polarised source L through the plate of crystal to a screen S, the streams on arrival being supposed to be reduced to a common plane of polarisation. For the sake of simplicity we may assume that L and S are parallel to the faces of the plate.

Let us first consider a single point O of the source. Since the two streams from O that meet at a point P of the screen pass through the crystal in different directions, their planes of polarisation after traversing the plate are not strictly at right-angles, but this effect of the double refraction on

the polarisation may be left out of account in most practical cases, and to the same degree of approximation we may also suppose that these planes of polarisation and the initial and final planes of polarisation intersect one another in the same straight line.

Hence if α, β, ψ be the angles that the primitive and final planes of polarisation and the plane of polarisation of the quicker wave within the plate make with a fixed plane, and if δ be the relative retardation of the streams at P measured in length in air, the effect at P for light of wave-length λ is represented by the vector

$$\{a_\lambda \cos(\psi-\alpha)\cos(\psi-\beta) + a_\lambda \sin(\psi-\alpha)\sin(\psi-\beta)\,e^{-\iota\kappa\delta}\}\,e^{\iota nt},$$

where $\kappa = 2\pi/\lambda$, $n = 2\pi/\tau$, the change of amplitude due to the refractions being neglected, and if a_λ^2 represent the primitive intensity, that at the point P will be

$$\begin{aligned} I &= a_\lambda^2 \cos^2(\psi-\alpha)\cos^2(\psi-\beta) + a_\lambda^2 \sin^2(\psi-\alpha)\sin^2(\psi-\beta) \\ &\qquad + 2a_\lambda^2 \sin(\psi-\alpha)\cos(\psi-\alpha)\sin(\psi-\beta)\cos(\psi-\beta)\cos\kappa\delta \\ &= a_\lambda^2\cos^2(\beta-\alpha) - a_\lambda^2 \sin 2(\psi-\alpha)\sin 2(\psi-\beta)\sin^2(\pi\delta/\lambda) \quad\text{.........(1).}\end{aligned}$$

Hence when the light that passes is white, the intensity is

$$I = \cos^2(\beta-\alpha)\,\Sigma_\lambda a_\lambda^2 - \Sigma_\lambda a_\lambda^2 \sin 2(\psi-\alpha)\sin 2(\psi-\beta)\sin^2(\pi\delta/\lambda) \quad\text{...(2),}$$

the summation extending to all the constituents of the composite stream.

The first term in these expressions represents what may be called the fundamental intensity, that is the intensity when the plate of crystal is removed, and in the second case has no effect in producing colour at P; but in the second summation δ/λ and in general ψ depend upon the wave-length, so that the different constituents of white light enter in different degrees, and this summation is the representation of a stream of more or less coloured light.

173. We must now determine the relative retardation at P of the two streams emanating from O. Let

T be the thickness of the plate,
h, h' the distances of its surfaces from O and P respectively,
i_1, i_2 the angles of incidence on the plate,
r_1, r_2 the corresponding angles of refraction,
μ_1, μ_2 the refractive indices: then

$$\delta = (h+h')(\sec i_2 - \sec i_1) + T(\mu_2 \sec r_2 - \mu_1 \sec r_1) \quad\text{...........(3),}$$

with the condition

$$0 = (h+h')(\tan i_2 - \tan i_1) + T(\tan r_2 - \tan r_1) \quad\text{..............(4);}$$

whence multiplying (4) by $\sin i_1$ and subtracting it from (3), we have

$$\delta = (h + h')\frac{1 - \cos(i_2 - i_1)}{\cos i_2} + T\left\{\frac{\mu_2 - \sin i_1 \sin r_2}{\cos r_2} - \frac{\mu_1 - \sin i_1 \sin r_1}{\cos r_1}\right\}.$$

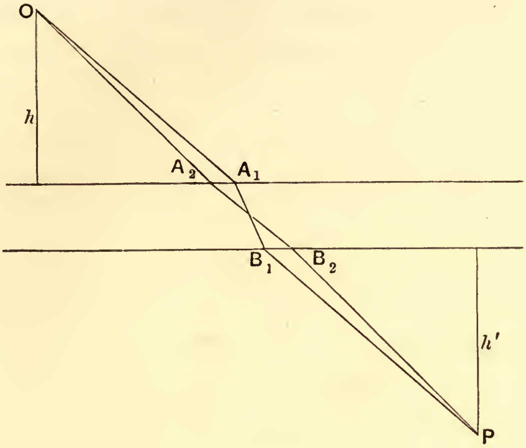


Fig. 43.

Let $\qquad i_1 = I - e, \quad i_2 = I + e, \quad r_1 = R_1 - \eta_1, \quad r_2 = R_2 + \eta_2,$

where $\qquad \sin I = \mu_1 \sin R_1 = \mu_2 \sin R_2,$

e, η_1, η_2 being small quantities, since the doubly refracting energy is in most cases weak; then

$$\frac{\mu_2 - \sin i_1 \sin r_2}{\cos r_2}$$

$$\fallingdotseq \frac{\sin I \cos^2 R_2 + e \cos I \sin^2 R_2 - \eta_2 \sin I \sin R_2 \cos R_2}{\sin R_2 \cos R_2}\left(1 + \eta_2 \frac{\sin R_2}{\cos R_2}\right)$$

$$\fallingdotseq \sin I \cot R_2 + e \cos I \tan R_2,$$

$$\frac{\mu_1 - \sin i_1 \sin r_1}{\cos r_1}$$

$$\fallingdotseq \sin I \cot R_1 + e \cos I \tan R_1;$$

whence neglecting e^2

$$\delta = T \sin I(\cot R_2 - \cot R_1) + Te\cos I(\tan R_2 - \tan R_1)$$

$$= T\sin I(\cot R_2 - \cot R_1) + Te\cos I\frac{\sin(R_2 - R_1)}{\cos R_1 \cos R_2}$$

$$\fallingdotseq T\sin I(\cot R_2 - \cot R_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

since $e \sin(R_2 - R_1)$ is of the order of the terms neglected.

To this approximation then the relative retardation is independent of the distances of O and P from the plate, and depends upon the mean of the

angles of incidence on the plate, or on the mean of the angles at which the interfering streams meet the screen at P.

174. Suppose now that the colour due to the light from O appears to be the same over a circle B surrounding the point P, then for all points of this circle the second term in the expression (2) must represent a stream of light of a practically constant tint, which may be expressed by the inequality

$$M + \epsilon > \Sigma_\lambda a_\lambda^2 \sin 2(\psi - \alpha) \sin 2(\psi - \beta) \sin^2(\pi\delta/\lambda) > M - \epsilon \ldots\ldots(6),$$

the colour represented by M being independent of the actual value of a_λ, provided Σa_λ^2 constitutes a stream of white light, and ϵ representing a very small variation of tint.

Let us now limit the source to a circle A round the point O, such that the rays from any point of the contour of A that meet at P are parallel to the rays from O that intersect at some point on the edge of B. Then the intensity at any point of the area B will be

$$\Sigma\{\cos^2(\beta - \alpha)\Sigma_\lambda a_\lambda^2 - \Sigma_\lambda a_\lambda^2 \sin 2(\psi - \alpha) \sin 2(\psi - \beta) \sin^2(\pi\delta/\lambda)\} \ldots (7),$$

the summation Σ extending to all points of A that contribute to the illumination of the point in question. But the rays from any point of A to any point of B are parallel to those from O to some point of this circle B and therefore the relation (6) holds for each term of the summation Σ and each contributes to the illumination light of practically the same tint, so that the colour of the area B is unaffected by an extension of the source to the amount assumed, this result being independent of the distance of the screen from the plate.

Now the relative retardation of the streams from O at any point of the screen is, as we have seen, independent of the distance of the point from the plate and depends upon the mean angle of incidence on the screen and therefore the area B, for which the relation (6) holds, increases as the screen is moved parallel to itself away from the plate. From this it follows that with a source of given size there is a limiting distance of the screen from the plate, at which the interference is first seen and beyond which it is always visible.

The expression for the retardation shows that for an uniform tint over the whole field the light must consist of nearly parallel rays. In this case the colour is perceptible on the surface of the plate itself, and the fringes seen with crystalline wedges and the patterns given by the superposition of plates of varying thickness are localised on the crystalline surface, since in other planes the streams to a given point from the different available points of the source traverse the crystal at places where the thickness is different and the relation (6) no longer holds for the different constituents of the summation (7).

On the other hand with a conical pencil, giving the phenomena of rings and brushes, there is partial localisation, as an extension of the source leads to an obliteration of the interference until the screen is at a considerable distance from the plate, and in most cases, particularly when the light is monochromatic, the rings and brushes are only seen in the principal focus of the observing system of lenses, an indefinite extension of the source being then permissible.

175. The two classes of interference, produced by crystalline plates, though very different in appearance, are in reality explained by the same principles: in the case of nearly parallel light, the expressions (1) and (2) have the same value over the whole extent of the field, while in the case of the rings and brushes, observed with conical pencils of light, the intensity varies from point to point of the plane, in which the phenomenon is observed.

Let us first consider the simpler case, in which the light is nearly parallel. When the light is monochromatic, the intensity is given by

$$I = a^2 \{\cos^2(\beta - \alpha) - \sin 2(\psi - \alpha)\sin 2(\psi - \beta)\sin^2(\pi\delta/\lambda)\},$$

and if the polariser and analyser remain fixed, while the plate is rotated in its own plane, there are eight positions given by

$$\psi = \alpha, \quad \pi/2 + \alpha, \quad \pi + \alpha, \quad 3\pi/2 + \alpha, \quad \beta, \quad \pi/2 + \beta, \quad \pi + \beta, \quad 3\pi/2 + \beta,$$

in which the intensity is the same as before the introduction of the plate, and between these positions the intensity becomes a maximum or a minimum.

If however $\delta = n\lambda$, the intensity is unaltered by the rotation.

In the special case in which the planes of polarisation and analysation are parallel, the intensity is

$$a^2 \{1 - \sin^2 2(\psi - \alpha)\sin^2(\pi\delta/\lambda)\},$$

and is a maximum when the planes of polarisation of the streams within the plate are parallel and perpendicular to the primitive plane of polarisation and is a minimum when they are inclined at 45° to this plane, the field being completely dark in this case, when $\delta = (2n + 1)\lambda/2$.

When the planes of polarisation and analysation are crossed

$$I = a^2 \sin^2 2(\psi - \alpha)\sin^2(\pi\delta/\lambda),$$

and the field is dark if $\delta = n\lambda$, and in other cases the light is entirely cut off when the planes of polarisation of the transmitted streams coincide with the initial and final planes of polarisation.

When the initial light is white, the intensity is given by the expression (2). In strictness, the angle ψ is dependent upon the wave-length, but when the dispersion is weak relatively to the double refraction, the product

$\sin 2(\psi - \alpha) \sin 2(\psi - \beta)$ has sensibly the same value for all terms of the summation, and we may take as the expression for the intensity

$$I = \cos^2(\beta - \alpha)\,\Sigma a_\lambda^2 - \sin 2(\psi - \alpha)\sin 2(\psi - \beta)\,\Sigma a_\lambda^2 \sin^2(\pi\delta/\lambda).$$

Since then the first term in this expression represents a stream of white light, the plate will appear uncoloured, when the plane of polarisation of either of the streams transmitted by it coincides with either the primitive or final plane of polarisation. In intermediate cases the field is coloured, the tint changing to its complementary as the plate passes through one of these eight positions, since the second term in the expression for the intensity then changes sign.

The plate exhibits only one colour during its revolution, when the planes of polarisation and analysation are either parallel or crossed; as the intensity in these cases is given by

$$I = \Sigma a_\lambda^2 - \sin^2 2(\psi - \alpha)\,\Sigma a_\lambda^2 \sin^2(\pi\delta/\lambda),$$

and $$I = \sin^2 2(\psi - \alpha)\,\Sigma a_\lambda^2 \sin^2(\pi\delta/\lambda),$$

respectively, the colours being thus complementary.

If the polariser and plate remain fixed, while the analyser is turned, the plate exhibits no colour for four positions of the analyser given by

$$\beta = \psi, \quad \psi + \pi/2, \quad \psi + \pi, \quad \psi + 3\pi/2,$$

and the colour changes to its complementary tint, as the analyser passes through one of these positions.

176. The crystalline plate shows no colour when it is very thin and also when its thickness exceeds a moderate amount. The reason for this is obvious: in the former case, the retardation of phase varies so little with the wave-length, that the resulting intensity is practically the same for all colours; in the latter case it alters so rapidly that for a slight change in the wave-length the intensity passes from a maximum to a minimum, and consequently so many constituents of the white light are weakened and these are so close to one another in colour that the light presents to the eye the appearance of being white. The true character of the light may be ascertained by analysing it with a spectroscope, when a spectrum is obtained traversed by dark bands corresponding to the tints that are weakened or annulled.

It is however possible, even with thick plates, to obtain the phenomenon of colour by combining two of them in a suitable manner between a polariser and an analyser: in order that this may be effected, the retardations of phase introduced by the two plates must nearly balance one another.

Making the same assumptions with respect to the polarisations as in the case of a single plate and neglecting the effect of the refraction from the

first into the second crystal, let the primitive and final planes of polarisation and the planes of polarisation of the quicker waves in the two plates make angles α, β, ψ_1, ψ_2 respectively with some fixed plane, and let δ_1 be the relative retardation in length in air for the two streams of given wave-length that traverse the second plate with the same speed, δ_2 that for the two streams that pass through the first plate at the same rate; then after traversing the analyser, the resultant stream will be represented by the vector

$$\begin{aligned}
a\,[\{&\cos(\psi_1-\alpha)\cos(\psi_2-\psi_1)-\sin(\psi_1-\alpha)\sin(\psi_2-\psi_1)\,e^{-\iota\kappa\delta_1}\}\cos(\psi_2-\beta)\\
&+\{\cos(\psi_1-\alpha)\sin(\psi_2-\psi_1)\,e^{-\iota\kappa\delta_2}\\
&+\sin(\psi_1-\alpha)\cos(\psi_2-\psi_1)\,e^{-\iota\kappa(\delta_1+\delta_2)}\}\sin(\psi_2-\beta)]\,e^{\iota nt}\\
&=\{A+Be^{-\iota\kappa\delta_1}+Ce^{-\iota\kappa\delta_2}+De^{-\iota\kappa(\delta_1+\delta_2)}\}\,e^{\iota nt},\ \text{(say)},
\end{aligned}$$

and the intensity, obtaining by multiplying this by the conjugate expression, is

$$\begin{aligned}
&A^2+B^2+C^2+D^2+2\,(AB+CD)\cos\kappa\delta_1+2\,(AC+BD)\cos\kappa\delta_2\\
&\quad+2BC\cos\kappa\,(\delta_1-\delta_2)+2AD\cos\kappa\,(\delta_1+\delta_2)\\
&=(A+B+C+D)^2-4\,(AB+CD)\sin^2(\pi\delta_1/\lambda)-4\,(AC+BD)\sin^2(\pi\delta_2/\lambda)\\
&\quad-4BC\sin^2\{\pi\,(\delta_1-\delta_2)/\lambda\}-4AD\sin^2\{\pi\,(\delta_1+\delta_2)/\lambda\}\\
&=a^2\left\{\cos^2(\beta-\alpha)+\sin 2\,(\psi_1-\alpha)\cos 2\,(\psi_2-\beta)\sin 2\,(\psi_2-\psi_1)\sin^2\frac{\pi\delta_1}{\lambda}\right.\\
&\quad-\cos 2\,(\psi_1-\alpha)\sin 2\,(\psi_2-\beta)\sin 2\,(\psi_2-\psi_1)\sin^2\frac{\pi\delta_2}{\lambda}\\
&\quad+\sin 2\,(\psi_1-\alpha)\sin 2\,(\psi_2-\beta)\sin^2(\psi_2-\psi_1)\sin^2\frac{\pi\,(\delta_1-\delta_2)}{\lambda}\\
&\quad\left.-\sin 2\,(\psi_1-\alpha)\sin 2\,(\psi_2-\beta)\cos^2(\psi_2-\psi_1)\sin^2\frac{\pi\,(\delta_1+\delta_2)}{\lambda}\right\}\ \ldots\ldots\ldots(8).
\end{aligned}$$

It follows from this that the combination acts as if only the first plate were present, when the plane of polarisation of the quicker wave in the second is parallel or perpendicular to the plane of analysation ($\psi_2=\beta$ or $\beta\pm\pi/2$) and that the first plate is inoperative, when the plane of polarisation of the quicker stream within it is parallel or perpendicular to the primitive plane of polarisation ($\psi_1=\alpha$ or $\alpha\pm\pi/2$). According as the planes of polarisation of the quicker waves in the plates are parallel or crossed ($\psi_2=\psi_1$ or $\psi_1\pm\pi/2$), we have

$$I=a^2\left\{\cos^2(\beta-\alpha)-\sin 2\,(\psi_1-\alpha)\sin 2\,(\psi_1-\beta)\sin^2\frac{\pi\,(\delta_1+\delta_2)}{\lambda}\right\}$$

and

$$I=a^2\left\{\cos^2(\beta-\alpha)-\sin 2\,(\psi_1-\alpha)\sin 2\,(\psi_1-\beta)\sin^2\frac{\pi\,(\delta_1-\delta_2)}{\lambda}\right\},$$

that is, the combination acts as a thicker or a thinner plate than either of the two constituent plates.

From the foregoing investigation we obtain a very delicate test for slight traces of double refraction in a plate. When the retardation of phase for light of mean wave-length amounts to π or to a small multiple of π, a crystalline plate between a crossed polariser and analyser shows in white light a distinctive greyish violet colour, known as a sensitive tint from the fact that it changes rapidly for a slight alteration in the retardation, becoming blue or red according as the retardation is increased or diminished. If then the plate to be tested be combined with a plate giving the sensitive tint, a slight trace of double refraction will be made manifest by a change of colour. The test is rendered still more delicate by cutting the sensitive plate in two and reuniting the halves after the one has been turned in its own plane through a right-angle: since now the planes of polarisation of the quicker waves in the two halves of the plate are perpendicular to one another, the tint of the one half will be raised and that of the other will be lowered, if the compound plate be combined with a second plate giving double refraction*.

177. If instead of a parallel beam of light a conical pencil be incident on a crystalline plate, the intensity varies from point to point of the field and the complete discussion of the phenomenon becomes very complicated. If however we confine our attention to directions making no great angle with the axis of the pencil, we may simplify the investigation by assuming that the planes of polarisation and analysation are constant over the field and that the planes of polarisation of the streams within the plate are at right-angles to one another and intersect the planes of primitive polarisation and of analysation along the axis of the pencil. The intensity is then given in the case of monochromatic light by the expression (1), namely

$$I = a^2 \left\{ \cos^2(\beta - \alpha) - \sin 2(\psi - \alpha) \sin 2(\psi - \beta) \sin^2 \frac{\pi\delta}{\lambda} \right\} \quad \ldots\ldots(9),$$

wherein ψ and δ alone depend upon the direction under consideration.

The interference phenomenon is thus characterised by three systems of curves; the curves of constant retardation, $\delta = \text{const.}$; the curves of like polarisation, $\psi = \text{const.}$; and the curves of constant intensity, $I = \text{const.}$

At all points of the field, for which

$$\sin 2(\psi - \alpha) \sin 2(\psi - \beta) \sin^2(\pi\delta/\lambda) = 0,$$

the intensity is the same as when the plate is removed. This equation defines (1) a system of curves, for which the relative retardation is an integral number of wave-lengths $\delta = n\lambda$, (2) lines of like polarisation $\psi = \alpha$

* Bravais, *Ann. de Ch. et de Phys.* (3) XLIII. 129 (1885).

or $\alpha + \pi/2$, $\psi = \beta$ or $\beta + \pi/2$, that is lines joining the points for which the streams within the plate are polarised in planes parallel and perpendicular to the planes of primitive and final polarisation. These systems of curves are called respectively the principal curves of constant retardation and the principal lines of like polarisation. The latter lines divide the field into regions in which the intensity is alternately greater and less than the fundamental intensity, but when the planes of polarisation and analysation are parallel or crossed ($\beta = \alpha$ or $\pi/2 + \alpha$), the two pairs of lines coincide and the intensity of the field except on these lines and on the principal curves of constant retardation, is in the former case less and in the latter case greater than it was before the introduction of the plate.

Unless the dispersion of the optic axes is considerable, the principal lines of like polarisation vary but slightly with the wave-length and thus with white light appear practically uncoloured: they are hence often called the achromatic lines. The principal curves of constant retardation on the other hand depend upon the wave-length, but in general each curve within a region bounded by principal lines of like polarisation, has the same colour in white light throughout its length, which changes into the complementary tint on passing into the adjacent region: when however the pairs of principal lines of like polarisation coincide, the hue is the same along the whole curve of constant retardation, whence they are sometimes called the isochromatic curves. When the dispersion of the optic axes is large, this is not the case and the curves of constant retardation are far from isochromatic.

178. The principal curves of constant retardation and the principal lines of like polarisation divide the field into spaces, in which there is a point of maximum or of minimum intensity surrounded by curves of constant intensity

$$\cos^2(\beta - \alpha) - \sin 2(\psi - \alpha)\sin 2(\psi - \beta)\sin^2(\pi\delta/\lambda) = \text{const.}$$

in which ψ and δ are regarded as variables.

The points of maximum and minimum intensity are situated at the intersection of the curves of constant retardation $\delta = (2n+1)\lambda/2$ with the two pairs of lines of like polarisation

$$\psi = (\alpha + \beta)/2, \qquad \psi = (\alpha + \beta + \pi)/2,$$

and

$$\psi = (\alpha + \beta)/2 + \pi/4, \qquad \psi = (\alpha + \beta)/2 + 3\pi/4$$

respectively.

Let us consider the region of the field between the lines $\psi = \alpha$ and $\psi = \beta$ and write

$$-\sin 2(\psi - \alpha)\sin 2(\psi - \beta)\sin^2(\pi\delta/\lambda) = k;$$

then curves in this region with intensity-constant k are touched by the lines of like polarisation $\psi = (\alpha + \beta)/2 \pm \eta$, where

$$-\sin(2\eta + \beta - \alpha)\sin(2\eta - \beta + \alpha) = k,$$

or
$$\sin 2\eta = \sqrt{\sin^2(\beta-\alpha)-k},$$

at the points in which they are cut by the curves of constant retardation $\delta=(2n+1)\lambda/2$, and are also touched by the curves

$$\delta = n\lambda + \Delta, \quad \delta = (n+1)\lambda - \Delta,$$

where
$$\sin^2(\beta-\alpha)\sin^2(\pi\Delta/\lambda) = k,$$

at the points in which they are intersected by the line of like polarisation

$$\psi = (\alpha+\beta)/2.$$

Further each curve with intensity-constant k is cut by a curve of constant retardation $\delta = n\lambda + \epsilon$, $(\epsilon > \Delta)$ in points on the lines of like polarisation

$$\psi = (\alpha+\beta)/2 \pm \eta',$$

where
$$\sin 2\eta' = \sqrt{\sin^2(\beta-\alpha) - k\,\text{cosec}^2(\pi\epsilon/\lambda)},$$

and the other points in which these lines meet the curve are on the curves of constant retardation $\delta=(n+1)\lambda-\epsilon$. Hence if δ_n, δ'_n be the relative retardations at the points in which a line of like polarisation meets an intensity-curve k in a space of order n, $\delta'_n - \delta_n = \lambda - 2\epsilon$ is independent of the order of the space and $\delta'_n + \delta_n = (2n+1)\lambda$ is independent of ϵ and hence of the intensity-curve k. Moreover if δ_{n-1}, δ'_{n-1} be the relative retardations at the points in which the same line of like polarisation meets the curve with the same intensity-constant k in the space of order $(n-1)$, we have $\delta'_{n-1} = n\lambda - \epsilon$, whence $\delta_n - \delta'_{n-1} = 2\epsilon$ is independent of the order-number n and $\delta_n + \delta'_{n-1} = 2n\lambda$ is independent of the intensity-constant k.

Similar results will clearly hold for the other regions of the field*.

179. We have seen that the relative retardation of the interfering streams at a given point of the pattern is $T\sin i(\cot r_2 - \cot r_1)$, where T is the thickness of the plate, i the mean of the angles of incidence of the streams, and r_1, r_2 the corresponding angles of refraction. Since by Huygens' principle the traces of the incident and refracted waves on the face of the crystal travel with the same speed, this expression is equal to $\Omega T(n_2-n_1)$, where Ω is the distance traversed by the light in air in unit time, and n_1, n_2 are the reciprocals of the intercepts on the normal to the plate through a point O of the surface made by the refracted waves in unit time after passing through O.

Hence if the axis of ζ be normal to the plate and

$$l\xi + m\eta + n_1\zeta = 1, \qquad l\xi + m\eta + n_2\zeta = 1,$$

* Lommel, *Pogg. Ann.* CXX. 69 (1863); *Wied. Ann.* XXXIX. 258 (1890). Niven, *Quart. J. of Math.* XIII. 174 (1874). Glazebrook, *Proc. Camb. Phil. Soc.* IV. 299 (1883). Spurge, *ibid.* V. 74 (1885); *Camb. Phil. Trans.* XIV. 63 (1884).

be the equations of the waves in unit time after passing through O, their relative retardation is

$$\delta = \Omega T (n_2 - n_1);$$

also θ being the azimuth of the plane of incidence with respect to that of $\xi\zeta$

$$l = \sin i \cos \theta/\Omega, \qquad m = \sin i \sin \theta/\Omega.$$

Let the plane of $\xi\zeta$ be chosen so as to contain the greatest axis Oz of the ellipsoid of polarisation, the plate lying on the side of positive ζ, then if Ox, Oy, Oz be the axes of the ellipsoid, and $yO\eta = \phi$, $zO\zeta = \chi$, the transformation from the axes of optical symmetry to the new axes, may be effected by the following successive transformations, each in one plane:

(1) through an angle ϕ in the plane of xy from Ox, Oy to Ox_1, $O\eta$,

(2) through an angle χ in the plane of zx_1 from Oz, Ox_1 to $O\zeta$, $O\xi$.

The formulæ for these transformations are

$$x = x_1 \cos\phi - \eta \sin\phi, \qquad y = x_1 \sin\phi + \eta \cos\phi,$$

$$x_1 = \xi \cos\chi + \zeta \sin\chi, \qquad z = -\xi \sin\chi + \zeta \cos\chi,$$

from which we obtain

$$x = \xi \cos\phi \cos\chi - \eta \sin\phi + \zeta \cos\phi \sin\chi,$$

$$y = \xi \sin\phi \cos\chi + \eta \cos\phi + \zeta \sin\phi \sin\chi,$$

$$z = -\xi \sin\chi + \zeta \cos\chi.$$

Now the equation to the wave-surface referred to the axes of optical symmetry is

$$\frac{a^2x^2}{\sigma^2 - a^2} + \frac{b^2y^2}{\sigma^2 - b^2} + \frac{c^2z^2}{\sigma^2 - c^2} = 0,$$

and the condition that the plane $lx + my + nz = 1$ should be a tangent plane to it, is obtained by eliminating ω between the equations

$$\omega^2 = (l^2 + m^2 + n^2)^{-1} \text{ and } \frac{l^2}{a^2 - \omega^2} + \frac{m^2}{b^2 - \omega^2} + \frac{n^2}{c^2 - \omega^2} = 0.$$

Hence the condition that in the new system of coordinates the plane

$$l\xi + m\eta + n\zeta = 1$$

should touch the wave-surface is found by eliminating ω between the equations

$$\omega^2 = (l^2 + m^2 + n^2)^{-1} \qquad \text{.............................(10)},$$

$$\frac{(l\cos\phi\cos\chi - m\sin\phi + n\cos\phi\sin\chi)^2}{a^2 - \omega^2} + \frac{(l\sin\phi\cos\chi + m\cos\phi + n\sin\phi\sin\chi)^2}{b^2 - \omega^2}$$

$$+ \frac{(l\sin\chi - n\cos\chi)^2}{c^2 - \omega^2} = 0 \qquad \text{.........(11)}.$$

The result of this elimination is a biquadratic in n, that from the nature of

the problem has two real positive and two real negative roots, and if n_1, n_2 be the positive roots, the relative retardation is

$$\delta = \Omega T (n_2 - n_1).$$

Writing $b = a$ for the case of an uniaxal crystal, equation (11) gives the two equations

$$\omega^2 - a^2 = 0 \text{ and } \{(l \cos\chi + n \sin\chi)^2 + m^2\} (\omega^2 - c^2) + (l \sin\chi - n \cos\chi)^2 (\omega^2 - a^2) = 0,$$

and the values of n are given by

$$a^2 (l^2 + m^2 + n^2) = 1,$$

$$c^2 (l^2 + m^2 + n^2) - 1 + (a^2 - c^2)(l \sin\chi - n \cos\chi)^2 = 0,$$

whence
$$n_1 = \frac{1}{a}\sqrt{1 - a^2 \frac{\sin^2 i}{\Omega^2}},$$

$$n_2 = \{\sqrt{(a^2 \cos^2\chi + c^2 \sin^2\chi)(1 - c^2 \sin^2 i/\Omega^2) - c^2 (a^2 - c^2) \sin^2\chi \cos^2\theta \sin^2 i/\Omega^2} + (a^2 - c^2) \sin\chi \cos\chi \cos\theta \sin i/\Omega\} \div (a^2 \cos^2\chi + c^2 \sin^2\chi)$$

and

$$\frac{\delta}{T} = -\frac{\sqrt{\Omega^2 - a^2 \sin^2 i}}{a} + \frac{(a^2 - c^2) \sin\chi \cos\chi \cos\theta \sin i}{a^2 \cos^2\chi + c^2 \sin^2\chi} + \frac{\sqrt{(a^2 \cos^2\chi + c^2 \sin^2\chi)(\Omega^2 - c^2 \sin^2 i) - c^2 (a^2 - c^2) \sin^2\chi \cos^2\theta \sin^2 i}}{a^2 \cos^2\chi + c^2 \sin^2\chi} \quad \text{......(12).}$$

In the special case of a plate perpendicular to the optic axis $(\chi = 0)$

$$\delta / T = \{\sqrt{\Omega^2 - c^2 \sin^2 i} - \sqrt{\Omega^2 - a^2 \sin^2 i}\}/a,$$

and when the plate is parallel to the optic axis $(\chi = \pi/2)$

$$\delta / T = \sqrt{\Omega^2 - (a^2 \cos^2\theta + c^2 \sin^2\theta) \sin^2 i}/c - \sqrt{\Omega^2 - a^2 \sin^2 i}/a.$$

Taking now the case of a biaxal plate perpendicular to the greatest axis of the ellipsoid of polarisation, the biquadratic becomes

$$(b^2c^2l^2 + c^2a^2m^2 + a^2b^2n^2)(l^2 + m^2 + n^2) - (b^2 + c^2) l^2 - (c^2 + a^2) m^2 - (a^2 + b^2) n^2 + 1 = 0,$$

or

$$a^2b^2n^4 - \{(a^2 + b^2) - b^2 (c^2 + a^2) l^2 - a^2 (b^2 + c^2) m^2\} n^2 + \{1 - c^2 (l^2 + m^2)\}\{1 - b^2l^2 - a^2m^2\} = 0,$$

and n_1, n_2 being the positive roots

$$(n_2 - n_1)^2 a^2b^2 = a^2 + b^2 - b^2 (c^2 + a^2) l^2 - a^2 (b^2 + c^2) m^2 - 2ab \sqrt{\{1 - c^2 (l^2 + m^2)\}\{1 - b^2l^2 - a^2m^2\}},$$

whence

$$a^2b^2 \frac{\delta^2}{T^2} = (a^2 + b^2)\Omega^2 - \{b^2 (c^2 + a^2) \cos^2\theta + a^2 (b^2 + c^2) \sin^2\theta\} \sin^2 i - 2ab \sqrt{(\Omega^2 - c^2 \sin^2 i)\{\Omega^2 - (b^2 \cos^2\theta + a^2 \sin^2\theta) \sin^2 i\}} \quad \text{......(13).}$$

In the general case of a biaxal plate cut in any direction, the relative retardation cannot be expressed in finite terms, and it is necessary to have recourse to an approximate solution*.

To obtain the Cartesian equations of the curves of constant retardation, we have to write

$$\xi = f\cos\theta\tan i \risingdotseq f\cos\theta\sin i, \qquad \eta = f\sin\theta\tan i \risingdotseq f\sin\theta\sin i,$$

since i is supposed small. In these expressions f is a constant and when the interference is observed in the principal focal plane of a system of lenses, is equal to the focal length of the optical system.

180. Let us now determine the lines of like polarisation†. If OA, OA' be the directions of the optic axes of the crystal, we know that the planes of polarisation of the waves, that travel in a direction OM within the crystal, bisect the angles between the planes MOA, MOA', that is they are tangent planes to the cones through OM that have the optic axes for their focal lines.

Referred to the principal axes of the crystal, let the general equation of the cones be

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0;$$

then the equation of the focal lines is

$$\frac{x^2}{A-B} + \frac{z^2}{C-B} = 0,$$

so that 2Ψ being the angle between the optic axes, A, B, C are connected by the relation

$$\tan^2\Psi = (A-B)/(B-C).$$

The tangent plane to the cone along the line $x/x' = y/y' = z/z'$ is

$$\frac{xx'}{A} + \frac{yy'}{B} + \frac{zz'}{C} = 0,$$

and if this be perpendicular to the plane $\lambda x + \mu y + \nu z = 0$ we have

$$\lambda x'/A + \mu y'/B + \nu z'/C = 0.$$

Eliminating then A, B, C between this equation and

$$x'^2/A + y'^2/B + z'^2/C = 0 \text{ and } \tan^2\Psi = (A-B)/(B-C),$$

we obtain the equation of a cone, such that the planes of polarisation of the waves, that travel along its generating lines, are parallel and perpendicular to a given plane.

* *Proc. R. S.* LXIII. 83 (1898): cf. also Zech, *Pogg. Ann.* XCVII. 129 (1856); CII. 354 (1857).

† Macé de Lépinay, *J. de Phys.* (2) II. 162 (1883). Lommel, *Wied. Ann.* XVIII. 56 (1883); *Pogg. Ann.* CXX. 69 (1863). Pitsch, *Wien. Ber.* XCI. (2) 527 (1885).

First we have

$$\frac{1/A}{y'z'(\mu z' - \nu y')} = \frac{1/B}{z'x'(\nu x' - \lambda z')} = \frac{1/C}{x'y'(\lambda y' - \mu x')},$$

or
$$A : B : C :: x'/(\mu z' - \nu y') : y'/(\nu x' - \lambda z') : z/(\lambda y' - \mu x');$$

but
$$A \cos^2 \Psi - B + C \sin^2 \Psi = 0,$$

whence substituting for A, B, C,

$$\frac{x' \cos^2 \Psi}{\mu z' - \nu y'} - \frac{y'}{\nu x' - \lambda z'} + \frac{z' \sin^2 \Psi}{\lambda y' - \mu x'} = 0 \quad \text{..................(14)},$$

a cone of the third degree passing through the two optic axes.

Now to the approximation adopted in finding the expression for the intensity, the direction (λ, μ, ν) is parallel to the surface of the plate, whence these direction-cosines are connected by the relation

$$\cos \phi \sin \chi \,.\, \lambda + \sin \phi \sin \chi \,.\, \mu + \cos \chi \,.\, \nu = 0,$$

and the form of the lines of like polarisation is determined by finding the section of the cone (14) by the surface of the plate

$$\cos \phi \sin \chi \,.\, x + \sin \phi \sin \chi \,.\, y + \cos \chi \,.\, z = T.$$

181. In the case of an uniaxal plate, $\Psi = 0$, and the surface of like polarisation becomes the plane

$$\mu x' - \lambda y' = 0,$$

and the cone of the second degree

$$\nu (x'^2 + y'^2) - z'(\lambda x' + \mu y') = 0.$$

Taking the same axes as in the case of the curves of constant retardation, we have to write $\xi \cos \chi + \zeta \sin \chi$ for x', η for y', $-\xi \sin \chi + \zeta \cos \chi$ for z', and $\lambda' \cos \chi + \nu' \sin \chi$ for λ, μ' for μ, $-\lambda' \sin \chi + \nu' \cos \chi$ for ν: but the line, for which the original direction-cosines were λ, μ, ν, is parallel to the surface of the plate, hence $\nu' = 0$ and if it make an angle ψ with the axis of ξ,

$$\mu' = \sin \psi, \qquad \lambda' = \cos \psi.$$

Making these substitutions and writing $\zeta = T$, the equations of the lines of like polarisation become

$$\xi - \eta \cot \psi + T \tan \chi = 0,$$

$$\eta^2 \tan \chi - \xi\eta \tan \psi \tan \chi + T(\xi + \eta \tan \psi + T \tan \chi) = 0.$$

The first of these equations represents a series of straight lines through the extremity of the optic axis ($\xi = -T \tan \chi$, $\eta = 0$), and the second a system of hyperbolas through the same point, the asymptotes of which are the trace of the principal section of the plate and the line

$$\eta - \xi \tan \psi = 0.$$

The curves of constant retardation are in this case given by (12) or to terms of the second order by

$$\frac{\delta}{T} = \frac{\Omega}{d} - \frac{\Omega}{a} + \frac{1}{2}\frac{a^2 - c^2}{d^2} \sin 2\chi \cos\theta \sin i$$
$$+ \frac{1}{2}\left(\frac{a}{\Omega} - \frac{c^2}{\Omega d}\right) \sin^2\theta \sin^2 i + \frac{1}{2}\left(\frac{a}{\Omega} - \frac{a^2c^2}{\Omega d^3}\right) \cos^2\theta \sin^2 i,$$

where d^2 is written for $a^2\cos^2\chi + c^2\sin^2\chi$. Hence their equation is

$$\frac{\delta}{T} = \frac{\Omega}{d} - \frac{\Omega}{a} + \frac{1}{2}\frac{a^2 - c^2}{d^2} \sin 2\chi \frac{\xi}{f} + \frac{1}{2}\left(\frac{a}{\Omega} - \frac{c^2}{\Omega d}\right)\frac{\eta^2}{f^2} + \frac{1}{2}\left(\frac{a}{\Omega} - \frac{a^2c^2}{\Omega d^3}\right)\frac{\xi^2}{f^2}.$$

The centre of the curves is at the point

$$\xi = -\frac{f}{2}\frac{\Omega d\,(a^2 - c^2)}{a\,(d^3 - ac^2)} \sin 2\chi, \qquad \eta = 0$$

and is at infinity, if $d^3 = ac^2$, and as the coefficient of ξ^2 is then zero, the curves are parabolas with their axes in the principal section of the plate: the value of χ corresponding to this case is given by

$$\cos^2\chi = \frac{c^{\frac{4}{3}}\,(a^{\frac{2}{3}} - c^{\frac{2}{3}})}{a^2 - c^2} = \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}} + a^{\frac{2}{3}}c^{\frac{2}{3}} + c^{\frac{4}{3}}},$$

which always gives a possible value of χ. According as χ is less or greater than this value, the curves will be ellipses or hyperbolas.

The curves can never become straight lines, as the coefficients of ξ^2 and η^2 cannot simultaneously vanish.

182. When the uniaxal plate is perpendicular to the optic axis, the curves of constant retardation are given by

$$\frac{\delta}{T} = \frac{\sqrt{\Omega^2 - c^2\sin^2 i}}{a} - \frac{\sqrt{\Omega^2 - a^2\sin^2 i}}{a} = \frac{(a^2 - c^2)\sin^2 i}{2a\Omega},$$

when i is small. For the principal curves of constant retardation $\delta = n\lambda$ and with small fields the squares of the sines of the angular radii of these curves form an arithmetic progression.

The lines of like polarisation in this case are

$$\xi - \eta\cot\psi = 0, \qquad \xi + \eta\tan\psi = 0,$$

and the principal lines of like polarisation are two pairs of straight lines parallel and perpendicular to the planes of polarisation and analysation.

The illumination of a small area $\rho d\rho d\psi$ of the field is

$$a^2\{\cos^2(\beta - \alpha) - \sin 2(\psi - \alpha)\sin 2(\psi - \beta)\sin^2(\pi\rho^2/F)\}\,\rho d\rho d\psi,$$

if we take the approximate value of δ and write F for $\dfrac{2a\Omega\lambda f^2}{(a^2 - c^2)\,T}$. Integrating

this with respect to ρ, we obtain for the amount of light that falls on the area of a curve of constant intensity included between the radii ψ and $\psi + d\psi$

$$\tfrac{1}{2}a^2 \cos^2(\beta - \alpha)(\rho_2^2 - \rho_1^2)\, d\psi - \tfrac{1}{4}a^2 \sin 2(\psi - \alpha) \sin 2(\psi - \beta)$$
$$\times \left\{\rho_2^2 - \rho_1^2 - \frac{F}{2\pi}\left(\sin\frac{2\pi\rho_2^2}{F} - \sin\frac{2\pi\rho_1^2}{F}\right)\right\} d\psi,$$

where ρ_1, ρ_2 are the distances from the centre of the field to the points in which the radius-vector cuts the curve. But we have (§ 178)

$$\rho_1^2/F = n + \epsilon/\lambda, \qquad \rho_2^2/F = (n+1) - \epsilon/\lambda,$$

whence the expression becomes

$$\tfrac{1}{2}a^2 \cos^2(\beta - \alpha)\, F(1 - 2\epsilon/\lambda)\, d\psi - \tfrac{1}{4}a^2 \sin 2(\psi - \alpha) \sin 2(\psi - \beta)$$
$$\times \left\{F(1 - 2\epsilon/\lambda) + \frac{F}{\pi} \sin\frac{2\pi\epsilon}{\lambda}\right\} d\psi,$$

which is independent of the order n of the space in which the curve is situated, and is the same for all curves having the same intensity-constant. This is true for each strip of the curves, and as all curves in a given region that have the same intensity-constant, are touched by the same pair of radii, it follows that the total illuminations of the areas bounded by each of the curves of a given intensity are the same.

183. With an uniaxal plate parallel to the optic axis, the curves of constant retardation are to terms of the second order

$$\frac{\delta}{T} = \frac{\Omega}{ac}(a - c) + \frac{a - c}{2\Omega}\left(\sin^2\theta - \frac{a}{c}\cos^2\theta\right)\sin^2 i.$$

Hence their equation is

$$c\eta^2 - a\xi^2 = \frac{2\Omega^2 f^2}{a}\left\{\frac{ac\delta}{\Omega(a - c)T} - 1\right\},$$

which represents a series of hyperbolas with asymptotes $\eta/\xi = \pm\sqrt{a/c}$.

If the field be small, there are no lines of like polarisation, as the polarisation of the streams within the plate is practically constant over the field, and when the principal section is parallel or perpendicular to either the plane of polarisation or that of analysation, the intensity is uniform. The isochromatic curves are most marked, when the planes of primitive and final polarisation are crossed and the principal section of the plate inclined to them at an angle of 45°.

With larger fields, the lines of like polarisation are a system of hyperbolas

$$\eta^2 - \xi\eta \tan\psi + T^2 = 0,$$

having for asymptotes the lines $\eta=0$, $\eta=\xi\tan\psi$. The real axis of the hyperbolas is inclined at an angle $\psi/2$ to the axis of ξ and the real semi-axis is

$$T\sqrt{2\cos\psi/(1-\cos\psi)}.$$

The principal lines of like polarisation can be best observed in white light with plates so thick that the coloured bands are invisible: with a crossed polariser and analyser they are seen as a black cross with its arms in the planes of primitive and final polarisation, when the principal section of the plate is in either of these planes, and in other cases they are a pair of hyperbolas with their real axes in the quadrant containing the optic axis and equally inclined to this direction, only one hyperbola being in the field, when the angle between the principal section and the plane of analysation or polarisation is small.

This affords a ready method of determining the direction of the optic axis in the case of a plate parallel to it. Starting with the case in which the achromatic curve is a black cross, the optic axis is parallel to one of the arms of the cross: if the plate be now slightly turned in its own plane, the cross becomes an hyperbola with its real axis in the quadrant, into which the optic axis has turned*.

184. Taking now the case of biaxal plates, let us first suppose that the surfaces of the plate are perpendicular to the mean line. The curves of constant retardation are then determined from (13), which may be written

$$\begin{aligned}&(a^2-c^2)^2\{\Omega^2\sin^2\Psi+b^2\cos^2\theta\sin^2 i+a^2\cos^2\Psi\sin^2\theta\sin^2 i\}^2\\&\quad-4(a^2-c^2)^2 b^2\Omega^2\sin^2\Psi\cos^2\theta\sin^2 i\\&\quad-2\frac{a^2b^2\delta^2}{T^2}\{(a^2+b^2)\Omega^2-b^2(a^2+c^2)\cos^2\theta\sin^2 i-a^2(b^2+c^2)\sin^2\theta\sin^2 i\}\\&\quad+\frac{a^4b^4\delta^4}{T^4}=0,\end{aligned}$$

where 2Ψ is the angle between the optic axes, and their equation is

$$\begin{aligned}&(a^2-c^2)^2\{b^2\xi^2+a^2\cos^2\Psi\,.\,\eta^2+\Omega^2f^2\sin^2\Psi\}^2-4\xi^2(a^2-c^2)^2b^2\Omega^2f^2\sin^2\Psi\\&-\frac{2a^2b^2f^2\delta^2}{T^2}\{(a^2+b^2)\Omega^2f^2-b^2(a^2+c^2)\xi^2-a^2(b^2+c^2)\eta^2\}+\frac{a^4b^4f^4\delta^4}{T^4}=0\end{aligned}\qquad\text{......(15).}$$

The lines of like polarisation are given by

$$\xi^2\cos^2\Psi-(\cos^2\Psi\cot\psi-\tan\psi)\,\xi\eta-\eta^2=T^2\sin^2\Psi\qquad\text{......(16).}$$

A system of hyperbolas through the points corresponding to the optic axes ($\xi=\pm T\tan\Psi$, $\eta=0$) having for their asymptotes the lines

$$\xi\cos^2\Psi+\eta\tan\psi=0,\quad \xi-\eta\cot\psi=0.$$

* Lommel, *Wied. Ann.* XVIII. 56 (1883).

When the angle between the optic axes is very small, the equations take simpler forms, as we may write $a=b$ in the small terms: neglecting δ^4 and $\delta^2 \sin^2 i$, the curves of constant retardation become

$$(\sin^2 i + \sin^2 \Psi')^2 - 4 \sin^2 \Psi' \cos^2 \theta \sin^2 i = \frac{2a^2 (a^2 + b^2) \Omega^2 \delta^2}{b^2 (a^2 - c^2)^2 T^2},$$

where $2\Psi'$ is the apparent angle between the optic axes. In Cartesian coordinates if the points corresponding to the optic axes be $\pm \alpha$, 0, the equation becomes

$$(\xi^2 + \eta^2 + \alpha^2)^2 - 4\alpha^2 \xi^2 = \frac{2a^2 (a^2 + b^2) \Omega^2 f^4 \delta^2}{b^2 (a^2 - c^2)^2 T^2},$$

a system of Cassini's ovals.

If i', i'' be the angles of incidence corresponding to the points in which a given curve cuts the plane of the optic axes, the complete equation gives

$$\sin^2 i' + \sin^2 i'' = 2 \sin^2 \Psi' - \frac{2a^2 (a^2 + c^2) \delta^2}{(a^2 - c^2)^2 T^2},$$

and if the curves be Cassini's ovals

$$\sin^2 i + \sin^2 i'' = 2 \sin^2 \Psi'.$$

When the angle between the optic axes is small, the lines of like polarisation become the equilateral hyperbolas

$$\xi^2 - 2\xi\eta \cot 2\psi - \eta^2 = T^2 \sin^2 \Psi,$$

and since $d\eta/d\xi = \tan 2\psi$ for $\eta = 0$, the angle that an hyperbola makes with the trace of the plane of the optic axes at the point corresponding to an optic axis is 2ψ, and hence at this point the angle between the principal lines of like polarisation is twice that between the planes of primitive and final polarisation. When these planes are crossed, the principal hyperbolas coincide, and if either of these planes coincide with the plane of the optic axes, the principal lines of like polarisation are straight lines in and perpendicular to this plane.

185. With a plate parallel to the plane of the optic axes, the relative retardation is obtained from (13) by changing c, b, a in cyclical order, if we take the axis of ζ perpendicular to its faces, and that of ξ parallel to the greatest axis of the ellipsoid of polarisation, and as far as terms of the second order

$$a^2 c^2 \delta^2 / T^2 = (a - c)^2 \Omega^2 + (a - c)(ac - b^2)(a \cos^2 \theta - c \sin^2 \theta) \sin^2 i;$$

whence the curves of constant retardation are the system of hyperbolas

$$a\xi^2 - c\eta^2 = \frac{f^2}{(a-c)(ac-b^2)} \left\{ \frac{a^2 c^2 \delta^2}{T^2} - (a - c)^2 \Omega^2 \right\},$$

having the lines $\eta/\xi = \sqrt{a/c}$ for their asymptotes.

Making the same changes in (16), the lines of like polarisation are given by

$$-\sin^2\Psi\,.\,\xi^2+\xi\eta\,\frac{1-\cos 2\Psi\cos 2\psi}{\sin 2\psi}-\cos^2\Psi\,.\,\eta^2=T^2,$$

and are thus hyperbolas with asymptotes

$$\xi\sin\psi-\eta\cos\psi=0,\quad \xi\cos\psi\sin^2\Psi-\eta\sin\psi\cos^2\Psi=0.$$

The real axes of the hyperbolas make angles

$$\tfrac{1}{2}\tan^{-1}\{(1-\cos 2\Psi\cos 2\psi)/(\cos 2\Psi\sin 2\psi)\}$$

with the axis of ξ and referred to their principal axes their equation becomes

$$\{\sqrt{1+\cos^2 2\Psi-2\cos 2\Psi\cos 2\psi}-\sin 2\psi\}\,x^2$$
$$-\{\sqrt{1+\cos^2 2\Psi-2\cos 2\Psi\cos 2\psi}+\sin 2\psi\}\,y^2=2T^2\sin 2\psi.$$

The principal lines of like polarisation are however scarcely visible, as the polarisation of the streams within the plate varies but slightly within the field of view.

186. The interference phenomenon, observed with a biaxal plate perpendicular to one of the optic axes, has a certain resemblance to that produced by an uniaxal plate similarly cut; the rings, when the field is small, being circles round the point corresponding to the optic axis.

This may be seen from the following approximate calculation. The relative retardation of the interfering streams is

$$\delta=T\sin i\,(\cot r_2-\cot r_1)=\Omega T\,(\cos r_2/\omega_2-\cos r_1/\omega_1).$$

When the double refraction is weak and the field is small, we may replace the angles r_1, r_2 by their mean value r and regard ω_1, ω_2 as the propagational speeds of the waves travelling in a given direction making angles ϕ, ϕ' with the optic axes, then

$$\delta=\Omega T\cos r\,\frac{{\omega_1}^2-{\omega_2}^2}{\omega_1\omega_2(\omega_1+\omega_2)}=\Omega T\cos r\,\frac{a^2-c^2}{2b^3}\sin\phi\sin\phi',$$

if we write for ω_1, ω_2 in the denominator their common value b corresponding to normal incidence: but

$$\sin\phi'\fallingdotseq\sin 2\Psi=2\sqrt{(a^2-b^2)(b^2-c^2)}/(a^2-c^2),$$
$$\sin\phi=\sin r\fallingdotseq b\sin i/\Omega,$$

whence retaining only the first power of $\sin i$

$$\delta=T\,\{\sqrt{(a^2-b^2)(b^2-c^2)}/b^2\}\sin i.$$

To this approximation the curves of constant retardation are circles with radii proportional to the relative retardation, instead of the square root of this quantity, as in the case of the uniaxal plate.

To obtain the lines of like polarisation, we have to write in (14)

$$x' = \xi \cos \Psi + T \sin \Psi, \qquad y' = \eta, \qquad z' = -\xi \sin \Psi + T \cos \Psi,$$

$$\lambda = \cos \psi \cos \Psi, \qquad \mu = \sin \psi, \quad \nu = -\cos \psi \sin \Psi,$$

which gives

$$\eta (\xi \sin \psi - \eta \cos \psi)^2 - 2T \cot 2\Psi (\xi \sin \psi - \eta \cos \psi)(\xi \cos \psi + \eta \sin \psi) - T^2 (\xi \sin 2\psi - \eta \cos 2\psi) = 0,$$

and when the field is very small

$$\xi \sin 2\psi - \eta \cos 2\psi = 0.$$

Thus in this case, the principal lines of like polarisation are two straight lines through the centre of the field inclined to one another at an angle equal to twice that between the planes of polarisation and analysation. When these planes are at right-angles, these lines coincide and the planes of polarisation and analysation bisect the angles between this line and the trace of the plane of the optic axes. Hence as the plate is turned in its own plane, the principal line of like polarisation turns at the same rate in the opposite direction. In the case of the uniaxal plate perpendicular to the optic axis between crossed polariser and analyser, the principal lines of like polarisation form a black cross, that remains fixed as the plate is rotated about its normal.

187. When a conical pencil of white light is employed, coloured rings or bands are obtained, provided, as in all cases of interference, the relative retardation of the interfering streams is not too great. The isochromatic curves follow more or less the course of the curves of constant retardation, unless the dispersion is excessive, and the principal lines of like polarisation become, when the polariser and analyser are crossed, dark brushes fringed with colour. The phenomena may however be considerably modified, if the axes of the ellipsoid of polarisation not only vary in magnitude, but also change their position, as the wave-length alters.

An uniaxal crystal has its optic axis in an invariable direction for all colours, determined by the principal axis of the crystallographic system to which it belongs, and the dispersion only affects the law of distribution of colour in the coloured curves. There are however a small number of crystals, such as the uniaxal Apophylite and Brucite of Texas, that are of opposite sign for the extreme spectral colours, becoming isotropic for some intermediate wave-length.

In biaxal crystals the optic axes have, in the majority of cases, different directions for the different colours and the plane in which they are situated may also vary, and crystals may exhibit both dispersion of the optic axes and dispersion of the mean line. The different cases of dispersion may be most conveniently examined with plates perpendicular to the first mean line placed

in the diagonal position between crossed polariser and analyser, that is with the plane of the optic axes for light of mean wave-length bisecting the angle between the planes of primitive and final polarisation.

With crystals of the prismatic system, the axes of the ellipsoid of polarisation coincide with the crystallographic axes for all wave-lengths and are fixed in direction, but in some few cases, such as Brookite, their order of magnitude changes, so that the plane of the optic axes for red light is at right-angles to the plane of the optic axes for blue light, the crystal being uniaxal for some intermediate colour. More generally the optic axes are in the same plane for all colours, the dispersion only affecting the angle between them, which may be the greater for blue light as in Aragonite, or for red light as is the case with Topaz. In these more common cases the system of rings is coloured symmetrically with respect to the plane of the optic axes and to the perpendicular line, while the colour for which the apparent separation of the axes is the least is that on the concave side of the summit of the hyperbolic brushes and on the concave side of the first ring round the point corresponding to the optic axis at the part nearest the centre of the field.

Crystals of the monoclinic system have one axis of the ellipsoid of polarisation fixed in direction, while the other two may have any directions at right-angles to one another in the perpendicular plane—the one plane of symmetry of the system. In a few rare instances, as is the case with crystals of Magnesium Ammonium Chromate, the fixed axis changes with the colour of the light from the greatest to the mean or to the least axis of the ellipsoid, but we need only consider the more ordinary cases, in which for all colours the fixed axis is (1) the first mean-line, (2) the second mean-line, (3) the intermediate axis of the ellipsoid of polarisation.

(1) When the first mean-line is in the direction of the fixed axis, the dispersion affects the angle between the optic axes and, if the position of the second mean-line vary in the plane of symmetry of the system, the plane in which the optic axes lie. This is known as "crossed dispersion," of which Borax affords an example, and is recognised by a symmetrical distribution of colour in the interference pattern with respect to the centre alone.

(2) If the fixed axis determine the second mean-line, and the position of the first mean-line in the perpendicular plane be dependent upon the colour, the optic axes for the different colours are in planes that intersect in the normal to the plane of symmetry. We then have dispersion of the optic axes accompanied with dispersion of the mean-line in the perpendicular direction, with respect to which the colour of the fringes is symmetrical. This is called "horizontal dispersion" and is exhibited by Adularia.

(3) "Inclined dispersion" occurs when the intermediate axis of the ellipsoid of polarisation is given by the fixed direction, while the other axes change their direction in the perpendicular plane. The plane of the optic

axes is then the same for all colours, but there is dispersion of the mean-line in this plane. This variety of dispersion is shown by Gypsum, Diopside and Sphene: it results in a symmetrical distribution of colour with respect to the trace of the plane of the optic axes on the plane of the interference pattern.

In crystals of the anorthic system none of the axes of the ellipsoid has a prescribed direction and nothing can be inferred *à priori* respecting the character of the dispersion of the optic axes.

188. The rings and brushes obtained with crystalline plates undergo considerable alteration and even entirely change their character, if the light employed be elliptically or circularly polarised and be subsequently either plane, circularly or elliptically analysed.

All varieties of polarisation from plane to circular may be obtained by associating an ordinary polariser with a plate of crystal of such a thickness that a relative retardation of $\lambda/4$ is introduced between the components, into which it divides a stream of polarised light. Such plates are termed quarter-wave plates and are usually of mica or selenite, on account of the facility with which these crystals can be split into extremely thin laminæ.

In the case of mica, which is a negative biaxal crystal, the laminæ are perpendicular to the plane of the optic axes and the stream polarised in this plane is less retarded than that polarised in the perpendicular plane: while with selenite, a positive biaxal crystal, the plates are parallel to the optic axes and the stream polarised in a plane parallel to the first mean-line is the one that is least retarded.

Let us then suppose that we place before a plate of crystal an ordinary polariser followed by a quarter-wave plate and that the light after passing through the crystal traverses a second quarter-wave plate and an analyser: we will further assume that though the light converges on the plate of crystal, it is so nearly parallel during its passage through the polariser, quarter-wave plates and analyser, that the polarisations may be regarded as constant over the whole beam for each of these portions of its course.

Let α be the angle between the primitive plane of polarisation and the plane of least retardation of the first quarter-wave plate,

ψ_1 the angle between this plane and the plane of polarisation of the quicker wave in the plate of crystal,

ψ_2 the angle between this plane of polarisation and the plane of least retardation of the second quarter-wave plate, and finally

β the angle between this plane and the final plane of analysation.

Then resolving in turn along directions parallel and perpendicular to the planes of polarisation of the quicker waves in the successive plates and

remembering that a retardation of $\lambda/4$ is introduced by a factor $-\iota$, the light emergent from the analyser is represented by the vector

$$\begin{aligned} a\,[\{&\cos\psi_2(\cos\alpha\cos\psi_1+\iota\sin\alpha\sin\psi_1) \\ &+\sin\psi_2(-\cos\alpha\sin\psi_1+\iota\sin\alpha\cos\psi_1)\,e^{-\iota\kappa\delta}\}\cos\beta \\ &+\iota\,\{\sin\psi_2(\cos\alpha\cos\psi_1+\iota\sin\alpha\sin\psi_1) \\ &-\cos\psi_2(-\cos\alpha\sin\psi_1+\iota\sin\alpha\cos\psi_1)\,e^{-\iota\kappa\delta}\}\sin\beta]\,e^{\iota nt} \\ &=a\,\{A+B\iota+Ce^{-\iota\kappa\delta}+\iota De^{-\iota\kappa\delta}\}\,e^{\iota nt},\ \text{(say)}, \end{aligned}$$

where

$$\begin{aligned} A&=\cos\alpha\cos\beta\cos\psi_1\cos\psi_2-\sin\alpha\sin\beta\sin\psi_1\sin\psi_2, \\ B&=\cos\alpha\sin\beta\cos\psi_1\sin\psi_2+\sin\alpha\cos\beta\sin\psi_1\cos\psi_2, \\ C&=-\cos\alpha\cos\beta\sin\psi_1\sin\psi_2+\sin\alpha\sin\beta\cos\psi_1\cos\psi_2, \\ D&=\cos\alpha\sin\beta\sin\psi_1\cos\psi_2+\sin\alpha\cos\beta\cos\psi_1\sin\psi_2. \end{aligned}$$

Whence the intensity, obtained by multiplying this by its conjugate expression, is

$$\begin{aligned} I&=a^2\,\{A^2+B^2+C^2+D^2+2\,(AC+BD)\cos\kappa\delta+2\,(AD-BC)\sin\kappa\delta\} \\ &=a^2\left\{(A+C)^2+(B+D)^2-4\,(AC+BD)\sin^2\frac{\pi\delta}{\lambda}\right. \\ &\left.+4\,(AD-BC)\sin\frac{\pi\delta}{\lambda}\cos\frac{\pi\delta}{\lambda}\right\} \\ &=a^2\left\{\cos^2(\alpha-\beta)\cos^2(\psi_1+\psi_2)+\sin^2(\alpha+\beta)\sin^2(\psi_1+\psi_2)\right. \\ &-(\sin 2\alpha\sin 2\beta-\cos 2\alpha\cos 2\beta\sin 2\psi_1\sin 2\psi_2)\sin^2\frac{\pi\delta}{\lambda} \\ &\left.+(\cos 2\alpha\sin 2\beta\sin 2\psi_1+\sin 2\alpha\cos 2\beta\sin 2\psi_2)\sin\frac{\pi\delta}{\lambda}\cos\frac{\pi\delta}{\lambda}\right\} \quad \text{......(17)}, \end{aligned}$$

in which expression the first two terms represent the intensity when the plate of crystal is removed.

We have then, as in the case of plane polarisation and analysation, principal curves of constant retardation $\delta=n\lambda$, along which the intensity is the same as before the introduction of the plate, but there are in general no principal lines of like polarisation. The intensity is a maximum or a minimum when

$$\tan 2\pi\frac{\delta}{\lambda}=\frac{\cos 2\alpha\sin 2\beta\sin 2\psi_1+\sin 2\alpha\cos 2\beta\sin 2\psi_2}{\sin 2\alpha\sin 2\beta-\cos 2\alpha\cos 2\beta\sin 2\psi_1\sin 2\psi_2},$$

having then the values

$$\begin{aligned} I=\frac{a^2}{2}\{1&+\cos 2\alpha\cos 2\beta\cos 2\psi_1\cos 2\psi_2 \\ &\pm\sqrt{(1-\cos^2 2\alpha\cos^2 2\psi_1)(1-\cos^2 2\beta\cos^2 2\psi_2)}\}. \end{aligned}$$

189. The above equations contain the solution of all the cases that may arise, but the most interesting ones are those in which either the coefficient of $\sin^2(\pi\delta/\lambda)$ or that of $\sin(2\pi\delta/\lambda)$ in equation (17) vanishes*.

The coefficient of $\sin^2(\pi\delta/\lambda)$ is zero, when $\alpha = \pm\pi/4$, $\beta = 0$ or $\pi/2$, and when $\alpha = 0$ or $\pi/2$, $\beta = \pm\pi/4$.

In the first case the light is plane analysed and circularly polarised in a right- or left-handed direction according as $\alpha = +\pi/4$ or $= -\pi/4$; and calling ψ the angle between the plane of polarisation of the least retarded stream in the crystalline plate and the final plane of polarisation, the intensity is in the two cases

$$I = \frac{a^2}{2}\{1 \pm \sin 2\psi \sin(2\pi\delta/\lambda)\}.$$

In the second case the light is plane polarised and circularly analysed in a right- or left-handed direction, according as $\beta = +\pi/4$ or $= -\pi/4$, if we understand by a right-handed circular analyser one that permits the transmission of a right-handed circularly polarised stream, in other words a combination of a quarter-wave plate and a plane analyser that, used as a polariser, produces a right-handed circularly polarised stream. Calling ψ' the angle between the original plane of polarisation and the plane of polarisation of the quicker wave in the plate of crystal, the intensity is

$$I = \frac{a^2}{2}\{1 \pm \sin 2\psi' \sin(2\pi\delta/\lambda)\},$$

according as the circular analysation is right- or left-handed.

Let us suppose the light to be plane polarised and circularly analysed in a right-handed direction, then γ and η being the angles that the primitive plane of polarisation and the plane of polarisation of the quicker wave in the plate make respectively with a fixed plane in the crystal, the intensity is given by

$$I = \frac{a^2}{2}\left\{1 + \sin 2(\eta - \gamma)\sin 2\pi\frac{\delta}{\lambda}\right\}.$$

Hence the illumination is the same as before the insertion of the plate along the curves of constant retardation $\delta = n\lambda/2$ and along the principal lines of like polarisation $\eta = \gamma$, $\eta = \pi/2 + \gamma$: while in the region for which

$$\gamma < \eta < \pi/2 + \gamma$$

the dark curves are given by

$$\delta = (n + 3/4)\lambda, \quad n = 0, 1, 2 \ldots,$$

the absolute minima occurring on the line $\eta = \gamma + \pi/4$, and outside this region the curves

$$\delta = (n + 1/4)\lambda, \quad n = 0, 1, 2 \ldots$$

* Bertin, *Ann. de Ch. et de Phys.* (3) LVII. 257 (1859); (5) XVIII. 495 (1879).

are dark, the absolute minima being on the line $\eta = \gamma + 3\pi/4$. Thus the interposition of the quarter-wave plate has had the effect of pushing the bands outwards by a quarter of an interval in the first region and pulling them in by the same amount in the remainder of the field.

Similar results are obtained when the analysation is left-handed, and when the polarisation is circular and the analysation plane.

In order to fix our ideas, let us assume that the primitive and final planes of polarisation are at right angles; then the principal lines of like polarisation are determined by these planes and the dark curves are expanded in the region bounded by these lines that contains the line of like polarisation, for which the plane of polarisation of the quicker wave in the crystalline plate is parallel to that of the quicker wave in the quarter-wave plate.

Moreover we see that there is a distinction between the behaviour of positive and negative crystals; for η being the angle between the fixed plane and the plane of polarisation of the least retarded stream in the crystalline plate, the region for which $\gamma < \eta < \gamma + \pi/2$ in the case of a positive plate is that for which $\gamma + \pi/2 < \eta < \gamma + \pi$ for a similarly orientated negative plate and *vice versâ*, so that on passing from the one case to the other the parts of the field, in which contraction and expansion occur, are interchanged.

Now in the case of a positive uniaxal crystal the stream polarised in the principal section is the least retarded: hence with a plate perpendicular to the optic axis, the primitive and final planes of polarisation being crossed, expansion or contraction of the rings will occur in the quadrants that contain the plane of least retardation of the quarter-wave plate, according as the crystal is positive or negative.

Turning now to the case of a biaxal plate cut perpendicularly to the first mean line, let us again suppose the primitive and final planes of polarisation to be at right angles, and the plate placed in the diagonal position, so that the plane of least retardation of the quarter-wave plate is either parallel or perpendicular to the plane of the optic axes of the biaxal plate under consideration. Then the principal line of like polarisation is a rectangular hyperbola with its asymptotes inclined at 45° to the trace of the plane of the optic axes and the contraction of the rings will occur in the region of the field bounded by this hyperbola that contains the line of like polarisation, for which the plane of polarisation of the quicker wave in the crystal is perpendicular to the plane of least retardation of the quarter-wave plate.

Now if the biaxal crystal be positive, the greatest axis of the ellipsoid of polarisation coincides with the first mean line, and it follows that corresponding to points on the trace of the plane of the optic axes lying on the concave side of the hyperbola, the plane of polarisation of the quicker wave is parallel to the plane of the optic axes, while it is perpendicular to this plane for the other

points of this trace and for points on the line perpendicular to it through the centre of the field. Hence with a positive plate, expansion or contraction of the rings will occur on the concave side of the hyperbola according as the plane of the optic axes of the biaxal plate is parallel or perpendicular to the plane of polarisation of the least retarded stream in the quarter-wave plate, the reverse being the case with a negative biaxal plate.

190. The coefficient of $\sin(2\pi\delta/\lambda)$ in equation (17) vanishes when

(1) $\alpha = 0$ or $\pi/2$ and $\beta = 0$ or $\pi/2$,

(2) $\alpha = \pm\pi/4$ and $\beta = \pm\pi/4$,

(3) $\psi_1 + \psi_2 = 0$ and $\beta = \alpha$ or $\alpha + \pi/2$,

or $\psi_1 + \psi_2 = \pi/2$ and $\beta = -\alpha$ or $-\alpha + \pi/2$.

The first of these three cases need not concern us further, as the light is then plane polarised and plane analysed.

In the second case the polarisation and analysation are both circular, the direction being the same if $\alpha = \beta = \pm\pi/4$ and opposite if $\alpha = -\beta = \pm\pi/4$, and the corresponding intensities are

$$I = a^2 \cos^2(\pi\delta/\lambda) \text{ and } I = a^2 \sin^2(\pi\delta/\lambda).$$

There are then no principal lines of like polarisation and the dark curves are continuous; circles with uniaxal plates perpendicular to the optic axis and Cassini's ovals with biaxal plates normal to the first mean line.

In the third case the light is polarised and analysed elliptically, the quarter-wave plates being parallel if $\psi_1 + \psi_2 = 0$ and crossed if $\psi_1 + \psi_2 = \pi/2$.

The character of elliptic polarisation is determined by the ellipse traced out by the extremity of the polarisation-vector that characterises the stream, and two elliptic polarisers are similar, if the ellipses be similar for the streams that they produce; they have the same sign, if the ellipses be traversed in the same direction; and the angle between them is that included between the planes of maximum polarisation of the emergent streams. An elliptic analyser is defined by the nature of the polarisation produced, when its position being retained the light is transmitted through it in the opposite direction and the position of the observer is reversed.

With these definitions we see that the polariser and analyser are similar, parallel and of the same sign, when $\psi_1 + \psi_2 = 0$, $\beta = \alpha$ and when $\psi_1 + \psi_2 = \pi/2$, $\beta = \pi/2 - \alpha$; that they are similar, crossed and of opposite sign, when $\psi_1 + \psi_2 = 0$, $\beta = \pi/2 + \alpha$ and when $\psi_1 + \psi_2 = \pi/2$, $\beta = -\alpha$.

The intensity in the first case is

$$I = a^2 \{1 - (1 - \cos^2 2\alpha \cos^2 2\psi_1) \sin^2(\pi\delta/\lambda)\},$$

and in the second case

$$I = a^2 \{1 - \cos^2 2\alpha \cos^2 2\psi_1\} \sin^2(\pi\delta/\lambda).$$

As these two expressions are complementary, we need only consider the phenomenon represented by the second. The principal curves of constant retardation are given by $\delta = n\lambda$ and these curves are black, the curves $\delta = n\lambda/2$ being bright: there are no principal lines of like polarisation, since the expression $1 - \cos^2 2\alpha \cos^2 2\psi_1$ is never zero, but the intensity of the bright curves is a minimum where they are cut by the lines $\psi_1 = 0$, $\psi_1 = \pi/2$, that is by the lines for which the plane of polarisation of the quicker wave in the plate is parallel or perpendicular to the planes of polarisation of the least retarded streams in the quarter-wave plates, and the intensity is a maximum along the lines $\psi_1 = \pi/4$, $\psi_1 = 3\pi/4$.

191. When a conical pencil of polarised light passes through a combination of two crystalline plates and is subsequently analysed, the intensity at any point of the field is given by (8) and since the angles ψ_1 and ψ_2 as well as the retardations δ_1 and δ_2 in general vary over the extent of the field, the interference pattern in monochromatic light becomes very complicated.

Thus writing the equation in the form

$$\begin{aligned} I = a^2 \Bigg[\cos^2(\beta - \alpha) &- \sin 2(\psi_1 - \alpha) \sin 2(\psi_1 - \beta) \sin^2 \frac{\pi\delta_1}{\lambda} \\ &- \sin 2(\psi_2 - \alpha) \sin 2(\psi_2 - \beta) \sin^2 \frac{\pi\delta_2}{\lambda} \\ &- 2 \sin 2(\psi_1 - \alpha) \sin 2(\psi_2 - \beta) \sin \frac{\pi\delta_1}{\lambda} \sin \frac{\pi\delta_2}{\lambda} \\ &\times \left\{ \cos \frac{\pi\delta_1}{\lambda} . \cos \frac{\pi\delta_2}{\lambda} - \cos 2(\psi_2 - \psi_1) \sin \frac{\pi\delta_1}{\lambda} \sin \frac{\pi\delta_2}{\lambda} \right\} \Bigg]. \end{aligned}$$

We see that when the polariser and analyser are crossed ($\beta - a = \pi/2$) the intensity is zero only if

$$\left. \begin{aligned} \sin 2(\psi_1 - \alpha) \sin \frac{\pi\delta_1}{\lambda} &= \pm \sin 2(\psi_2 - \alpha) \sin \frac{\pi\delta_2}{\lambda} \\ \cos \frac{\pi\delta_1}{\lambda} \cos \frac{\pi\delta_2}{\lambda} - \cos 2(\psi_2 - \psi_1) \sin \frac{\pi\delta_1}{\lambda} \sin \frac{\pi\delta_2}{\lambda} &= \mp 1 \end{aligned} \right\} \text{.........(18)},$$

so that there are no longer continuous dark curves, but only isolated dark spots given by the intersection of the systems of curves (18)*.

192. When however the field is very small or when the primitive light is white, so that the interference is visible only for small retardations, the problem may in many cases be reduced to one of much less complexity by

* Cf. Langberg, *Pogg. Ann.* Erg.-Bd. I. 529 (1842). Ohm, *Abhandl. Bayer. Akad.* VII. 43, 265 (1855). Van der Willigen, *Arch. du musée Teyler*, III. 241 (1873). Bertin, *Ann. de Ch. et de Phys.* (6) II. 485 (1884). Pockels, *Gött. Nachr.* (1890) 259. Hecht, *N. Jahrb. für Min.* Beil.-Bd. XI. 318 (1898).

the possibility of regarding the polarisations of the streams within the compound plate as constant over the part considered.

As an instance of this simplification of the investigation, let us consider the case of a Savart's plate, which consists of two plates of an uniaxal crystal of equal thickness, cut at the same inclination—about 45°—to the optic axis and superposed with their principal sections at right angles.

Taking as the fixed plane of reference the principal section of the first plate and regarding the polarisations of the streams within the plate as invariable over the field, the expression for the intensity reduces to

$$I = a^2 \left\{\cos^2(\beta - \alpha) - \sin 2\alpha \sin 2\beta \sin^2 \pi \frac{\delta_1 - \delta_2}{\lambda}\right\}.$$

Hence the interference pattern depends upon the single system of curves of constant retardation $\delta_1 - \delta_2 = \text{const.}$, and the principal curves of constant retardation are given by $\delta_1 - \delta_2 = n\lambda$, and these are bright, black or of intermediate intensity, according as the polariser and analyser are parallel, crossed or inclined at some other angle.

To determine the form of the curves, we have from § 181

$$\frac{\delta_1}{T} = \Omega\left(\frac{1}{d} - \frac{1}{a}\right) + \tfrac{1}{2}\frac{a^2 - c^2}{d^2}\sin 2\chi \cos\theta \sin i$$
$$+ \tfrac{1}{2}\left(\frac{a}{\Omega} - \frac{c^2}{\Omega d}\right)\sin^2\theta \sin^2 i + \tfrac{1}{2}\left(\frac{a}{\Omega} - \frac{a^2c^2}{\Omega d^3}\right)\cos^2\theta \sin^2 i,$$

where χ is the angle that the optic axis makes with the normal to the plate and $d^2 = a^2\cos^2\chi + c^2\sin^2\chi$; while δ_2 is obtained from this expression by writing $\theta - 90°$ for θ. Hence the principal curves of constant retardation are given by

$$\frac{m\lambda}{T} = \frac{(a^2 - c^2)\sin\chi\cos\chi}{a^2\cos^2\chi + c^2\sin^2\chi}(\cos\theta - \sin\theta)\sin i$$
$$- \frac{c^2(a^2 - c^2)\sin^2\chi}{2\Omega(a^2\cos^2\chi + c^2\sin^2\chi)^{\frac{3}{2}}}(\cos^2\theta - \sin^2\theta)\sin^2 i.$$

The curves are thus equilateral hyperbolas seen at a considerable distance from their vertices, with asymptotes parallel to the bisectors of the angles between the principal sections of the plates. They appear as a system of parallel straight lines bisecting the angle between the principal sections that is related in the same manner to the directions of the optic axes of the two plates.

The terms of the first order attain their maximum importance when $\cos 2\chi = -(a^2 - c^2)/(a^2 + c^2)$, which corresponds to an inclination of the optic axes of nearly 45° and in this case the bands are very close together.

193. Returning to the general formula (8), let us consider the interference phenomena exhibited by a twin uniaxal plate*.

Taking the principal section of the plate as the fixed plane of reference, let OL represent the normal to the plate, OA_1, OA_2 the optic axes of the first and second constituents respectively, OM the direction of the wave-normal in the plates, neglecting as we have done in obtaining (8) the effect of the refraction at the twin plane: then we have

$$\sin\psi_1 = \sin r \sin\theta / \sin A_1M,$$

$$\sin\psi_2 = \sin r \sin\theta / \sin A_2M,$$

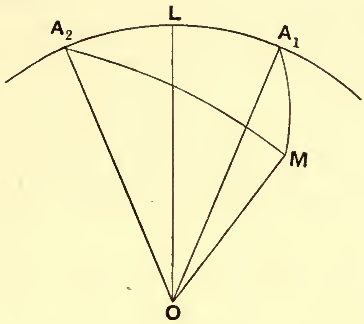


Fig. 44.

and

$$\cos A_1M = \cos r \cos\chi + \sin r \sin\chi \cos\theta,$$

$$\cos A_2M = \cos r \cos\chi - \sin r \sin\chi \cos\theta.$$

The character of the interference pattern depends upon the four systems of curves of constant retardation

$$\delta_1 = \text{const.},\quad \delta_2 = \text{const.},\quad \delta_1 + \delta_2 = \text{const.},\quad \delta_1 - \delta_2 = \text{const.},$$

of which the first two are called the primary curves of the first and second kind; while the last two may be termed the secondary curves of the first and second kind respectively. There are in general no principal lines of like polarisation.

The primary curves of the first kind are given by the equation

$$\left(\frac{a}{\Omega} - \frac{a^2c^2}{\Omega d^3}\right)\cos^2\theta \sin^2 i + \left(\frac{a}{\Omega} - \frac{c^2}{\Omega d}\right)\sin^2\theta\sin^2 i$$
$$+ \frac{a^2 - c^2}{d^2}\sin 2\chi\cos\theta\sin i + 2\left(\frac{\Omega}{d} - \frac{\Omega}{a}\right) = \frac{h\lambda}{T_1},$$

where T_1 is the thickness of the first plate, and those of the second kind are given by an equation obtained from this by changing T_1 into T_2 and writing $\theta \pm \pi$ for θ. Hence the equations of the secondary curves of the first and second kind are given by

$$\left(\frac{a}{\Omega} - \frac{a^2c^2}{\Omega d^3}\right)\cos^2\theta \sin^2 i + \left(\frac{a}{\Omega} - \frac{c^2}{\Omega d}\right)\sin^2\theta\sin^2 i$$
$$+ \frac{T_1 \mp T_2}{T_1 \pm T_2}\cdot\frac{a^2 - c^2}{d^2}\sin 2\chi\cos\theta\sin i + 2\left(\frac{\Omega}{d} - \frac{\Omega}{a}\right) = \frac{h\lambda}{T_1 \pm T_2},$$

the upper and lower signs referring to the curves of the first and second kind respectively.

* Pockels, *loc. cit.*

Thus in general the curves of the four systems differ from one another only in their dimensions and in the positions of their centres in the trace of the principal section of the plate on the screen of observation. If the plates have the same thickness, the secondary curves of the second kind become a system of straight lines perpendicular to the principal section; those of the first kind have their centre at the middle point of the field and when the primary curves are parabolas, become straight lines parallel to the principal section.

Let us now suppose that the planes of polarisation and analysation are crossed and parallel and perpendicular to the principal section of the plate—the so-called normal position. Then the expression for the intensity becomes

$$I = a^2 \left\{ a_1 \sin^2 \frac{\pi \delta_1}{\lambda} + a_2 \sin^2 \frac{\pi \delta_2}{\lambda} + a_3 \sin^2 \frac{\pi (\delta_1 + \delta_2)}{\lambda} + a_4 \sin^2 \frac{\pi (\delta_1 - \delta_2)}{\lambda} \right\},$$

where

$$a_1 = \sin 2\psi_1 \cos 2\psi_2 \sin 2(\psi_1 - \psi_2), \quad a_2 = \cos 2\psi_1 \sin 2\psi_2 \sin 2(\psi_2 - \psi_1),$$

$$a_3 = \sin 2\psi_1 \sin 2\psi_2 \cos^2(\psi_1 - \psi_2), \quad a_4 = -\sin 2\psi_1 \sin 2\psi_2 \sin^2(\psi_2 - \psi_1).$$

Even in this case the equation is too complicated for general treatment and it is necessary to obtain an idea of the principal features of the interference pattern by considering different parts of the field, at which one or other of the four systems of curves attains a primary importance.

(*a*) Along the trace of the principal section of the plate, $\psi_1 = 0$ or π, $\psi_2 = \pi$ or 0 and a_1, a_2, a_3, a_4 are all zero: hence through the centre of the field and parallel to the principal section there is a black brush, that has its greatest width near the centre, where the increase in the value of the coefficients is slowest.

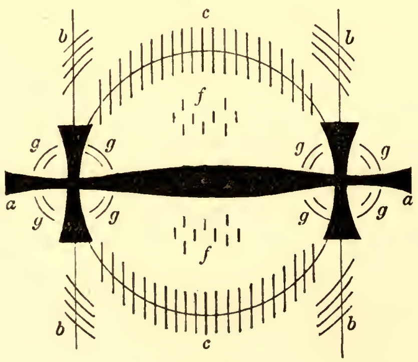


Fig. 45.

(*b*) On lines perpendicular to the principal section through the points corresponding to the optic axes, either $\psi_2 = \pm \pi/2$ or $\psi_1 = \pm \pi/2$ and in these two cases respectively only $a_1 = \sin^2 2\psi_1$ or $a_2 = \sin^2 2\psi_2$ differ from zero. Consequently on these lines the primary curves of the first or second kind are

alone visible, and the intensity of the bright curves of these systems increases with their distance from the axial points, while near these points there is a short dark brush perpendicular to the principal section, $\sin 2\psi_1$ and $\sin 2\psi_2$ being there very small.

(*c*) On the circle having the line joining the axial points as diameter, $\psi_1 - \psi_2 = \pm \pi/2$ and all the coefficients vanish except $a_4 = \sin^2 2\psi_1$, so that on this circle the secondary curves of the second kind alone appear; but in no part of the field are the secondary curves of the first kind isolated, since the case, in which all the coefficients except a_3 are zero, cannot occur.

(*d*) For points near the black brush in the centre of the field,

$$\psi_2 = \pm(\pi - \epsilon), \quad \psi_1 = \pm \epsilon,$$

where ϵ is a small angle, and neglecting powers of ϵ above the second

$$I = 4\epsilon^2 a^2 \left\{2 \sin^2 \frac{\pi\delta_1}{\lambda} + 2 \sin^2 \frac{\pi\delta_2}{\lambda} - \sin^2 \frac{\pi(\delta_1 + \delta_2)}{\lambda}\right\},$$

the maximum value of which is $I_m = 16\epsilon^2 a^2$, whence

$$I = I_m \left\{\tfrac{1}{2} \sin^2 \frac{\pi\delta_1}{\lambda} + \tfrac{1}{2} \sin^2 \frac{\pi\delta_2}{\lambda} - \tfrac{1}{4} \sin^2 \frac{\pi(\delta_1 + \delta_2)}{\lambda}\right\}.$$

When $\delta_1 + \delta_2 = (2n+1)\lambda/2$

$$I = I_m \left\{\tfrac{1}{2} \sin^2 \frac{\pi\delta_1}{\lambda} + \tfrac{1}{2} \cos^2 \frac{\pi\delta_1}{\lambda} - \tfrac{1}{4}\right\} = \tfrac{1}{4} I_m,$$

and thus the secondary curves of the first kind are here of uniform intensity and will appear relatively dark, as their intensity is only one-fourth of the maximum.

On the secondary curves of the first kind between these dark curves, $\delta_1 + \delta_2 = n\lambda$ and for n even, $I = I_m \sin^2\{\pi(\delta_1 - \delta_2)/(2\lambda)\}$ while for n odd $I = I_m \cos^2\{\pi(\delta_1 - \delta_2)/(2\lambda)\}$. Hence on these curves the intensity varies between 0 and I_m, and the parts for which $0 < I < \frac{1}{2} I_m$ have the same width as those for which $\frac{1}{2} I_m < I < I_m$, so that since the parts appear bright when $I = \frac{1}{2} I_m$, the dark parts of these curves are narrower than the bright.

When $\quad \delta_1 - \delta_2 = n\lambda, \; I = I_m \sin^4\{\pi(\delta_1 + \delta_2)/(2\lambda)\}$

or $\quad = I_m \cos^4\{\pi(\delta_1 + \delta_2)/(2\lambda)\}$,

according as n is even or odd and consequently the portions of the secondary curves of the second kind cut off by the curves of constant intensity

$$\delta_1 + \delta_2 = (2n+1)\lambda/2$$

are alternately bright of intensity between $I_m/4$ and I_m and dark of intensity between 0 and $I_m/4$.

Hence the general appearance near the centre of the field will be relatively dark continuous curves of the secondary system of the first kind and between

these dark spots arranged in chess-board fashion in a direction perpendicular (the plates having equal thickness) to the principal section of the plate.

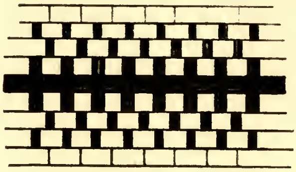


Fig. 46.

(*e*) For points near the circle through the axial points, we may write

$$\psi_2 - \psi_1 = \pm (\pi/2 + \epsilon),$$

whence

$$a_1 = \mp \epsilon \sin 4\psi_1 + 4\epsilon^2 \sin^2 2\psi_1, \quad a_2 = \pm \epsilon \sin 4\psi_1 + 4\epsilon^2 \cos^2 2\psi_1,$$
$$a_3 = - \epsilon^2 \sin^2 2\psi_1, \quad a_4 = \sin^2 2\psi_1 \pm \epsilon \sin 4\psi_1 - 3\epsilon^2 \sin^2 2\psi_1,$$

and

$$\begin{aligned} I = a^2 \Bigg[& \left(\sin^2 2\psi_1 \pm \epsilon \sin 4\psi_1 - 3\epsilon^2 \sin^2 2\psi_1\right) \sin^2 \frac{\pi(\delta_1 - \delta_2)}{\lambda} \\ & \mp \epsilon \sin 4\psi_1 \left(\sin^2 \frac{\pi\delta_1}{\lambda} - \sin^2 \frac{\pi\delta_2}{\lambda}\right) \\ & + \epsilon^2 \left(4 \sin^2 2\psi_1 \sin^2 \frac{\pi\delta_1}{\lambda} + 4 \cos^2 2\psi_1 \sin^2 \frac{\pi\delta_2}{\lambda} \right. \\ & \left. - \sin^2 2\psi_1 \sin^2 \frac{\pi(\delta_1 + \delta_2)}{\lambda}\right) \Bigg]. \end{aligned}$$

But $\sin^2 \frac{\pi\delta_1}{\lambda} - \sin^2 \frac{\pi\delta_2}{\lambda} = \sin \frac{\pi(\delta_1 + \delta_2)}{\lambda} \sin \frac{\pi(\delta_1 - \delta_2)}{\lambda}$, and hence on the secondary curves of the second kind $\delta_1 - \delta_2 = n\lambda$ the intensity is given by the term containing ϵ^2 as a factor, which is very small throughout the region in question. Hence these curves will appear dark for a certain distance on each side of the circle on the line joining the axial points as diameter.

(*f*) In order to determine the nature of the transition from the chess-board pattern (*d*) to the dark secondary curves of the second kind in the vicinity of the circle through the axial points, let us consider the part of the field for which

$$\psi_1 = \pi/8, \quad \psi_2 = \pi - \pi/8.$$

We have in this case

$$a_1 = 1/2, \quad a_2 = 1/2, \quad a_3 = -1/4, \quad a_4 = 1/4,$$

and

$$I = \tfrac{1}{2} a^2 \left\{ \sin^2 \frac{\pi\delta_1}{\lambda} + \sin^2 \frac{\pi\delta_2}{\lambda} - \tfrac{1}{2} \sin^2 \frac{\pi(\delta_1 + \delta_2)}{\lambda} + \tfrac{1}{2} \sin^2 \frac{\pi(\delta_1 - \delta_2)}{\lambda} \right\}.$$

Hence for

$$\delta_1 + \delta_2 = (2n+1)\lambda/2, \quad I = \tfrac{1}{4}a^2\left\{1 + \sin^2\frac{\pi(\delta_1 - \delta_2)}{\lambda}\right\}$$

and these curves, instead of being continuous and dark, have an intensity varying between $a^2/4$ and $a^2/2$ and thus appear less marked.

Further for $\delta_1 + \delta_2 = n\lambda$

$$I = a^2 \sin^2 \frac{\pi(\delta_1 - \delta_2)}{2\lambda}\left\{1 + \cos^2\frac{\pi(\delta_1 - \delta_2)}{2\lambda}\right\} \text{ when } n \text{ is even,}$$

$$I = a^2 \cos^2 \frac{\pi(\delta_1 - \delta_2)}{2\lambda}\left\{1 + \sin^2\frac{\pi(\delta_1 - \delta_2)}{2\lambda}\right\} \text{ when } n \text{ is odd;}$$

hence on passing along the curve $\delta_1 + \delta_2 = n\lambda$ from a dark spot the intensity increases more rapidly than in the former case (d), so that the dark portions of these curves in the direction of the secondary curves of the second kind are narrower than they are near the centre of the field.

Finally for $\delta_1 - \delta_2 = n\lambda$,

$$I = a^2 \sin^4 \frac{\pi(\delta_1 + \delta_2)}{2\lambda}, \text{ when } n \text{ is even, } I = a^2 \cos^4 \frac{\pi(\delta_1 + \delta_2)}{2\lambda} \text{ when } n \text{ is odd,}$$

and hence the changes of intensity on the secondary curves of the second kind follow the same law as before.

The result of this is that in the region under consideration, the main features are isolated portions of the secondary curves of the second kind.

(g) For $\psi_1 = \pm\frac{\pi}{4}$, or $\pm\frac{3\pi}{4}$ and ψ_2 nearly π, the coefficient a_1 is the most important, and for $\psi_2 = \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$ and ψ_1 very small, the principal coefficient is a_2; hence the primary curves of the first and second kind occur most strongly near the first and second axial point respectively in the middle of the sectors between the arms of the black brushes.

CHAPTER XV.

THE STUDY OF POLARISED LIGHT.

194. WE have assumed in what precedes the possibility of obtaining a stream of polarised light without giving any definite specification of the means by which this may be effected. It has, however, been shown in the course of our investigations that the light reflected at the polarising angle from the surface of an isotropic medium is at any rate nearly plane polarised, and that the polarisation of the stream transmitted by a pile of transparent isotropic plates tends to become perfect as the number of the plates is indefinitely increased and when the angle of incidence on the pile approximates to the polarising angle.

Certain crystals, conspicuous among which is Tourmaline, also polarise the light that they transmit,—a property that is due to the variation of the absorption of light with the position of the plane of polarisation in the crystal. Thus tourmaline absorbs light polarised in a plane parallel to the optic axis more energetically than light polarised in the perpendicular plane, and a moderate thickness of a plate, cut parallel to the axis, transmits sensibly the extraordinary stream alone. The polarisation is however seldom quite perfect and the intensity of the extraordinary stream is also much weakened by absorption in its passage through the plate.

195. By far the most effectual mode of obtaining a plane polarised stream of strong intensity is to separate a beam of common light into two oppositely polarised streams by double refraction and to subsequently isolate one of the streams. This is done by what are termed polarising prisms, of which there are two types; in the one such a lateral separation of the streams is produced, that it is possible to block off one of the emergent pencils by a screen; in the other the second stream is prevented from emerging by total reflection*.

Of the first class of polarising prisms there are three principal forms.

* For a discussion of polarising prisms, see Feussner, *Zeitschr. f. Instrumkd.* IV. 41 (1884). Grosse, *Die gebräuchlichen Polarisationsprismen*, Clausthall, 1887; *Verhandl. d. Ges. deutsch. Naturf.* XI. (2) 33 (1891).

The earliest example is a polariser devised by Rochon* and exhibited to the Paris Academy in 1777. This consists of two prisms of Iceland spar, having the same angle and so cut that in the one the optic axis is perpendicular to one of the faces, while in the other it is parallel to the refracting edge. The prisms are mounted together with their edges in opposite directions, and in such a way that the light is incident on the face that is normal to the optic axis. The ordinary stream passes through the polariser without deviation and is achromatic, while the extraordinary stream is deviated towards the edge of the second prism by an amount dependent upon the colour of the light.

For the second prism of Rochon's combination, Senarmont† substituted a prism in which the optic axis is parallel to the face of emergence and perpendicular to the refracting edge. The separation of the streams is less than in Rochon's form, but the prism is easier to make and less costly of material.

On the other hand in a combination due to Wollaston‡ there is a larger angle between the emergent streams, but neither pencil is achromatic. This polariser, as in Rochon's and Senarmont's forms, consists of two prisms of equal angles, but the optic axis is parallel to the face of entry and perpendicular to the edge of the first prism, and parallel to the refracting edge of the second prism. Both streams are deflected by their passage through the polariser, the deviations being in opposite directions.

196. Polarisers that depend upon the total reflection of one of the streams may be divided into two groups, according as it is the ordinary or the extraordinary pencil that is prevented from passing.

In the prisms of the first group§, which are modifications of a type devised by Nicol, a prismatic piece of Iceland spar is divided into two halves by a cut and the pieces are joined together again with a thin layer of some medium between them, the refractive index of which is less than that corresponding to the ordinary stream in the spar. If then the angle of incidence on this layer be in excess of a certain value, the ordinary stream

* *Recueil de Mém. sur la Mécanique et sur la Physique*, Brest, 1783; *Gilb. Ann.* XL. 141 (1812); *J. de Phys.* LIII. 192 (1801); *Acta nova Acad. Petropolitanæ*, VI. Part I. 37 (1788).

† *Ann. de Ch. et de Phys.* (3) L. 480 (1857).

‡ *Phil. Trans.* CX. 126 (1820).

§ Nicol, *Edin. New Phil. Journ.* VI. 83 (1828); XXVII. 332 (1839). Spassky, *Pogg. Ann.* XLIV. 168 (1838). Radicke, *ibid.* L. 25 (1840). Hasert, *ibid.* CXIII. 188 (1861). Potter, *Phil. Mag.* (4) XIV. 452 (1857); XVI. 419 (1858). Foucault, *C. R.* XLV. 238 (1857); *Pogg. Ann.* CII. 642 (1857). Hartnack and Prazmowski, *C. R.* LXII. 149 (1866); *Pogg. Ann.* CXXVII. 494 (1866); *Carl Report*, I. 325 (1866); II. 217 (1867); *Ann. de Ch. et de Phys.* (4) VII. 181 (1866). Glan, *Carl Report*, XVI. 570 (1880); XVII. 195 (1881). Glazebrook, *Phil. Mag.* (5) X. 247 (1880); XV. 352 (1883). S. P. Thompson, *ibid.* (5) XII. 349 (1881); XV. 435 (1883); XXI. 476 (1886). Madan and Ahrens, *Nature*, XXXI. 371 (1885).

will be totally reflected and absorbed by the blackened sides of the spar, but the extraordinary stream, the refractive index for which is in the case of spar always less than that for the ordinary stream, will in general pass through the layer and emerge from the prism.

Prisms of this class differ in the crystallographic orientation of the faces and of the cut and in the medium placed between the two halves. In the earliest form, devised by Nicol, a natural crystal of spar is taken, the length of which is about $3\frac{1}{2}$ times its breadth, and instead of the end faces, that make an angle of about $70° 52'$ with the side edges, new faces are cut inclined to these edges at an angle of $68°$. The crystal is then cut in half by a section perpendicular to these new faces and to the principal section of the prism, that is to the shorter diagonal of the faces, and the two halves are cemented together with canada balsam.

Considering only the principal section of the prism, the total reflection of the ordinary stream commences from a certain direction IK of the incident light, that of the extraordinary stream from the direction JL of the incident pencil.

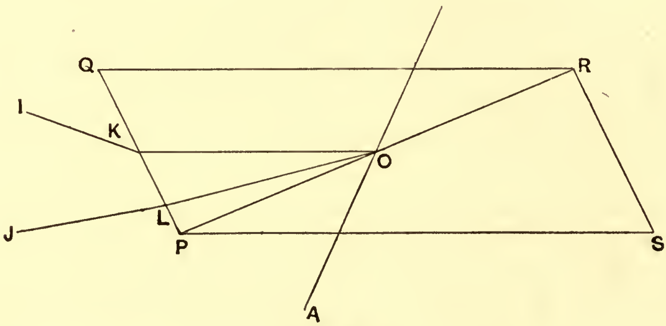


Fig. 47.

For light incident in intermediate directions, only the extraordinary stream is transmitted and the angle between IK and JL defines the field of the prism.

Looking at an uniformly illuminated surface along the axis of the prism, so that the cut slopes away from the eye from left to right, the appearance is as follows:—on the left is a black space bounded by a violet edge, then a bright space due to the extraordinary stream alone, and lastly on the right a space illuminated by both streams, terminated on the left by an orange border and traversed by coloured bands. These bands are fringes of transmission analogous to Herschel's bands in the neighbourhood of total reflection. The blue border of the black space shows that the total reflection of the extraordinary stream commences first for red light, which is explained by the fact that the extraordinary dispersion of spar is less than that of the balsam, so that the relative refractive index increases from violet to red.

To determine the field of the prism, suppose that $PQRS$ represents the principal section and OK, OL the limiting wave-normals of the ordinary and extraordinary streams. Let μ_0, μ_e be the principal refractive indices of the spar and n the index of the balsam.

Then for the total reflection of the ordinary stream, if the angle $POK = \beta$, we have

$$\cos\beta = n/\mu_0 \quad\text{.................................(1)},$$

and the corresponding angle of incidence i_0 is given by

$$\sin i_0 = \mu_0 \sin\beta = n \tan\beta \quad\text{..........................(2)}.$$

For the extraordinary stream, if the angle $POL = \gamma$, and the optic axis OA make an angle α with the cut, we have

$$\cos\gamma = n/\mu \quad\text{...............................(3)},$$

where $$\mu^{-2} = \mu_0^{-2}\cos^2(\gamma+\alpha) + \mu_e^{-2}\sin^2(\gamma+\alpha) \quad\text{..............(4)},$$

and the corresponding angle of incidence i_e is found from

$$\sin i_e = \mu \sin\gamma = n \tan\gamma,$$

whence from (3) and (4)

$$\sin i_e = \frac{-n(\mu_0^2-\mu_e^2)\sin\alpha\cos\alpha + \mu_0\mu_e\sqrt{\mu_0^2\cos^2\alpha + \mu_e^2\sin^2\alpha - n^2}}{\mu_0^2\cos^2\alpha + \mu_e^2\sin^2\alpha} \quad\text{...(5)}.$$

The optic axis makes an angle of about 45° 23′ 30″ with the original end faces of the rhomb, so that for the Nicol described above we have $\alpha = 41° \, 44' \, 30''$ nearly. Hence taking $n = 1{\cdot}548$, $\mu_0 = 1{\cdot}65846$, $\mu_e = 1{\cdot}48654$, we obtain

$$i_0 = 36° \, 31' \, 30'', \quad \gamma = 6° \, 10' \, 0'', \quad i_e = 9° \, 37' \, 40''.$$

Thus the field of polarised light is 26° 53′ 50″, and the ratio of the length to the breadth of the prism, which is nearly the cotangent of the angle between OL and the axis of the prism, is 3·53.

Without discussing the various forms that have been suggested for a Nicol's prism, let us determine under what circumstances a prism on this principle with its end faces at right angles to the axis has a maximum field symmetrical with respect to the axis.

From equations (1) and (3) the field within the prism is

$$\chi = \beta - \gamma = \cos^{-1}\frac{n}{\mu_0} - \cos^{-1}\{n\sqrt{\mu_0^{-2} + (\mu_e^{-2} - \mu_0^{-2})\sin^2\theta}\},$$

θ being the angle between the optic axis and the extraordinary wave-normal at the limit of total reflection. For this to be as large as possible, we must have

(1) $\theta = \pi/2$, $\chi = \cos^{-1}(n/\mu_0) - \cos^{-1}(n/\mu_e)$.

(2) $n = \mu_e$, since for a greater value of n, β decreases while γ cannot be less than zero, and for a less value of n, β increases less rapidly than γ.

Thus the maximum field within the prism is

$$\chi = \cos^{-1}(\mu_e/\mu_0).$$

In order that the external field may be symmetrical with respect to the axis of the prism, we must have

$$\sin i = \mu_e \sin\phi = \mu_0 \sin(\chi - \phi),$$

where ϕ is the angle between the cut and the normal to the end faces.

This gives

$$\tan\phi = \frac{\mu_0 \sin\chi}{\mu_e + \mu_0 \cos\chi} = \frac{\mu_0 \sin\chi}{2\mu_e}.$$

The field is then $2i$ and the ratio of the length to the breadth of the prism is $\cot\phi$. In the case of Iceland spar we have

$$\chi = 26^\circ\, 19'\, 10'', \quad \phi = 13^\circ\, 53'\, 30'', \quad 2i = 41^\circ\, 49',$$

the ratio of the length to the breadth is 4·04. Since the optic axis must be at right-angles to the limiting extraordinary wave-normal, it can have one of two directions:—either it may be in the normal section of the prism at an angle of $13^\circ\, 53'\, 30''$ to the end faces, or it may be perpendicular to the normal section and parallel to the cut.

By selecting the second position for the optic axis, a prism is obtained in which several defects of an ordinary Nicol's prism are considerably reduced. In the first place it gives no lateral displacement of a stream of light directly transmitted through it; secondly a conical pencil incident directly on the prism emerges with a polarisation that is more nearly constant over its whole extent; and thirdly the error in the determination of the plane of polarisation of a parallel pencil slightly inclined to the axis of rotation is reduced to a minimum*.

197. The second group of prisms† depending upon the total reflection of one of the polarised streams is made by fixing a thin crystalline plate between the two equal prisms of glass, turned in opposite directions, by means of a cement with a refractive index equal to that of the glass.

Considering only the normal section of the prisms, an investigation similar to that of the last section shows that for the maximum field, the refractive index of the glass and cement should be equal to the greatest index of the plate, and that with a biaxal plate the mean axis of the ellipsoid of polarisation should be parallel to its faces and the plate arranged so that this axis is in the normal section of the prisms: while with uniaxal plates the optic axis should be in a plane perpendicular to the normal section, which can

* Glazebrook, *loc. cit.*

† This type of prism was first suggested by Sang in 1837: cf. *Proc. R. S. Edin.* XVIII. 337 (1891). Jamin, *C. R.* LXVIII. 221 (1869); *Pogg. Ann.* CXXXVII. 174 (1869). Zenker, cf. *Zeitschr. f. Instrumkd.* IV. 50 (1884). Bertrand, *C. R.* XCIX. 538 (1884).

always be managed with a plate of any crystallographic orientation by turning it in its own plane. In order that the field may be symmetrical, the angles of the glass prisms must be equal to

$$90^\circ - \tfrac{1}{2}\cos^{-1}\frac{\mu_a}{\mu_c},$$

μ_a and μ_c being the least and the greatest indices of the plate.

With these prisms (considering only the normal section) light polarised in a plane perpendicular to the greatest axis of the ellipsoid of polarisation is totally reflected, when the inclination to the plate is less than $\cos^{-1}(\mu_a/\mu_c)$: the angular field on emergence is 2χ, where

$$\sin\chi = \mu_c \sin\{\tfrac{1}{2}\cos^{-1}(\mu_a/\mu_c)\}$$

and the ratio of the length to the breadth is $\cot\{\tfrac{1}{2}\cos^{-1}(\mu_a/\mu_c)\}$.

With a plate of Iceland spar, such prisms give a field of 44° 22′ and the ratio of the length to the breadth is 4·28.

With a plate of sodium nitrate ($\mu_0 = 1{\cdot}587$, $\mu_e = 1{\cdot}336$), the field is increased to 53° and the ratio of the length to the breadth is reduced to 3·42.

198. Any one of the above forms of polarisers may be employed as an analyser and will work sufficiently well, provided the analysation merely consists in bringing streams of light to a common plane of polarisation; but when it is a question of the exact determination of the plane of polarisation of a stream of plane polarised light, an analyser that works by extinction has not the required accuracy, as after the illumination has been reduced to a small quantity, the eye is unable to perceive any further diminution in the intensity and there is in consequence considerable uncertainty in the determination of the position of the analyser, at which the light is entirely quenched.

A delicate test of the existence of polarisation in a stream of light is afforded by the rings and brushes obtained when a conical pencil of polarised light traverses a crystalline plate and is subsequently analysed. This is the principle of a sensitive analyser due to Savart*. It consists of a Savart's plate (§ 192) connected with a Nicol's prism, the principal section of which bisects the angle between the principal sections of the double plate. We have seen that when a slightly convergent stream of polarised light is viewed through this combination, a series of parallel straight bands is in general perceived, but it is clear that these bands will vanish, when the analyser is so turned that the plane of polarisation of the incident light is coincident with either of the principal sections of the plate, as then there is no separation of the light into two oppositely polarised streams and consequently no interference.

* *Pogg. Ann.* XLIX. 292 (1840).

199. A still more sensitive plan is that adopted in the "half-shade" analysers* which depend upon the readiness with which the eye can compare the intensities of two streams seen in juxtaposition, when the illumination is slight.

Suppose that the field of view is divided along a straight line into two parts of equal intensity by some apparatus that introduces a small angle between the planes of polarisation of the two halves, so that the plane of polarisation of the one is parallel to OP, that of the other is parallel to OP'. Then if the field be viewed through a Nicol's prism, the whole will appear uniformly illuminated only when the principal section of the Nicol is the interior or exterior bisector of the angle POP'. Conversely the apparatus that produces this difference in the polarisations of the two halves of the field may be employed as an analyser, and if it be turned until the whole field has the same illumination, the plane of polarisation of the incident light will then bisect the interior or the exterior angle between the planes of polarisation of the analyser. When however the plane of polarisation of the incident light bisects the acute angle between the planes of polarisation of the analyser, the two parts of the field are too bright to admit of an accurate comparison of the intensities.

The first analyser of this type was devised by Jellett. A long rhomb of spar is taken and the ends are cut off by planes perpendicular to the longitudinal edges: the prism thus obtained is then divided into two by a plane perpendicular to its ends and making a small angle with the longer diagonals of these faces, and the parts are joined together along the plane of section after one of the halves has been reversed.

Suppose now that a cylindrical stream of polarised light falls normally on the end face of the prism, so as to be equally divided by the plane of section: on entry into the prism the extraordinary streams in the two parts will be deviated and if the prism have a sufficient length, can be blocked by a diaphragm, but the ordinary streams will pass undeviated and the planes of polarisation of the two halves of the emergent pencil will make equal small angles with the normal to the plane of section. Hence in order to render the two halves of the field equally dark, the prism must be turned until the plane of section coincides with the primitive plane of polarisation of the light.

A similar result is obtained with Cornu's analyser. A Nicol's prism is

* Jellett, *B. A. Report*, 1860, II. 13; *Proc. Ir. Acad.* VIII. 279 (1863). Cornu, *Bull. Soc. Chem.* (2) XIV. 140 (1870). Righi, *Mem. dell' Acc. R. di Bologna*, (4) VI. 599 (1885). Lippich, *Zeitschr. f. Instrumkd.* II. 167 (1882); XII. 333 (1892); *Wien. Ber.* LXXXV. (2) 268 (1882); XCI. (2) 1059 (1885); XCIX. (2ª) 695 (1890). Laurent, *Dingler Polytechnisches Journal*, CCXXIII. 608; *J. de Phys.* III. 183 (1874); *C. R.* LXXVIII. 349 (1874). Dufet, *J. de Phys.* (2) I. 552 (1882). Poynting, *Phil. Mag.* (5) X. 18 (1880). Macé de Lépinay, *J. de Phys.* (3) IX. 585 (1900); *C. R.* CXXXI. 832 (1900).

cut in half by a plane through the short diagonals of its end faces: and a wedge-shaped piece is removed from one half, its edge being parallel to the length of the prism and its angle about 3°. The two parts are then re-united, forming a prism that consists of two half-Nicols with their principal sections inclined to one another at a small angle.

A different method of obtaining a half-shade analyser has been adopted by Laurent. This depends upon the action of a half-wave plate of quartz cut parallel to the axis, in traversing which a stream of plane polarised light has its polarisation changed, the new plane of polarisation being inclined to the principal section at the same angle as the primitive plane, but on the opposite side of the axis. Half the field of view is covered by the plate, to which is attached a Nicol's prism with its principal section inclined at a small angle to that of the plate. In examining a stream of light the eye looking through the Nicol's prism is focussed on the edge of the plate, and the instrument is rotated until both halves of the field are equally dark: when this is the case, the principal section of the plate is parallel to the plane of polarisation of the stream.

200. Half-shade analysers present the same appearance when the light examined is partially or elliptically polarised, as when it is plane polarised, the direction determined in these two cases being the plane of partial polarisation and that of maximum polarisation. An ellipticity, even though slight, in the polarisation of a stream of white light may however be readily detected by means of a Bravais' plate* (§ 176).

Let us suppose that a stream of elliptically polarised light traverses the plate and is subsequently analysed in a plane inclined at an angle γ to the principal section of one of its halves. We may represent any one of the constituents of the primitive composite stream by its components polarised in planes parallel and perpendicular to this principal section with the polarisation-vectors

$$ae^{\iota nt} \text{ and } be^{\iota(nt+\Delta)},$$

respectively, and if δ be the relative retardation of phase introduced by the plate, the polarisation-vector for the stream emerging from the analyser will be for the one half of the field

$$\{a\cos\gamma + b\sin\gamma e^{\iota(\Delta-\delta)}\}\, e^{\iota nt},$$

and for the other

$$\{a\cos\gamma e^{-\iota\delta} + b\sin\gamma e^{\iota\Delta}\}\, e^{\iota nt},$$

giving as the intensity in the two cases

$$(a\cos\gamma + b\sin\gamma)^2 - 2ab\sin 2\gamma \sin^2\frac{\Delta \mp \delta}{2},$$

* *Ann. de Ch. et de Phys.* (3) XLIII. 129 (1855). Quincke, *Pogg. Ann.* CXXVII. 199 (1866).

and for the intensity of the composite stream

$$\Sigma (a \cos \gamma + b \sin \gamma)^2 - 2 \sin 2\gamma \Sigma ab \sin^2 \frac{\Delta \mp \delta}{2}.$$

When the incident light is plane polarised, $\Delta = 0$, and the two halves of the field have the same colour, and this is the sensitive tint, if the analyser be set for extinction of the light, when the plate is away. If however the initial stream be elliptically polarised, the tint of one half is raised and that of the other is lowered and under no circumstances can these be made alike, unless the analyser be set so as to render the whole field free from colour.

201. Before proceeding to the study of the means employed for the investigation of an elliptically polarised stream, we will first consider a method of representing geometrically the state of polarisation of a train of waves of light*.

Taking the axis of z in the direction of propagation, a stream of polarised light may be represented by its components polarised in planes parallel respectively to the axes of x and y with the polarisation-vectors

$$\xi = \bar{a} e^{\iota nt}, \qquad \eta = \bar{b} e^{\iota nt},$$

wherein $\bar{a}$ and $\bar{b}$ are in general complex, and their ratio

$$\bar{b}/\bar{a} = (b/a) e^{\iota \Delta'} = (b/a) \cos \Delta' + \iota (b/a) \sin \Delta' = u + v\iota, \text{ say,}$$

a and b being the amplitudes of the components, and Δ' the acceleration of phase of the second relatively to that of the first.

This ratio defines the form and the orientation of the elliptic vibration of the extremity of the polarisation-vector of the stream; and we may therefore represent the state of polarisation by a point on a plane, for which the abscissa is u and the ordinate is v, the length of the radius-vector to the representative point giving the ratio of the amplitudes and the angle that it makes with the axis of abscissæ being the difference of phase. Since the polarisation is right- or left-handed according as Δ' lies between 0 and π or between π and 2π, the vibrations in the stream will be right- or left-handed according as the representative point is above or below the axis of u.

When the point is on the axis of u, the stream is plane polarised in an azimuth $\tan^{-1} u$ with respect to the plane of xz; if the point be on the axis of v, the difference of phase is $\pi/2$ and the planes of maximum and minimum polarisation are parallel to the axes of x and y.

Points p, p' on the axis of ordinates at unit distance from the origin represent circular polarisation.

Now if θ be the angle that the planes of maximum and minimum

* Poincaré, *Théorie Math. de la Lumière*, II. p. 276.

polarisation make with the coordinate axes, $\tan\beta$ be the ratio of the axes of the elliptic vibrations and $\tan\sigma = b/a$, we have

$$\tan 2\theta = \cos\Delta' \tan 2\sigma, \quad \sin 2\beta = \sin\Delta' \sin 2\sigma,$$

which give, since

$$u = \cos\Delta' \tan\sigma, \quad v = \sin\Delta' \tan\sigma,$$

$$u^2 + v^2 + 2u \cot 2\theta - 1 = 0, \quad u^2 + v^2 - 2v \operatorname{cosec} 2\beta + 1 = 0.$$

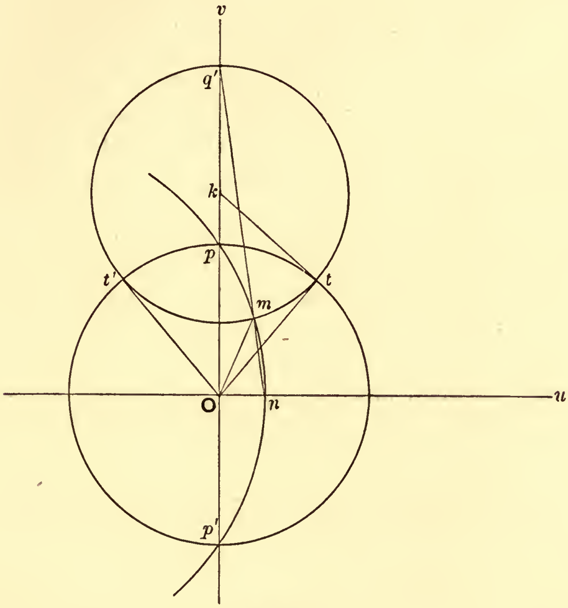


Fig. 48.

Thus if θ be constant, the points representing the different states of polarisation lie on a circle through p and p', and if the ratio of the axes of the elliptic vibrations be constant, the points corresponding to different orientations of the axes are on a circle cutting the first system of circles orthogonally.

Any point is the intersection of a circle of the one (θ) system with a circle of the second (β) system: the distance from the origin of the point, in which the circle of the first (θ) system cuts the axis of u, is the tangent of the angle that the plane of maximum or of minimum polarisation makes with the axis of x, according as the representative point is within or without the circle of radius equal to unity with its centre at the origin.

The polarisation-vectors of the component streams being

$$\xi = a e^{\iota(nt+\phi)}, \qquad \eta = b e^{\iota(nt+\phi+\Delta')},$$

referred to the axes of x and y, let the polarisation-vectors of the components polarised in the planes of maximum and minimum polarisation be

$$\xi' = c \cos \beta e^{\iota(nt+\phi+\epsilon)}, \qquad \eta' = \iota c \sin \beta e^{\iota(nt+\phi+\epsilon)},$$

then the angle between ξ and ξ' being θ

$$c \cos \beta e^{\iota\epsilon} = a \cos \theta + b \sin \theta e^{\iota\Delta'}, \quad \iota c \sin \beta e^{\iota\epsilon} = - a \sin \theta + b \cos \theta e^{\iota\Delta'},$$

whence

$$c \cos \beta \sin \epsilon = b \sin \theta \sin \Delta', \qquad c \sin \beta \cos \epsilon = b \cos \theta \sin \Delta',$$

and $$\tan \epsilon = \tan \beta \tan \theta = On/Oq' = \tan Oq'n,$$

n being the point in which the circle (θ) cuts the axis of u and q' the point furthest from the origin in which the circle (β) intersects the axis of v. But it is easy to show that the line $q'n$ passes through the representative point m, so that ϵ is the angle $Oq'm$ or the complement of the angle Onm.

The effect of changing from one set of rectangular axes to another is to move the representative point along the circle (β). Now the difference of phase between the component streams polarised in planes parallel to the axes is the angle that the line joining O to the representative point makes with the axis Ou: hence if Ot, Ot' be the tangents from O to the circle (β), the difference of phase between any two rectangular components of the stream lies between tOu and $t'Ou$. Now $Ot = 1$, $Ok = \operatorname{cosec} 2\beta$, where k is the centre of the circle (β); therefore

$$\sin tOu = \cos tOk = Ot/Ok = \sin 2\beta \text{ and } tOu = 2\beta,$$

so that the difference of phase varies between 2β and $\pi - 2\beta$.

It is now easy to represent the effect on the polarisation that is produced by passing the stream normally through a crystalline plate. Let α be the angle between the plane of xz and the plane of polarisation of the most retarded stream in the plate, Δ the relative retardation of phase that the plate introduces; then we have first to find the polarisation-vectors of the components polarised in planes parallel to those of the streams in the plate; secondly, to introduce the difference of phase Δ between these components; and finally, to determine the nature of the polarisation when the stream is again referred to the original axes.

The primitive polarisation being represented by the point m of the circles (θ) and (β), the final state of polarisation of the stream is, therefore, found by the following successive operations:—Firstly, a motion of m along the circle (β) to the point m' in which it cuts the circle ($\theta - \alpha$); secondly, a rotation of Om' through the angle Δ, bringing m' to m'', the point of intersection (say) of the circles (θ') and (β'); thirdly, a motion of m'' along the circle (β') to the point in which it intersects the circle ($\theta' + \alpha$).

202. The final polarisation may, however, be determined in a far more simple fashion by another method of representation that is derived from the foregoing by a stereographic projection.

Describe a sphere of unit diameter touching the plane of uv at the origin of the coordinates, and let the points of the plane be projected on the surface of this sphere by joining them to O', the extremity of the diameter of the sphere through O.

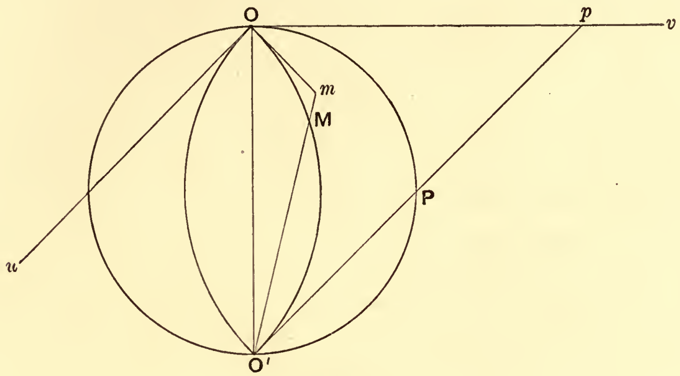


Fig. 49.

Then the axes of u and v project into great circles at right-angles to one another, the former of which may be called the equator; the points p, p' become the poles; the circles (θ) will be represented by the meridians 2θ and the circles (β) by the parallels of latitude 2β; and if $\tan\sigma$ be the length of the radius-vector Om and the angle $mOu = \Delta'$, the point m will be projected into M, where the arc $OM = 2\sigma$ and the angle $MOO' = \Delta'$.

Thus any point on the sphere will represent the state of polarisation of a stream of light, the azimuth of its plane of maximum polarisation being half the longitude, and the ratio of the axes of the elliptic vibrations being the tangent of half the latitude of the point; and the polarisation is right- or left-handed according as the point is in the northern or the southern hemisphere.

This being the case, the effect of the transmission through a crystalline plate is represented by three successive rotations:—(1) round the polar axis PP' through an angle -2α, (2) round the diameter OO' in a left-handed direction through the angle Δ, (3) round the polar axis through an angle 2α; and these three operations are clearly equivalent to a single rotation in a left-handed direction through an angle Δ round an equatorial diameter AA' where the arc $OA = 2\alpha$.

203. The investigation of a stream of elliptically polarised light consists in determining the elliptic path traced by the extremity of the polarisation-vector, and this may be done in two ways:—

(1) by finding the ratio of the amplitudes and the difference of phase of the vibrations of the polarisation-vectors of the component streams polarised in two rectangular directions;

(2) by determining the ratio of the axes of the ellipse, their position with respect to some given line and the direction in which the path is traversed.

The principle adopted in both cases is the same and consists in the reduction of the pencil of light to a plane polarised stream and the subsequent determination of the plane of polarisation of the pencil.

204. In the first of these methods of studying a stream of elliptically polarised light, the reduction to plane polarisation is effected by means of a compensator, which introduces an adjustable relative retardation between the rectangular components of the stream. Compensators are of two kinds: the first class of instruments introduces a retardation that is variable over the whole extent of the field and gives rise to interference fringes that are localised on the surface of the compensator; in the second class, the retardation introduced is the same for the whole field, which is of uniform colour or intensity according as the light is white or monochromatic.

The compensator of the first type, known as Babinet's compensator, consists of two prisms of quartz having the same very small angle, mounted together to form a plate and cut so that the outer surfaces are parallel to the optic axis of the crystal, which is in one prism perpendicular and in the other parallel to the refracting edge. One of the prisms is fixed, while the other can be moved over it by means of a micrometer screw, and the prisms should be so arranged that the one with its edge perpendicular to the optic axis receives the incident light.

When this is the case, a stream of light falling normally on the compensator traverses the first prism with a speed Ω/μ_0, and the second with a speed Ω/μ_e, if it be polarised in a plane perpendicular to the edge, while these speeds will be interchanged in the case of polarisation in a plane parallel to the edge: consequently passage through the compensator will retard the second stream relatively to the first by an amount $(\mu_e - \mu_0)(d_1 - d_2)$ measured in length in air, where d_1, d_2 are the distances traversed in the first and second prisms respectively, since, the phenomenon under consideration being localised at the compensator, these distances may be regarded as sensibly the same for the two streams.

Thus the retardation, that is introduced, is the same along each line parallel to the edges of the prisms, but is different along the length of the compensator: hence if a stream of light polarised in an azimuth α with respect to the principal section of the first prism fall normally on the instru-

ment and be subsequently analysed, a series of coloured bands will be seen, when the light is white, and a set of bright and dark bands, when it is monochromatic.

Since the difference of phase is not constant along the whole compensator, it is necessary to confine the attention to a small portion of the field, within which the relative retardation may be regarded as practically constant. This is marked off by two spider-lines parallel to the edges of the prisms. If the prisms be of equal thickness at the place thus indicated, the emergent light is plane polarised in the same azimuth α as the incident stream, since the changes of phase and amplitude due to passage into or out of each prism are sensibly the same for both component streams. On moving one of the prisms over the other a varying retardation is introduced: the emergent light is elliptically polarised and cannot be quenched by a rotation of the analyser, though for two positions the intensity becomes a maximum and a minimum respectively. By a further motion of the prism the relative retardation becomes $\pm \lambda/2$ and the light is again plane polarised in an azimuth of $\pi - \alpha$: if the distance that the prism has to be shifted between the two positions that give plane polarised light be w, then a shift of w' from the initial position of zero-retardation gives a relative retardation of $\pm \frac{w'}{w}\frac{\lambda}{2}$, and the compensator is thus graduated.

Before using the compensator it has to be ascertained which of the two prisms is the one that can be moved, and which is the direction of its motion that increases or diminishes its thickness at the point between the spider-lines. We must also find out whether the prism, that receives the incident light, has its edge perpendicular or parallel to the optic axis, as in the latter case the sign of the retardation is the opposite to that given above.

This may be determined in the following manner*:—Light from a slit parallel to the edges of the prisms falls on a Billet's divided lens arranged to give two real images of the slit on the surface of the compensator at the part opposite the spider-lines, and the compensator is set so that the prisms have the same thickness at this place. The light from these images after traversing the compensator gives rise to two systems of interference fringes polarised in perpendicular planes, and these can be separated from one another by examining them with a double-image prism. Now it is easy to see that of these systems of fringes the one that has its centre nearest the edge of the first prism is due to light polarised in the principal section of that prism: and hence the edge of the first prism will be parallel or perpendicular to the optic axis, according as the system of bands nearest to or furthest from it is due to light polarised in the parallel plane.

* Quincke, *Pogg. Ann.* CXXVII. 211 (1866).

205. Suppose now that a stream of elliptically polarised light falls normally upon the compensator, arranged so that the light enters by the prism that has its edge perpendicular to the optic axis, this also being the prism that can be moved. Then starting from the zero position of the instrument, in which $d_1 = d_2$ at the place marked out by the spider-lines, the micrometer screw is turned in one direction or the other until the least motion is found that renders the light, emergent between the lines, plane polarised, and the azimuth α of its plane of polarisation with respect to the plane at right-angles to the edges of the prisms is observed.

Now the polarisation-vectors of the components of the incident light polarised in planes perpendicular and parallel respectively to the edges of the prisms may be represented by

$$\xi = a \cos nt, \quad \eta = b \cos (nt + \Delta),$$

the polarisation of the stream being right-handed if Δ be between 0 and π, or between $-\pi$ and -2π, and left-handed when Δ is between π and 2π or between 0 and $-\pi$.

Suppose that the phase of the second component is retarded relatively to that of the first in its passage through the compensator by an amount Δ', this being positive or negative according as the motion of the prism has increased or diminished d_1, and being in all cases less than π. Then on emergence from the compensator the components may be represented by

$$\xi = ka \cos (nt + \phi), \quad \eta = kb \cos (nt + \Delta - \Delta' + \phi),$$

and hence the azimuth of the plane of polarisation α is $\tan^{-1}(b/a)$ or $\pi - \tan^{-1}(b/a)$, according as $\Delta - \Delta' = \pm\, 2n\pi$ or $\pm\,(2n+1)\,\pi$.

Hence if α be less than $\pi/2$, the polarization is right- or left-handed according as Δ' is positive or negative, and if α exceed $\pi/2$, the polarisation is right- or left-handed, according as Δ' is negative or positive: thus the polarisation is right- or left-handed according as $\tan \alpha \sin \Delta'$ is positive or negative.

Since the numerical value of $\tan \alpha$ gives the ratio of the amplitudes of the polarisation-vectors of the component streams polarised in planes parallel and perpendicular to the edges of the prisms, the elliptic polarisation is completely determined.

206. Babinet's compensator may also be used for a direct determination of the position and ratio of the axes of the elliptic vibrations in a stream of light.

A beam of elliptically polarised light may be represented by the polarisation-vectors

$$\xi = c \cos \beta \cos nt \quad \text{and} \quad \eta = -c \sin \beta \sin nt$$

corresponding to the component streams polarised in the planes of maximum

and minimum polarisation of the beam, β being less than $\pi/4$ and positive or negative according as the polarisation is right- or left-handed.

Now suppose the compensator is so set that it retards the phase of the stream polarised in a plane perpendicular to the edges of the prisms by an amount $\pi/2$ relatively to that of the stream polarised in a plane parallel to the edges, and let us first suppose that it is turned until the former of these planes coincides with the plane of maximum polarisation of the incident light, then the polarisation-vectors of the emergent streams will be

$$\xi = kc\cos\beta\cos(nt+\phi), \quad \eta = -kc\sin\beta\sin\left(nt+\phi+\frac{\pi}{2}\right) = -kc\sin\beta\cos(nt+\phi),$$

and the emergent light will be plane polarised in an azimuth α with respect to the plane perpendicular to the edges of the prisms, given by $\tan\alpha = -\tan\beta$, and $\tan\alpha$ will be positive or negative according as the polarisation is left- or right-handed.

Similarly if the compensator be so turned that the plane perpendicular to the edges of the prisms coincides with the plane of minimum polarisation of the incident light, the azimuth α of the plane of polarisation of the emergent stream, measured from this plane, is given by $\tan\alpha = -\cot\beta$.

Hence when the emergent light is plane polarised, the plane perpendicular to the edges of the prisms will give the plane of maximum or minimum polarisation of the stream according as $\tan\alpha$ is numerically less or greater than unity; its numerical value is the ratio of the axes of the elliptic vibration; and the polarisation is left- or right-handed according as $\tan\alpha$ is positive or negative.

207. The chief objection to the use of Babinet's compensator is that the fringes are localised at the instrument*, and it is therefore necessary to focus the eye on its surface, which renders it difficult to fix the direction of the stream of light that is studied. This disadvantage is overcome by employing a compensator of the second kind, that introduces the same relative retardation over the whole field.

An instrument of this type was devised by Biot, and consists of a plate of quartz cut parallel to the optic axis, followed by a second plate of adjustable thickness also parallel to the optic axis, and so placed that its axis is at right-angles to that of the first plate: in order that the thickness of the second plate may be capable of adjustment it is formed of two quartz wedges with the edges parallel to the optic axis, one of which can be moved over the other by means of a micrometer screw.

208. Instead of using the compensator in the second of the two methods described above, it is perhaps more convenient to employ a quarter-wave

* Schmidt, *Wied. Ann.* xxxv. 360 (1888). Macé de Lépinay, *J. de Phys.* (2) x. 204 (1891).

plate for the determination of the elements of the elliptic polarisation. Unfortunately it is difficult to obtain quarter-wave plates that are absolutely correct, and if perfect for one wave-length, they are of necessity imperfect for light of a different frequency. It is, however, possible to use an imperfect plate for the investigation of the polarisation*, provided the relative retardation of phase that it introduces lies between 2β and $\pi - 2\beta$, the limiting differences of phase between the rectangular components of the stream of elliptically polarised light.

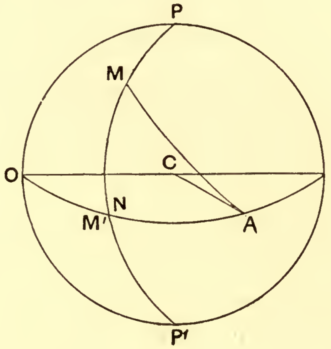


Fig. 50.

Let the point M, representative of the polarisation of the stream to be investigated, be determined by its longitude $ON = 2\theta$ and its latitude $NM = 2\beta$: then the effect of transmission through the plate is given by a rotation through an angle Δ round the axis AA' in the plane of the equator, where Δ is the relative retardation of phase introduced by the plate and OA is twice the angle that the plane of polarisation of the most retarded stream in the plate makes with the plane of reference. If the resulting polarisation be plane, this rotation must bring M into the equator to the point M', say, and the arc AM' is twice the angle σ that the resulting plane of polarisation makes with that of the most retarded stream in the plate.

If the arc $NA = 2\phi$, the spherical triangle ANM, in which

$$AM = AM' = 2\sigma,$$

gives

$$\sin 2\phi = \tan 2\beta \cot \Delta \qquad (6),$$

$$\cos 2\sigma = \cos 2\beta \cos 2\phi \qquad (7),$$

$$\cos \Delta = \tan 2\phi \cot 2\sigma \qquad (8).$$

Whence it follows that there are two possible positions of the axis AA', BB', such that $NA + NB = \pi$, and that the values of σ corresponding to these positions are complementary to one another. If the polarisation be right-

* MacCullagh, *Proc. R. I. Acad.* II. 384 (1843); *Collected Works*, pp. 238—242. Stokes, *B. A. Report*, 1851, Part II. 14; *Math. and Phys. Papers*, III. 197.

handed, A will lie within or without the arc ON, according as Δ is greater or less than $\pi/2$; the reverse being the case if the polarisation be left-handed.

If then χ_1 and $90^\circ + \chi_2$ be the azimuths, measured from a fixed plane of reference in a direction from right to left, of the plane of polarisation of the most retarded stream in the crystalline plate, when the emergent light is plane polarised, and if θ be the azimuth of the plane of maximum or of minimum polarisation of the primitive stream, we have

$$\chi_1 = \theta \pm \phi,\ \chi_2 = \theta \mp \phi, \text{ and } \theta = (\chi_1 + \chi_2)/2,\ \phi = (\chi_1 \sim \chi_2)/2 \quad \text{.........(9).}$$

Again if σ_1 and σ_2 be the azimuths of the plane of polarisation of the stream emerging from the plate in its first and second positions, measured from right to left from a plane of reference fixed in the plate, $(\sigma_1 + \sigma_2)/2$ gives a direction inclined at 45° to the plane of polarisation of the most retarded stream in the plate, and $\sigma_2 \sim \sigma_1 = \pi/2 \pm 2\sigma$, whence

$$\cos 2\beta = \sin(\sigma_2 \sim \sigma_1) \sec(\chi_2 \sim \chi_1) \quad \text{......................(10).}$$

Further it is easy to see that $\tan(\sigma_2 - \sigma_1)$ and $\tan(\chi_2 - \chi_1)$ have the same or opposite signs according as Δ is less or greater than $\pi/2$, and therefore

$$\cos\Delta = \tan(\sigma_2 - \sigma_1)\tan(\chi_2 - \chi_1) \quad \text{......................(11).}$$

To complete the specification of the state of polarisation of the primitive stream, we require to know the azimuths of the resulting plane of polarisation measured from the plane of polarisation of the most retarded stream in the crystalline plate. If σ_1' and σ_2' be these azimuths measured in a left-handed direction, the stream is right- or left-handed, according as σ_1' and σ_2' are greater or less than $\pi/2$, and the angle θ gives the plane of maximum or of minimum polarisation according as $\sin\sigma_2'$ is greater or less than $\sin\sigma_1'$.

209. We have seen in Chapter II that a stream of light may be one of seven different types: it is possible to have (1) common light, (2) polarised light, which may be either (*a*) elliptically, (*b*) circularly or (*c*) plane polarised, and (3) partially polarised light, the partial polarisation being (*a*) elliptical, or (*b*) circular, or (*c*) plane.

A stream of common light, when examined with a Nicol's prism, appears of constant intensity for all positions of the prism and it retains this characteristic after transmission through a quarter-wave plate, whatever may be its orientation.

Circularly polarised light resembles common light when it is viewed through an analyser, but after transmission through a quarter-wave plate it can be extinguished by a rotation of the analyser.

Elliptically polarised light, when observed through a Nicol, has an intensity dependent upon the orientation of the analyser, but in no case is the light entirely quenched.

Plane polarised light, when similarly investigated, can be entirely extinguished by rotating the analyser.

Partially plane polarised light resembles elliptically polarised light, when examined with a Nicol's prism, but the two kinds of light are distinguished by the fact that elliptically polarised light is converted into plane polarised light by transmission through a quarter-wave plate with its principal plane in the plane of maximum or of minimum polarisation of the stream.

Partially elliptically polarised light resembles partially plane polarised light, but may be differentiated from it in either of two ways: (1) by transmitting the stream through a quarter-wave plate with its principal plane in the plane of partial polarisation, then the light will be partially plane or partially elliptically polarised, according as the plane of partial polarisation remains the same or is altered: (2) by placing the quarter-wave plate with its principal plane at 45° to the plane of partial polarisation, then in the case of partial plane polarisation all traces of polarisation will disappear.

Partially circularly polarised light appears like common light, but is distinguished from it by transmission through a quarter-wave plate, which reduces it to partially plane polarised light*.

* Beer, *Höhere Optik*, 2nd ed. p. 176.

CHAPTER XVI.

ABSORBING MEDIA.

210. THE characteristic property of absorbing media is that they reduce the intensity of a stream of light in its progress through them by an amount that increases with the distance traversed, and it therefore follows that in these media the polarisation-vector of a train of plane waves of light must have a varying amplitude, so that, if we represent its components by the real parts of

$$u = \bar{A} e^{\iota(\bar{l}x + \bar{m}y + \bar{n}z - st)}, \quad v = \bar{B} e^{\iota(\bar{l}x + \bar{m}y + \bar{n}z - st)},$$

$$w = \bar{C} e^{\iota(\bar{l}x + \bar{m}y + \bar{n}z - st)},$$

one at least of the quantities $\bar{l}$, $\bar{m}$, $\bar{n}$ must be complex.

Now in the case of a transparent isotropic medium, l, m, n are connected by the relation

$$l^2 + m^2 + n^2 = s^2/\Omega^2,$$

where Ω is the propagational speed of light in the medium, and we can only retain this relation in the case of absorbing media, if we assume that Ω^2 then becomes a complex quantity. We are therefore led to extend the differential equations and hence also the boundary conditions obtained in Chapter XIII, so as to include absorbing anisotropic media, by assigning to Φ the value

$$2\bar{\Phi} = \bar{a}_{11}u^2 + \bar{a}_{22}v^2 + \bar{a}_{33}w^2 + 2\bar{a}_{23}vw + 2\bar{a}_{31}wu + 2\bar{a}_{12}uv \quad \text{.........(1)},$$

where $\bar{a}_{11}$... are complex quantities.

If we write $$\bar{a}_{hk} = a_{hk} + \iota a'_{hk},$$

we have $$\bar{\Phi} = \Phi + \iota\Phi',$$

where $$2\Phi = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + 2a_{23}vw + 2a_{31}wu + 2a_{12}uv \quad \text{.........(2)},$$

$$2\Phi' = a_{11}'u^2 + a_{22}'v^2 + a_{33}'w^2 + 2a_{23}'vw + 2a_{31}'wu + 2a_{12}'uv \quad \text{......(3)},$$

and by a proper choice of axes, we can bring either Φ into the form

$$2\Phi = a^2u^2 + b^2v^2 + c^2w^2 \quad \text{..............................(4)},$$

or Φ' into the form

$$2\Phi' = a'^2u^2 + b'^2v^2 + c'^2w^2 \quad \text{............................(5)};$$

in the first case the axes are termed the polarisation-axes, a, b, c being the principal polarisation-constants, and in the second case the axes are called the absorption-axes, a', b', c' being the principal constants of absorption.

In a crystal of the anorthic system the two sets of axes are independent of one another, and taking the axes in the direction of the polarisation-axes, the medium is characterised by nine constants: this number is reduced to seven in the case of a monoclinic crystal, as then one of the polarisation-axes is coincident with one of the absorption-axes. In crystals of the prismatic system, the two sets of axes are identical and there are six constants: these are reduced to four in the case of crystals of the tetragonal and hexagonal systems, while crystals of the cubic system and isotropic media possess only two constants*.

211. In order to determine the general characteristics of the propagation of light in absorbing media, let us take the axis of z in the direction of the wave-normal; then $w = 0$ and u, v are functions of z and t alone, so that the differential equations become

$$\ddot{u} = \bar{a}_{11}\frac{\partial^2 u}{\partial z^2} + \bar{a}_{12}\frac{\partial^2 v}{\partial z^2}, \qquad \ddot{v} = \bar{a}_{22}\frac{\partial^2 v}{\partial z^2} + \bar{a}_{12}\frac{\partial^2 u}{\partial z^2} \quad\text{..............(6).}$$

Let

$$(u, v) = (\bar{A}, \bar{B})\,\bar{D}e^{\iota\bar{\kappa}(z-\bar{\omega}t)} \quad\text{..........................(7),}$$

where $\quad \bar{A}^2 + \bar{B}^2 = 1, \qquad \bar{\kappa} = \kappa(1 - \iota\nu), \qquad \bar{\omega} = \omega/(1 - \iota\nu);$

then these equations give

$$(\bar{\omega}^2 - \bar{a}_{11})\bar{A} = \bar{a}_{12}\bar{B}, \qquad (\bar{\omega}^2 - \bar{a}_{22})\bar{B} = \bar{a}_{12}\bar{A} \quad\text{..............(8),}$$

whence

$$(\bar{\omega}^2 - \bar{a}_{11})(\bar{\omega}^2 - \bar{a}_{22}) = \bar{a}_{12}^2 \quad\text{..........................(9),}$$

and

$$\frac{\bar{B}^2}{\bar{A}^2} + \frac{\bar{a}_{11} - \bar{a}_{22}}{\bar{a}_{12}}\frac{\bar{B}}{\bar{A}} - 1 = 0 \quad\text{........................(10).}$$

Equation (9) determines two values of $\bar{\omega}^2$ and therefore two values of ω and ν, while (10) gives the corresponding values of the complex ratio $\bar{B}/\bar{A}$; denoting these by the suffixes (1), (2), we have

$$\bar{A}_1\bar{A}_2 + \bar{B}_1\bar{B}_2 = 0 \quad\text{..............................(11).}$$

Since the ratio $\bar{B}/\bar{A}$ is complex, it follows that the two waves thus determined are elliptically polarised. Now by a proper choice of the origin of time we can arrange that

$$\bar{A}\bar{D} = c(\cos\beta\cos\alpha - \iota\sin\beta\sin\alpha), \quad \bar{B}\bar{D} = c(\cos\beta\sin\alpha + \iota\sin\beta\cos\alpha),$$

where $\tan\beta$ is the ratio of the axes of the elliptic path of the end of the

* Drude, *Wied. Ann.* XXXII. 584 (1887); XL. 665 (1890). Winkelmann, *Handb. der Physik*, II. 807—819. Voigt, *Wied. Ann.* XXIII. 577 (1884); *Komp. der theor. Physik*, II. 708—725.

polarisation-vector and α is the angle that these axes make with the coordinate axes: whence (11) gives the two equations

$$\cos(\beta_2+\beta_1)\cos(\alpha_2-\alpha_1)=0, \qquad \sin(\beta_2-\beta_1)\sin(\alpha_2-\alpha_1)=0,$$

which are satisfied only if

$$\alpha_2=\alpha_1+\pi/2 \text{ and } \beta_2=\beta_1,$$

or

$$\alpha_2=\alpha_1 \qquad \text{and } \beta_2=\pi/2-\beta_1,$$

both of which conditions express that in the two waves the ellipses are similar and traversed in the same direction, while their major axes are at right-angles.

212. Unless the polarisation-axes and the absorption-axes are coincident, it is impossible to bring Φ and Φ' simultaneously into the forms (4) and (5) and by no real transformation of the coordinate axes can $\bar{\Phi}$ be made to assume the form

$$2\bar{\Phi}=\bar{a}^2u^2+\bar{b}^2v^2+\bar{c}^2w^2,$$

but this may be effected by the employment of a complex system of coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$.

Let the scheme of transformation be

	$\bar{x}$	$\bar{y}$	$\bar{z}$
x	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_3$
y	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_3$
z	$\bar{\gamma}_1$	$\bar{\gamma}_2$	$\bar{\gamma}_3$

where $\bar{\alpha}_1$, $\bar{\beta}_1$, ... represent complex direction-cosines, fulfilling the ordinary conditions of an orthogonal transformation; then

$$\left.\begin{array}{l}\bar{a}_{11}=\bar{\alpha}_1{}^2\bar{a}^2+\bar{\alpha}_2{}^2\bar{b}^2+\bar{\alpha}_3{}^2\bar{c}^2\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \bar{a}_{23}=\bar{\beta}_1\bar{\gamma}_1\bar{a}^2+\bar{\beta}_2\bar{\gamma}_2\bar{b}^2+\bar{\beta}_3\bar{\gamma}_3\bar{c}^2\\ \cdots\cdots\cdots\cdots\cdots\cdots\end{array}\right\}\quad\text{......(12)},$$

whence we obtain three sets of equations of the form

$$\left.\begin{array}{r}(\bar{a}_{11}-\bar{x}_h{}^2)\,\bar{\alpha}_h+\bar{a}_{12}\,\bar{\beta}_h+\bar{a}_{31}\,\bar{\gamma}_h=0\\ \bar{a}_{12}\,\bar{\alpha}_h+(\bar{a}_{22}-\bar{x}_h{}^2)\,\bar{\beta}_h+\bar{a}_{23}\,\bar{\gamma}_h=0\\ \bar{a}_{31}\,\bar{\alpha}_h+\bar{a}_{23}\,\bar{\beta}_h+(\bar{a}_{33}-\bar{x}_h{}^2)\,\bar{\gamma}_h=0\end{array}\right\}\quad\text{......(13)},$$

$h=1, 2, 3$ and $\bar{x}_1=\bar{a}$, $\bar{x}_2=\bar{b}$, $\bar{x}_3=\bar{c}$: and from these equations we obtain

$$\begin{vmatrix} \bar{a}_{11}-\bar{x}^2 & \bar{a}_{12} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22}-\bar{x}^2 & \bar{a}_{23} \\ \bar{a}_{31} & \bar{a}_{23} & a_{33}-\bar{x}^2 \end{vmatrix} = 0 \quad \text{.................(14)},$$

the three roots of which give the values of $\bar{a}^2$, $\bar{b}^2$, $\bar{c}^2$, and these being known, equations (13) give the values of $\bar{\alpha}_h$, $\bar{\beta}_h$, $\bar{\gamma}_h$.

213. The differential equations now become

$$(\ddot{u}, \ddot{v}, \ddot{w}) = \nabla^2\left(\frac{\partial\bar{\Phi}}{\partial u}, \frac{\partial\bar{\Phi}}{\partial v}, \frac{\partial\bar{\Phi}}{\partial w}\right)$$

$$-\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\frac{\partial\bar{\Phi}}{\partial u}+\frac{\partial}{\partial y}\frac{\partial\bar{\Phi}}{\partial v}+\frac{\partial}{\partial z}\frac{\partial\bar{\Phi}}{\partial w}\right) \text{............(15)},$$

with $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \text{..........................(16)},$$

where $$2\bar{\Phi} = \bar{a}^2u^2+\bar{b}^2v^2+\bar{c}^2w^2 \text{..............................(17)}.$$

Writing $$(u, v, w) = (\bar{A}, \bar{B}, \bar{C})\,\bar{D}e^{\iota\bar{\kappa}(\bar{l}x+\bar{m}y+\bar{n}z-\bar{\omega}t)} \text{..................(18)},$$

where $\bar{A}^2+\bar{B}^2+\bar{C}^2=1$ and $\bar{l}$, $\bar{m}$, $\bar{n}$ are the complex direction-cosines of the wave-normal, determined by

$$\bar{l}=\bar{\alpha}_1 l+\bar{\beta}_1 m+\bar{\gamma}_1 n, \quad \bar{m}=\bar{\alpha}_2 l+\bar{\beta}_2 m+\bar{\gamma}_2 n, \quad \bar{n}=\bar{\alpha}_3 l+\bar{\beta}_3 m+\bar{\gamma}_3 n \text{ ......(19)},$$

these equations give

$$(\bar{a}^2-\bar{\omega}^2)\bar{A} = (\bar{a}^2\bar{A}\bar{l}+\bar{b}^2\bar{B}\bar{m}+\bar{c}^2\bar{C}\bar{n})\,\bar{l} \text{.................(20)},$$

and two similar equations, with

$$\bar{A}\bar{l}+\bar{B}\bar{m}+\bar{C}\bar{n}=0 \text{..............................(21)}.$$

Whence $$\frac{\bar{l}^2}{\bar{a}^2-\bar{\omega}^2}+\frac{\bar{m}^2}{\bar{b}^2-\bar{\omega}^2}+\frac{\bar{n}^2}{\bar{c}^2-\bar{\omega}^2}=0 \quad \text{.....................(22)},$$

$$\bar{A} : \bar{B} : \bar{C} :: \frac{\bar{l}}{\bar{a}^2-\bar{\omega}^2} : \frac{\bar{m}}{\bar{b}^2-\bar{\omega}^2} : \frac{\bar{n}}{\bar{c}^2-\bar{\omega}^2} \text{.....................(23)}.$$

Separating (22) into its real and imaginary parts we obtain two simultaneous equations involving ω and ν. The results are very complicated, but it is clear that Fresnel's laws for transparent crystalline media no longer hold.

214. A notable simplification of the problem however occurs when we can regard the absorption as slight, and in that case the propagation is determined in accordance with Fresnel's laws.

Let us take the polarisation-axes as the coordinate axes, then

$$\bar{a}_{11}=a^2+\iota a'_{11}, \quad \bar{a}_{22}=b^2+\iota a'_{22}, \quad \bar{a}_{33}=c^2+\iota a'_{33},$$

$$\bar{a}_{23}=\iota a'_{23}, \quad \bar{a}_{31}=\iota a'_{31}, \quad \bar{a}_{12}=\iota a'_{12},$$

and regarding a'_{hk} as a small quantity of the first order, we have from (14) by neglecting terms of the second order

$$(\bar{a}_{11}-\bar{x}^2)(\bar{a}_{22}-\bar{x}^2)(\bar{a}_{33}-\bar{x}^2)=0,$$

which gives as the values of $\bar{x}^2$

$$a^2+\iota a'_{11}, \qquad b^2+\iota a'_{22}, \qquad c^2+\iota a'_{33};$$

whence, writing

$$\bar{\alpha}_h=\alpha_h+\iota\alpha_h', \qquad \bar{\beta}_h=\beta_h+\iota\beta_h', \qquad \bar{\gamma}_h=\gamma_h+\iota\gamma_h',$$

we obtain from (13)

$$\left.\begin{aligned} a'_{12}\beta_1+a'_{31}\gamma_1&=0\\ a'_{12}\alpha_1+(a'_{22}-a'_{11})\beta_1+(b^2-a^2)\beta'_1+a'_{23}\gamma_1&=0\\ a'_{31}\alpha_1+a'_{23}\beta_1+(a'_{33}-a'_{11})\gamma_1+(c^2-a^2)\gamma_1'&=0\\ \alpha_1\alpha_1'+\beta_1\beta_1'+\gamma_1\gamma_1'&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(24),$$

and

$$\left.\begin{aligned} a'_{12}\beta_1'+a'_{31}\gamma_1'&=0\\ a'_{12}\alpha_1'+(a'_{22}-a'_{11})\beta_1'-(b^2-a^2)\beta_1+a'_{23}\gamma_1'&=0\\ a'_{31}\alpha_1'+a'_{23}\beta_1'+(a'_{33}-a'_{11})\gamma_1'-(c^2-a^2)\gamma_1&=0\\ \alpha_1^2+\beta_1^2+\gamma_1^2-\alpha_1'^2-\beta_1'^2-\gamma_1'^2&=1 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(25),$$

which are satisfied as far as terms of the first order by

$$\alpha_1=1,\ \beta_1=0,\ \gamma_1=0,\ \alpha_1'=0,\ \beta_1'=\frac{a'_{12}}{a^2-b^2},\ \gamma_1'=\frac{a'_{31}}{a^2-c^2},$$

and α_2, β_2, ... γ_3' are obtained from these by a cyclical change of letters and subscripts. With these values we obtain

$$\left.\begin{aligned} \bar{l}&=l+\iota l'=l+\iota\left(m\frac{a'_{12}}{a^2-b^2}+n\frac{a'_{31}}{a^2-c^2}\right)\\ \bar{m}&=m+\iota m'=m+\iota\left(n\frac{a'_{23}}{b^2-c^2}+l\frac{a'_{12}}{b^2-a^2}\right)\\ \bar{n}&=n+\iota n'=n+\iota\left(l\frac{a'_{31}}{c^2-a^2}+m\frac{a'_{23}}{c^2-b^2}\right) \end{aligned}\right\}\quad\ldots\ldots\ldots(26),$$

and thence, since $\bar{\omega}^2 \risingdotseq \omega^2(1+2\iota\nu)$,

$$\Sigma\frac{\bar{l}^2}{\bar{a}^2-\bar{\omega}^2}\risingdotseq\Sigma\frac{l^2}{a^2-\omega^2}+\iota\Sigma\left\{\frac{2ll'}{a^2-\omega^2}-\frac{l^2}{a^2-\omega^2}\,\frac{a'_{11}-2\nu\omega^2}{a^2-\omega^2}\right\}.$$

Hence separating the real and imaginary parts, equation (22) gives

$$\frac{l^2}{a^2-\omega^2}+\frac{m^2}{b^2-\omega^2}+\frac{n^2}{c^2-\omega^2}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

and

$$2\nu\omega^2\left\{\frac{l^2}{(a^2-\omega^2)^2}+\frac{m^2}{(b^2-\omega^2)^2}+\frac{n^2}{(c^2-\omega^2)^2}\right\}$$

$$=a'_{11}\frac{l^2}{(a^2-\omega^2)^2}+a'_{22}\frac{m^2}{(b^2-\omega^2)^2}+a'_{33}\frac{n^2}{(c^2-\omega^2)^2}$$

$$+2a'_{23}\frac{mn}{(b^2-\omega^2)(c^2-\omega^2)}+2a'_{31}\frac{nl}{(c^2-\omega^2)(a^2-\omega^2)}+2a'_{12}\frac{lm}{(a^2-\omega^2)(b^2-\omega^2)}\ (28).$$

Now, if $\bar{A} = A + \iota A'$, $\bar{B} = B + \iota B'$, $\bar{C} = C + \iota C'$,

$$A : B : C :: l/(a^2 - \omega^2) : m/(b^2 - \omega^2) : n/(c^2 - \omega^2),$$

and therefore

$$\begin{aligned} 2\nu\omega^2 &= a'_{11} \cos^2 \theta_1 + a'_{22} \cos^2 \theta_2 + a'_{33} \cos^2 \theta_3 \\ &\quad + 2a'_{23} \cos \theta_2 \cos \theta_3 + 2a'_{31} \cos \theta_3 \cos \theta_1 + 2a'_{12} \cos \theta_1 \cos \theta_2 \\ &= a'^2 \cos^2 \theta_1' + b'^2 \cos^2 \theta_2' + c'^2 \cos^2 \theta_3' \quad \text{...............(29)}, \end{aligned}$$

where θ_1, θ_2, θ_3 and θ_1', θ_2', θ_3' are the angles that the major axis of the elliptic vibrations makes with the polarisation- and the absorption-axes respectively and a', b', c' are the principal constants of absorption.

215. When the polarisation- and the absorption-axes coincide, as is the case with prismatic crystals, a further simplification occurs, as then

$$a'_{23} = a'_{31} = a'_{12} = 0,$$

and writing a'^2, b'^2, c'^2 for a'_{11}, a'_{22}, a'_{33} respectively we have

$$2\nu\omega^2 = a'^2 A_1^2 + b'^2 B_1^2 + c'^2 C_1^2 \quad \text{...............(30)}.$$

This may be expressed in terms of the angles χ, χ' that the wave-normal makes with the optic axes. Through the centre of a sphere of unit radius, let us draw lines parallel to the optic axes, the wave-normal and the vectors (A_1, B_1, C_1) and (A_2, B_2, C_2) and let these meet the surface of the sphere in the points A, A', N, ω_1, ω_2 respectively: then if $\pm l_0$, 0, n_0 be the direction-cosines of the optic axes and i be the angle ANA',

$$\sin \chi \sin i/2 = \cos (\omega_1 A) = A_1 l_0 + C_1 n_0,$$

$$-\sin \chi' \sin i/2 = \cos (\omega_1 A') = -A_1 l_0 + C_1 n_0,$$

whence

$$A_1 l_0 = \sin \frac{\chi + \chi'}{2} \cos \frac{\chi - \chi'}{2} \sin \frac{i}{2},$$

$$C_1 n_0 = \cos \frac{\chi + \chi'}{2} \sin \frac{\chi - \chi'}{2} \sin \frac{i}{2},$$

and similarly

$$A_2 l_0 = \cos \frac{\chi + \chi'}{2} \sin \frac{\chi - \chi'}{2} \cos \frac{i}{2},$$

$$C_2 n_0 = \sin \frac{\chi + \chi'}{2} \cos \frac{\chi - \chi'}{2} \cos \frac{i}{2}.$$

Also if 2Ψ be the angle between the optic axes

$$\cos 2\Psi = \cos \chi \cos \chi' + \sin \chi \sin \chi' \cos i,$$

whence

$$\sin \chi \sin \chi' \sin^2 \frac{i}{2} = \cos^2 \frac{\chi - \chi'}{2} - \cos^2 \Psi = \sin^2 \Psi - \sin^2 \frac{\chi - \chi'}{2},$$

$$\sin \chi \sin \chi' \cos^2 \frac{i}{2} = \cos^2 \Psi - \cos^2 \frac{\chi + \chi'}{2} = \sin^2 \frac{\chi + \chi'}{2} - \sin^2 \Psi.$$

Making these substitutions we obtain

$$2\nu\omega^2 = \left[(c^2 - a^2)\sin^2\frac{\chi \mp \chi'}{2}\cos^2\frac{\chi \mp \chi'}{2}\left(\frac{a'^2 - b'^2}{a^2 - b^2}\sin^2\frac{\chi \pm \chi'}{2} + \frac{b'^2 - c'^2}{b^2 - c^2}\cos^2\frac{\chi \pm \chi'}{2}\right)\right.$$

$$\left. + a'^2 \sin^2\frac{\chi \pm \chi'}{2}\cos^2\frac{\chi \mp \chi'}{2} - c'^2\cos^2\frac{\chi \pm \chi'}{2}\sin^2\frac{\chi \mp \chi'}{2}\right]$$

$$\div\left(\cos^2\frac{\chi \mp \chi'}{2} - \cos^2\frac{\chi \pm \chi'}{2}\right) \qquad \text{......................(31)},$$

and

$$\omega^2 = a^2 - (a^2 - c^2)\sin^2\frac{\chi \mp \chi'}{2}.$$

When the wave-normal is very nearly coincident with one of the optic axes, so that χ, say, is very small, we may proceed to a further simplification*; for if ψ be the angle $A'AN$, we have approximately

$$\chi' + \chi = \chi' - \chi = 2\Psi, \qquad i = \pi - \psi, \qquad \omega_1 = \omega_2 = b,$$

whence

$$A_1 = \cos\Psi\cos\psi/2, \qquad B_1 = \sin\psi/2, \qquad C_1 = -\sin\Psi\cos\psi/2,$$
$$A_2 = -\cos\Psi\sin\psi/2, \qquad B_2 = \cos\psi/2, \qquad C_2 = \sin\Psi\sin\psi/2,$$

and (30) gives

$$\left.\begin{aligned} 2\nu_1 b^2 &= (a'^2\cos^2\Psi + c'^2\sin^2\Psi)\cos^2\frac{\psi}{2} + b'^2\sin^2\frac{\psi}{2} \\ 2\nu_2 b^2 &= (a'^2\cos^2\Psi + c'^2\sin^2\Psi)\sin^2\frac{\psi}{2} + b'^2\cos^2\frac{\psi}{2} \end{aligned}\right\} \qquad \text{.........(32)}.$$

When the wave-normal coincides with the optic axis, these formulæ become indeterminate, but we obtain from (30)

$2\nu_1'b^2 = b'^2$, when the plane of polarisation is parallel to the plane of the optic axes, and

$2\nu_2'b^2 = a'^2\cos^2\Psi + c'^2\sin^2\Psi$ for the stream polarised in the perpendicular plane.

There are two types of biaxal absorbing crystals: in those of the first type, such as Andalusite, Hornblende, Titanite, $\nu_2' > \nu_1'$, while in those of the second type, of which Cordierite, Epidote, Axinite are examples, $\nu_1' > \nu_2'$. Similarly with uniaxal crystals: in those of the first type (Magnesium platino-cyanide) $\nu_0 < \nu_e$, in those of the second type (Tourmaline) $\nu_0 > \nu_e$.

In traversing unit distance in the direction of the wave-normal, the amplitude of the vibrations is diminished in the proportion $\exp\{2\pi\nu/(\tau\omega)\} : 1$. If then we draw through a given point vectors equal to the absorption-coefficients ν_1/ω_1 and ν_2/ω_2, we shall obtain a surface of two sheets that has a certain analogy with the surface of wave-quickness. The sheets of this

* Voigt, *Wied. Ann.* XXIII. 595 (1884).

surface intersect, not in definite points but along portions of curves, that in the case of anorthic and monoclinic crystals are unsymmetrical with respect to the planes of optical symmetry and do not in general pass through the optic axes. Thus in weakly absorbing crystals, while there are at most two directions of equal wave-velocity, there is a series of axes of equal absorption*.

216. We are now in a position to consider the interference phenomena exhibited by plates of weakly absorbing crystals†.

Let α, β, η be the angles that the primitive and final planes of polarisation and the plane of polarisation of the quicker wave within the plate make respectively with some fixed plane of reference: then neglecting the ellipticity of the polarisation of the streams within the crystal, as this is very slight in the case of weak absorption, and making the same assumptions as in Chapter XIV, the polarisation-vectors of the streams emergent from the plate may be represented by

$$a\cos(\alpha-\eta)\,e^{-\nu_1\sigma_1}\,e^{\iota(nt-\kappa\delta_1)} \quad\text{and}\quad a\sin(\alpha-\eta)\,e^{-\nu_2\sigma_2}\,e^{\iota(nt-\kappa\delta_2)},$$

where $\sigma = 2\pi T/(\tau\omega\cos r)$, T being the thickness of the plate and r the angle of entry; and that of the stream leaving the analyser will be

$$a\{\cos(\alpha-\eta)\cos(\beta-\eta)\,e^{-\nu_1\sigma_1}\,e^{-\iota\kappa\delta_1} + \sin(\alpha-\eta)\sin(\beta-\eta)\,e^{-\nu_2\sigma_2}\,e^{-\iota\kappa\delta_2}\}\,e^{\iota nt},$$

and the intensity is

$$I = a^2\{\cos^2(\alpha-\eta)\cos^2(\beta-\eta)\,e^{-2\nu_1\sigma_1} + \sin^2(\alpha-\eta)\sin^2(\beta-\eta)\,e^{-2\nu_2\sigma_2}$$
$$+\,2\sin(\alpha-\eta)\cos(\alpha-\eta)\sin(\beta-\eta)\cos(\beta-\eta)\,e^{-(\nu_1\sigma_1+\nu_2\sigma_2)}\cos\kappa\delta\}\ldots(33),$$

where δ is the relative retardation of the streams as determined in Chapter XIV.

If the incident light be unpolarised, we may replace it by two independent streams of equal intensity polarised in any two rectangular planes, and the final intensity will be the sum of the final intensities of these streams: hence

$$I = \frac{a^2}{2}\{\cos^2(\beta-\eta)\,e^{-2\nu_1\sigma_1} + \sin^2(\beta-\eta)\,e^{-2\nu_2\sigma_2}\}\ldots\ldots\ldots\ldots(34).$$

When the light is neither polarised nor analysed, the intensity is

$$I = \frac{a^2}{2}(e^{-2\nu_1\sigma_1} + e^{-2\nu_2\sigma_2})\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

217. Let us first apply these formulæ to the case of an uniaxal plate perpendicular to the optic axis placed in convergent light between crossed Nicol's prisms. The intensity then is

$$I = \frac{a^2}{4}\sin^2 2(\alpha-\eta)\{e^{-2\nu_0\sigma_0} + e^{-2\nu_e\sigma_e} - 2\,e^{-(\nu_0\sigma_0+\nu_e\sigma_e)}\cos\kappa\delta\}\ (36),$$

* Drude, *Wied. Ann.* XL. 676 (1890).

† Voigt, *Wied. Ann.* XXIII. 587 (1884); *N. Jahrb. für Min.* (1885) I. 119. Drude, *Lehrb. der Optik*, pp. 345—351. Liebisch, *Phys. Kryst.* pp. 527—533.

where $$2\nu_0 = \frac{a'^2}{a^2}, \qquad 2\nu_e = \frac{a'^2\cos^2\chi + c'^2\sin^2\chi}{a^2\cos^2\chi + c^2\sin^2\chi}.$$

In the direction of the optic axis $\nu_0 = \nu_e$, $\sigma_0 = \sigma_e$, $\delta = 0$, whence $I = 0$, and the intensity also vanishes when $\eta = \alpha$ or $\alpha + \pi/2$, so that there is a black cross with its arms parallel to the principal planes of the Nicol's prisms. The second factor in (36) equated to zero gives a series of dark rings round the optic axis, but these are only completely black if $\nu_0\sigma_0 = \nu_e\sigma_e$.

The rings become less conspicuous the stronger the absorption, since the factor $\exp\{-(\nu_0\sigma_0 + \nu_e\sigma_e)\}$ becomes vanishingly small, and

$$I = \frac{a^2}{4}\sin^2(\alpha - \eta)(e^{-2\nu_0\sigma_0} + e^{-2\nu_e\sigma_e}).$$

Thus in crystals of the first type, such as magnesium platino-cyanide, for which a' is small and c' is large, the field is bright except for the dark cross; in crystals of the second type, such as tourmaline, for which a' is large and c' is small, the whole field is dark.

When the incident light is unpolarised

$$I = \frac{a^2}{2}\{\cos^2(\beta - \eta)\, e^{-2\nu_0\sigma_0} + \sin^2(\beta - \eta)\, e^{-2\nu_e\sigma_e}\},$$

and in the direction of the optic axis

$$I_0 = \frac{a^2}{2} e^{-2\nu_0\sigma_0}.$$

Thus with crystals of the first type, there is a dark brush perpendicular to the plane of analysation interrupted by a bright spot at the centre; while with crystals of the second type, the brush is parallel to the plane of analysation and is continuous.

When the light is neither polarised nor analysed

$$I = \frac{a^2}{2}(e^{-2\nu_0\sigma_0} + e^{-2\nu_e\sigma_e}),$$

and in crystals of the first type there is a bright spot surrounded by a dark field; with those of the second type there is a dark spot in the centre of a lighter field.

218. As a second example of the interference phenomena given by absorbing crystals, we will consider the case of a biaxal plate cut in a direction perpendicular to one of the optic axes.

Taking the plane of the optic axes as the plane of reference, we have for small angles of incidence $\eta = \psi/2$, where ψ is the azimuth of the plane of incidence: hence if the planes of polarisation and analysation be crossed

$$I = \frac{a^2}{4}\sin^2(2\alpha - \psi)\{e^{-2\nu_1\sigma} + e^{-2\nu_2\sigma} - 2e^{-(\nu_1+\nu_2)\sigma}\cos\kappa\delta\} \quad\text{......(37)},$$

where $\sigma = 2\pi T/(\tau b)$ and ν_1, ν_2 are given by (32), which formulæ, though strictly holding only for the cases of prismatic crystals, will afford with sufficient accuracy a qualitative explanation of the phenomena observed with crystals of other systems.

For the direction of the optic axis itself, we obtain by resolving the incident light into streams polarised in planes parallel and perpendicular to the plane of the optic axes

$$I_0 = \frac{a^2}{4} \sin 2\alpha \, (e^{-\nu_1'\sigma} - e^{-\nu_2'\sigma})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38),$$

where $$2\nu_1' = b'^2/b^2, \qquad 2\nu_2' = (a'^2 \cos^2 \Psi + c'^2 \sin^2 \Psi)/b^2 \quad \ldots\ldots\ldots\ldots (39).$$

The factor $\sin^2 (2\alpha - \psi)$ gives a principal line of like polarisation $\psi = 2\alpha$, which is black, but is interrupted by a brighter spot at the point corresponding to the optic axis, unless $\alpha = 0$ or $\pi/2$. Since $\cos \kappa\delta$ changes periodically as the angle of incidence increases, the last factor in (37) will give a series of dark rings. These however will be too faint to be observed, if the plate be of a thickness for the absorption to be marked, as the factor $\exp \{-(\nu_1 + \nu_2)\sigma\}$ then is very small and the term in question becomes

$$J = e^{-2\nu_1\sigma} + e^{-2\nu_2\sigma}.$$

Now $$\frac{\partial J}{\partial \psi} = \sigma \sin \psi \, (\nu_2' - \nu_1') \, (e^{-2\nu_1\sigma} - e^{-2\nu_2\sigma}),$$

and this is zero, if $\psi = 0$ or π, giving a maximum value of J, and if $\nu_1 = \nu_2$ or $\psi = \pi/2$ corresponding to a minimum value of J. Thus in addition to the black line of like polarisation $\psi = 2\alpha$, there is a dark line perpendicular to the plane of the optic axes.

If the planes of polarisation and analysation be parallel and the plate be of sufficient thickness

$$I = a^2 \left\{\cos^4 \left(\alpha - \frac{\psi}{2}\right) e^{-2\nu_1\sigma} + \sin^4 \left(\alpha - \frac{\psi}{2}\right) e^{-2\nu_2\sigma}\right\},$$

and the phenomenon is essentially the same as when the light is unanalysed, the intensity then being

$$I = a^2 \left\{\cos^2 \left(\alpha - \frac{\psi}{2}\right) e^{-2\nu_1\sigma} + \sin^2 \left(\alpha - \frac{\psi}{2}\right) e^{-2\nu_2\sigma}\right\}.$$

Taking this last case and supposing $\alpha = 0$, we have

$$I = a^2 \left(\cos^2 \frac{\psi}{2} \, e^{-2\nu_1\sigma} + \sin^2 \frac{\psi}{2} \, e^{-2\nu_2\sigma}\right),$$

$$I_0 = a^2 e^{-2\nu_2'\sigma}.$$

Now

$$\frac{\partial I}{\partial \psi} = a^2 \sin \psi \left\{\sigma (\nu_1' - \nu_2') \left(\sin^2 \frac{\psi}{2} e^{-2\nu_2\sigma} - \cos^2 \frac{\psi}{2} e^{-2\nu_1\sigma}\right) - \frac{e^{-2\nu_1\sigma} - e^{-2\nu_2\sigma}}{2}\right\}$$

$$= 0, \text{ if } \psi = 0 \text{ or } \pi, \text{ or if } \psi = \pm \pi/2.$$

But for $\psi = 0$ or π, $I = I_1 = a^2 e^{-2\nu_2'\sigma}$,

for $\psi = \pm \pi/2$, $I = I_2 = a^2 e^{-(\nu_1' + \nu_2')\sigma}$,

whence for crystals of the first type, for which $\nu_2' > \nu_1'$, $I_2 > I_1$ and the dark brush is in the plane of the optic axes and continues through the centre of the field: with crystals of the second type, $\nu_1' > \nu_2'$, $I_1 > I_2$ and there is a dark brush perpendicular to the plane of the optic axes interrupted by a brighter central spot. The reverse is the case when $\alpha = \pi/2$.

When the light is neither polarised nor analysed

$$I = \frac{a^2}{2} \{e^{-2\nu_1\sigma} + e^{-2\nu_2\sigma}\},$$

$$I_0 = \frac{a^2}{2} \{e^{-2\nu_1'\sigma} + e^{-2\nu_2'\sigma}\}.$$

This expression has already been discussed and it shows that there will be a dark brush perpendicular to the plane of the optic axes with a brighter spot at its centre.

219. Passing now to the problem of reflection and refraction* at the interface of absorbing media, we may at once apply the formulæ obtained for transparent substances, provided we replace by complex quantities the parameters that occur therein.

Thus in the first place the geometrical laws of the phenomenon follow from the fact that the boundary conditions are linear, homogeneous relations between the vectors characterising the incident, reflected and refracted streams: for the interface being the plane of yz and the vectors being proportional to

$$\exp \{\iota\kappa_h (\bar{l}_h x + \bar{m}_h y + \bar{n}_h z - \bar{\omega}_h t)\},$$

where

$$\bar{\kappa}_h \bar{\omega}_h = \kappa_h \omega_h = 2\pi/\tau_h,$$

it follows that the quantities

$$\bar{\kappa}_h \bar{\omega}_h, \quad \bar{\kappa}_h \bar{m}_h, \quad \bar{\kappa}_h \bar{n}_h,$$

must have the same value for each of the streams.

* Drude, *Wied. Ann.* XXXII. 584 (1887); XXXIV. 489; XXXV. 508 (1888); XXXVI. 532, 865 (1889); XXXIX. 481 (1890). Voigt, *ibid.* XXIII. 104 (1884); XXV. 95 (1885); XXXI. 233 (1887); XXXV. 76 1888); *Komp. der theor. Physik*, II. 730—747.

Let us as before adopt the light-vector ϖ as representative of the streams, and let us suppose that the normal to the planes of constant amplitude of the incident stream are in the plane of incidence, which we will take as the plane of xz, then $m = m' = 0$ and therefore $\bar{m} = 0$ and we may write

$$\left.\begin{aligned} &(\varpi_1, \varpi_2, \varpi_3) = (\bar{n}, \bar{k}, -\bar{l})\,\bar{D} \\ \text{where}\qquad &\bar{k} = \tan\bar{\phi}, \quad \bar{D} = \cos\bar{\phi}\,\bar{A}\exp\{\iota\kappa(\bar{l}x + \bar{n}z - \bar{\omega}t)\} \end{aligned}\right\} \quad \text{.........(40).}$$

$\bar{\phi}$ being the complex azimuth of the vector with respect to the plane of incidence.

Since the vector ϖ is independent of y, the differential equations give

$$\left.\begin{aligned} \ddot{\varpi}_1 &= \frac{\partial}{\partial z}\left\{-\bar{a}_{12}\frac{\partial\varpi_2}{\partial z} + \bar{a}_{22}\left(\frac{\partial\varpi_1}{\partial z} - \frac{\partial\varpi_3}{\partial x}\right) + \bar{a}_{23}\frac{\partial\varpi_2}{\partial x}\right\} \\ \ddot{\varpi}_2 &= -\frac{\partial}{\partial z}\left\{-\bar{a}_{11}\frac{\partial\varpi_2}{\partial z} + \bar{a}_{12}\left(\frac{\partial\varpi_1}{\partial z} - \frac{\partial\varpi_3}{\partial x}\right) + \bar{a}_{31}\frac{\partial\varpi_2}{\partial x}\right\} \\ &\quad + \frac{\partial}{\partial x}\left\{-\bar{a}_{31}\frac{\partial\varpi_2}{\partial z} + \bar{a}_{23}\left(\frac{\partial\varpi_1}{\partial z} - \frac{\partial\varpi_3}{\partial x}\right) + \bar{a}_{33}\frac{\partial\varpi_2}{\partial x}\right\} \\ \ddot{\varpi}_3 &= -\frac{\partial}{\partial x}\left\{-\bar{a}_{12}\frac{\partial\varpi_2}{\partial z} + \bar{a}_{22}\left(\frac{\partial\varpi_1}{\partial z} - \frac{\partial\varpi_3}{\partial x}\right) + \bar{a}_{23}\frac{\partial\varpi_2}{\partial x}\right\} \end{aligned}\right\} \quad \text{.........(41),}$$

whence, writing $\bar{\omega}/\bar{n} = \bar{h}$, $\bar{l}/\bar{n} = \bar{e}$, we obtain

$$(\bar{a}_{22}\bar{e}^2 + \bar{a}_{22} - \bar{h}^2)(\bar{a}_{33}\bar{e}^2 - 2\bar{a}_{31}\bar{e} + \bar{a}_{11} - \bar{h}^2) = (\bar{a}_{23}\bar{e} - \bar{a}_{12})^2(\bar{e}^2 + 1) \quad \text{...(42),}$$

$$\bar{k} = \frac{\bar{h}^2 - \bar{a}_{22} - \bar{a}_{22}\bar{e}^2}{(\bar{a}_{23}\bar{e} - \bar{a}_{12})\sqrt{\bar{e}^2 + 1}} \quad \text{..........................(43);}$$

similar equations with b written for a applying to the second medium.

Now the incident stream being given, the value of $\bar{h}$ is known and equation (42) determines the complex directions of the normals of the reflected and refracted waves and (43) determines the azimuths of the light-vectors. Care must be taken to select the values of $\bar{e}$ that correspond to streams leaving the interface with amplitudes that decrease as the distance therefrom increases.

Introducing now the surface conditions we obtain as in Chapter XIII,

$$\left.\begin{aligned} \Sigma\bar{l}\cos\bar{\phi}\bar{A} &= \Sigma\bar{l}_1\cos\bar{\phi}_1\bar{B} \\ \Sigma\bar{n}\cos\bar{\phi}\bar{A} &= \Sigma\bar{n}_1\cos\bar{\phi}_1\bar{B} \\ \Sigma\bar{k}\cos\bar{\phi}\bar{A} &= \Sigma\bar{k}_1\cos\bar{\phi}_1\bar{B} \\ \Sigma\kappa(-\bar{a}_{31}\bar{k}\bar{n} + \bar{a}_{23} + \bar{a}_{33}\bar{k}\bar{l})&\cos\bar{\phi}\bar{A} \\ &= \Sigma\bar{\kappa}_1(-\bar{b}_{31}\bar{k}_1\bar{n}_1 + \bar{b}_{23} + \bar{b}_{33}\bar{k}_1\bar{l}_1)\cos\bar{\phi}_1\bar{B} \end{aligned}\right\} \quad \text{.....................(44),}$$

the suffix ($_1$) and the letter B referring to the second medium.

If the first medium be isotropic, we have, denoting by $\bar{F}$, $\bar{G}$ and $\bar{F}'$, $\bar{G}'$ the amplitudes of the components of the light-vectors of the incident and reflected streams perpendicular and parallel respectively to the plane of incidence

$$\left.\begin{aligned} (\bar{G}-\bar{G}')\cos\bar{i} &= \bar{l}_1\cos\bar{\phi}_1\bar{B}_1+\bar{l}_2\cos\bar{\phi}_2\bar{B}_2\\ (\bar{G}+\bar{G}')\sin\bar{i} &= \bar{n}_1\cos\bar{\phi}_1\bar{B}_1+\bar{n}_2\cos\bar{\phi}_2\bar{B}_2\\ \bar{F}+\bar{F}' &= \sin\bar{\phi}_1\bar{B}_1+\sin\bar{\phi}_2\bar{B}_2\\ \bar{\kappa}\bar{\omega}^2(\bar{F}-\bar{F}')\cos\bar{i} &= \bar{\kappa}_1\{(\bar{b}_{33}\bar{l}_1-\bar{b}_{31}\bar{n}_1)\sin\bar{\phi}_1+\bar{b}_{23}\cos\bar{\phi}_1\}\bar{B}_1\\ &\quad+\bar{\kappa}_2\{(\bar{b}_{33}\bar{l}_2-\bar{b}_{31}\bar{n}_2)\sin\bar{\phi}_2+\bar{b}_{23}\cos\bar{\phi}_2\}\bar{B}_2 \end{aligned}\right\}\quad\ldots(45),$$

where $\bar{i}$ is the complex angle of incidence.

220. Without proceeding to the further developments of these equations, we will now take the more interesting case of metallic reflection, in which both media are isotropic. Then $\bar{r}$ being the complex angle of refraction, we have

$$\left.\begin{aligned} (\bar{G}-\bar{G}')\cos\bar{i} &= \bar{G}_1\cos\bar{r}\\ (\bar{G}+\bar{G}')\sin\bar{i} &= \bar{G}_1\sin\bar{r}\\ \bar{F}+\bar{F}' &= \bar{F}_1\\ (\bar{F}-\bar{F}')\sin\bar{i}\cos\bar{i} &= \bar{F}_1\sin\bar{r}\cos\bar{r} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(46),$$

whence

$$\left.\begin{aligned} \bar{G}' &= -\frac{\sin(\bar{i}-\bar{r})}{\sin(\bar{i}+\bar{r})}\bar{G},\quad \bar{G}_1=\frac{\sin 2\bar{i}}{\sin(\bar{i}+\bar{r})}\bar{G}\\ \bar{F}' &= \frac{\tan(\bar{i}-\bar{r})}{\tan(\bar{i}+\bar{r})}\bar{F},\quad \bar{G}_1=\frac{\sin 2\bar{i}}{\sin(\bar{i}+\bar{r})\cos(\bar{i}-\bar{r})}\bar{G} \end{aligned}\right\}\quad\ldots\ldots(47).$$

When the incident light is plane polarised and the first medium is transparent $\bar{F}$, $\bar{G}$ and $\bar{i}$ are real. Taking this case and writing the complex refractive index

$$\bar{\mu}=\mu(1-\nu\iota)=\theta e^{-\epsilon\iota}$$

and

$$\bar{\mu}^2\cos^2\bar{r}=\theta^2e^{-2\epsilon\iota}-\sin^2 i=U^2e^{-2u\iota},$$

we have

$$\mu=\theta\cos\epsilon,\quad \nu=\tan\epsilon\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48),$$

$$U^2\cos 2u=\theta^2\cos 2\epsilon-\sin^2 i,\quad U^2\sin 2u=\theta^2\sin 2\epsilon\ \ldots\ldots\ldots(49),$$

which give

$$\cot(2u-\epsilon)=\cot\epsilon\cos\left(2\tan^{-1}\frac{\sin i}{\theta}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(50),$$

$$\tan(u-2\epsilon)=\tan u\cos\left(2\tan^{-1}\frac{U}{\sin i}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(51).$$

Whence

$$\left.\begin{aligned} \frac{\bar{F}'}{\bar{F}} &= \frac{\theta^2e^{-2\epsilon\iota}\cos i-Ue^{-u\iota}}{\theta^2e^{-2\epsilon\iota}\cos i+Ue^{-u\iota}}=\tan\phi_1e^{\Delta_1\iota}\\ \text{where}\quad \cos 2\phi_1 &= \cos(u-2\epsilon)\sin\left(2\tan^{-1}\frac{U}{\theta^2\cos i}\right)\\ \tan\Delta_1 &= \sin(u-2\epsilon)\tan\left(2\tan^{-1}\frac{U}{\theta^2\cos i}\right) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(52);$$

and
$$\left.\begin{aligned} \frac{\bar{G}}{G} &= \frac{\cos i - Ue^{-u\iota}}{\cos i + Ue^{-u\iota}} = \tan \phi_2 e^{\Delta_2 \iota}, \\ \text{where} \quad \cos 2\phi_2 &= \cos u \sin \left(2 \tan^{-1} \frac{U}{\cos i}\right) \\ \tan \Delta_2 &= \sin u \tan \left(2 \tan^{-1} \frac{U}{\cos i}\right) \end{aligned}\right\} \quad \text{......(53).}$$

Also if the incident light be polarised at an azimuth of 45° to the plane of incidence, we have

$$\frac{\bar{F}'}{\bar{G}'} = \frac{\sin^2 i - U \cos i e^{-u\iota}}{\sin^2 i + U \cos i e^{-u\iota}} = \tan \phi \, e^{\Delta \iota} \quad \text{......(54)},$$

where Δ is the difference of phase between the components of the light-vector of the reflected stream perpendicular and parallel to the plane of incidence and $\tan \phi$ is the ratio of the amplitudes, and

$$\left.\begin{aligned} \cos 2\phi &= \cos u \sin \left(2 \tan^{-1} \frac{U}{\sin i \tan i}\right) \\ \tan \Delta &= \sin u \tan \left(2 \tan^{-1} \frac{U}{\sin i \tan i}\right) \end{aligned}\right\} \quad \text{......(55).}$$

Now as i increases from 0° to 90°, Δ decreases from π to 0: hence at a certain angle of incidence I, called the principal angle of incidence, $\Delta = \pi/2$ and if the corresponding value of ϕ be β, we have that when $i = I$

$$U = \tan I \sin I, \qquad u = 2\beta.$$

The angles I and β having been determined, the values of θ and ϵ may be obtained from (51) and (49), we have in fact

$$\left.\begin{aligned} \tan 2(\beta - \epsilon) &= \tan 2\beta \cos 2I \\ \theta^2 &= \sin^2 I \tan^2 I \sin 4\beta \operatorname{cosec} 2\epsilon \end{aligned}\right\} \quad \text{......(56)},$$

and these being known, equations (49), (50) give the values of u and U for any angle of incidence.

221. The simplest method of investigating the phenomenon of metallic reflection is to directly measure Δ and $\tan \phi$ by means of Babinet's compensator and an analysing prism, as described in § 205*. Other methods have however been employed. Thus Jamin† compared the intensity of the light reflected from a metal with that reflected at the same angle from a glass surface, when the light was polarised in planes parallel and perpendicular to the plane of incidence, and determined the relative difference of phase between these streams introduced by the metallic reflection by observations of the angles of incidence, for which the reflected light was plane polarised after 2, 4, 6... reflections at the surfaces of two mirrors of the metal

* Quincke, *Pogg. Ann.* CXXVIII. 541 (1866). Hennig, *Gött. Nachr.* (1887) 365.

† *Ann. de Ch. et de Phys.* (3) XIX. 296 (1847).

placed parallel to one another, the primitive light being polarised in a plane inclined to the plane of incidence.

The change of phase that metallic reflection introduces, when the light is polarised in a plane either parallel or perpendicular to the plane of incidence, may be compared with that caused by reflection at the surface of a transparent substance by aid of the phenomenon of interference, such as that produced by Fresnel's mirrors*, or by thin isotropic plates†. Of these methods the most satisfactory is that employed by Wernicke‡. A stream of white light falls upon a thin film of some transparent substance, the hinder surface of which is in part coated with the metal to be investigated, and the reflected light is analysed with a spectroscope. A channelled spectrum is thus obtained, and the relative difference of phase due to the metallic reflection is determined from the shift of the bands in the part of the spectrum given by the light reflected from the coated region of the film. This method has been improved by Drude§, who employed a wedge-shaped film and monochromatic light.

222. It has been pointed out that the optical constants of a metal θ and ϵ or μ and ν may be obtained from measures of the principal angle of incidence I and the principal azimuth β, but Drude‖ has shown that the most accurate plan is to deduce these constants from a series of measures of ϕ and Δ for angles of incidence near the principal incidence.

From equations (55) we have

$$\tan u = \sin\Delta \tan 2\phi, \qquad \cos\left(2\tan^{-1}\frac{U}{\sin i \tan i}\right) = \cos\Delta \sin 2\phi,$$

whence u and U are determined from the various observations and ϵ and θ are then deduced from (51) and (49).

Now in most cases $\theta = \mu\sqrt{1+\nu^2}$ is sufficiently large for powers of $1/\theta$ above the second to be neglected, and if this be so

$$U \risingdotseq \theta\left(1 - \frac{\cos 2\epsilon \sin^2 i}{2\theta^2}\right),$$

and since

$$U^2 \cos 2u = \theta^2 \cos 2\epsilon - \sin^2 i,$$

we have

$$U\cos u \risingdotseq \theta\cos\epsilon\left(1 - \frac{\sin^2 i}{2\theta^2}\right), \qquad U\sin u \risingdotseq \theta\sin\epsilon\left(1 + \frac{\sin^2 i}{2\theta^2}\right) \quad \ldots(57).$$

* Senarmont, *Ann. de Ch. et de Phys.* (2) LXXIII. 361 (1840). Quincke, *Pogg. Ann.* CXLII. 219 (1871).

† Quincke, *ibid.* CXLII. 380 (1871). Wiener, *Wied. Ann.* XXXI. 629 (1887).

‡ *Berl. Monatsber.* (1875) 673.

§ *Wied. Ann.* L. 595 (1893).

‖ *ibid.* XXXIX. 504 (1890).

To a first approximation then U is a constant $= \theta$, and denoting its mean value deduced from the observations by S, we may use this value for θ in the small terms on the right-hand sides of equations (57); whence the optical constants may be calculated from the formulæ

$$\mu = \theta \cos \epsilon = \frac{\Sigma U \cos u}{N}\left(1 + \frac{1}{2S^2}\frac{\Sigma \sin^2 i}{N}\right),$$

$$\mu\nu = \theta \sin \epsilon = \frac{\Sigma U \sin u}{N}\left(1 - \frac{1}{2S^2}\frac{\Sigma \sin^2 i}{N}\right),$$

where N is the number of observations.

In this manner Drude has determined the optical constants of a number of metals, some of his results being given in the following table.

	Sodium Light		Red Light	
	μ	ν	μ	ν
Aluminium	1·44	3·63	1·62	3·36
Antimony	3·04	1·63	3·17	1·56
Bismuth	1·90	1·93	2·07	1·90
Cadmium	1·13	4·43	1·31	4·05
Copper	0·641	4·09	0·580	5·24
Gold	0·366	7·71	0·306	10·2
Iron	2·36	1·36		
Steel	2·41	1·38	2·62	1·32
Lead	2·01	1·73	1·97	1·74
Magnesium	0·37	11·8	0·40	11·5
Mercury	1·73	2·87	1·87	2·78
Nickel	1·79	1·86	1·89	1·88
Platinum	2·06	2·06	2·16	2·06
Silver	0·181	20·3	0·203	19·5
Tin	1·48	3·55	1·66	3·30
Zinc	2·12	2·60	2·36	2·34

The first thing that we notice from these values is that copper, gold, magnesium and silver have refractive indices less than unity, so that the propagational speed of light in these metals is less than it is in free ether. This remarkable result has been completely confirmed by experiments with metallic prisms of small refracting angle*, which Kundt first succeeded in making, in most cases by electrolytic deposition on platinised glass. The

* Kundt, *Wied. Ann.* XXXIV. 469 (1888); XXXVI. 824 (1889): *Phil. Mag.* (5) XXVI. 1 (1888). Du Bois and Rubens, *Wied. Ann.* XLI. 507 (1890). Shea, *ibid.* XLVII. 177 (1892).

formula required for calculating the refractive index from the observations is easily deduced from the geometrical laws of refraction at the surface of absorbing media*.

Another interesting fact is that only in the cases of copper, lead and gold is the dispersion normal: in all other cases the index for red light is greater than that for sodium light.

The value of $\mu\nu$, on which the absorption depends, varies in the case of sodium light from 2·62 for copper to 5·48 for zinc. Copper is thus the most transparent of the metals, but even in this case the reduction of intensity in traversing unit thickness, which is given by exp $\{-4\pi\mu\nu/\lambda\}$, λ being the wave-length in free ether, is considerable.

The larger the value of ν, the greater is the intensity of the light reflected at the surface of an absorbing medium. Hence when a mixed stream is incident, the constituent that is most absorbed in the medium will have the greater importance in the reflected pencil, and this predominance will be increased at each subsequent reflection. In fact by repeated reflections it has been found possible to separate out waves of very large wave-length. Thus by five reflections at the surface of Sylvite, waves of length 0·061 mm. have been isolated†.

* Voigt, *Wied. Ann.* xxiv. 144 (1885). Drude, *ibid.* xlii. 666 (1891). H. A. Lorentz, *ibid.* xlvi. 244 (1892).

† Rubens and Nichols, *Wied. Ann.* lx. 418 (1897). Rubens and Aschkinass, *ibid.* lxv. 241 (1898).

CHAPTER XVII.

DISPERSION.

223. We have hitherto merely considered the propagation of trains of waves of definite period without taking into account the fact that in material media light travels in a given direction with a speed that depends upon the frequency of the waves. The equations obtained may of course be made to include the facts of dispersion by regarding the parameters of the medium as functions of the period, but this procedure leaves unexplained the unequal rate at which waves of different periods travel and gives no information respecting the law that connects the speed with the frequency of the luminous vibrations. Moreover it affords no explanation of the complex values of the parameters, that we have been led to adopt, in order to explain the phenomena presented by absorbing media.

Observations of Jupiter's satellites show that in free ether the velocity of light is independent of the frequency, for were this not the case, the satellites would appear to be coloured at the commencement and at the end of an eclipse. It thus becomes natural to attribute dispersion to the influence of the molecules of the material substance, and the fact that these occasion the phenomenon may be ascribed to either of two causes: it may be that the coarse-grainedness of the substance introduces "a geometrical dimension in the ponderable matter which is comparable with the wave-length," or it may be that there is "a definite interval of time somehow ingrained in the constitution of the ponderable matter which is comparable with period*."

Now so far as ordinary dispersion is concerned, the first of these hypotheses may be made to give a fairly satisfactory account of the facts, but other allied phenomena and especially that of abnormal or anomalous dispersion cannot be explained in this manner and it is therefore necessary to regard the second of the above assumptions as giving the actual cause of the influence of the molecules on the propagational speed of light.

* Lord Kelvin, *Baltimore Lectures*, p. 8, Camb. (1904).

224. The coincidence of many of the Fraunhofer lines in the solar spectrum with the bright lines of the spectra given with the same apparatus by the vapours of certain elements, has been shown by Kirchhoff to be an instance of a general law that may be enunciated as follows: "if a body emits in a given direction a beam propagating certain vibrations, defined by their period and their state of polarisation, it is capable of absorbing a beam propagating the same vibrations in the opposite direction*."

This important result has been explained by Stokes† by the aid of the well-known dynamical theorem that, if a system, capable of executing vibrations, be acted on by a periodic force, the amplitudes of the forced vibrations will be very large when the period and direction of the force are identical or nearly so with the period and direction of the free vibrations of the system. It follows then that if a stream, incident on a body, contain constituents that have periods and polarisations in agreement with those in the stream that the body emits, these components will excite within the molecules of the substance vibrations that have a considerable amplitude, and inasmuch as there can be no creation of energy, they must themselves be gradually extinguished during the passage of the stream through the medium.

Closely allied with intense selective absorption we have the phenomenon of anomalous dispersion. In the case of most transparent bodies the refrangibility of a stream of light increases with the frequency of the vibrations, so that when a stream of white light traverses a prism of the substance, the red rays are the least deviated and the deviation increases continuously as we pass from red to violet. With prisms formed of certain media however the ordinary distribution of colours in the spectrum is largely departed from, the least deviated being in some cases the green or the blue. This was first observed by Fox Talbot‡ about the year 1840, but we owe the first published account of the phenomenon to Leroux§, who discovered in 1862 that vapour of iodine, which absorbs all but the red and violet rays, refracts the latter less powerfully than the former.

Later experimental investigations by Christiansen‖, Kundt¶ and others have shown that there is an intimate connection between anomalous dispersion and the absorptive power of a substance, and have established the law that the propagational speed in the medium is abnormally decreased for waves of less frequency and abnormally increased for those of greater frequency than those that are absorbed by the body.

* Cotton, *Astrophys. J.* IX. 237 (1899).

† See *Phil. Mag.* (4) XX. 20 (1860). Lord Kelvin, *Baltimore Lectures*, p. 101, Camb. (1904).

‡ *Proc. R. S. Edin.* VII. 408 (1870).

§ *C. R.* LV. 126 (1862): *Phil. Mag.* (4) XXIV. 245 (1862).

‖ *Pogg. Ann.* CXLI. 479; CXLIII. 250 (1871): *Phil. Mag.* (4) XLI. 244 (1871).

¶ *Pogg. Ann.* CXLII. 163; CXLIII. 149, 259 (1871); CXLIV. 128; CXLV. 67, 164 (1872).

This relation between dispersion and absorption leads to the conclusion that both must be attributed to the same cause,—vibrations within the molecules of the material medium excited by the vibrations in the stream of light incident upon it.

225. It follows then that, in order to obtain the law of dispersion we must introduce into our equations additional vectors dependent upon the action of the molecules of the body and connected with the polarisation-vector for the pure ether by relations that express that the vibrations of the latter occasion forced vibrations of these new vectors*.

There are several ways, all *à priori* equally possible, in which these additional vectors may be introduced into the equations, but perhaps the most simple and natural is to retain the form of the equations that have been deduced for the case of the free ether, regarding therein the polarisation-vector for the material medium as the resultant of the vector d for the pure ether and of the vectors d_h, expressive of the action of the molecules of the substance.

We then have $$\dot{D} = -\operatorname{curl} \varpi, \quad \dot{\varpi} = \operatorname{curl} e \quad \ldots\ldots(1),$$

where $$D = d + \Sigma d_h \quad \ldots\ldots(2),$$

and the components of the vector e are given by

$$(e_1, e_2, e_3) = \tfrac{1}{2}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)(\Omega^2 d^2) \quad \ldots\ldots(3),$$

u, v, w being the components of d.

As regards the equations connecting the vectors d_h with the vector d, these, if we take the coordinate axes in the direction of the axes of symmetry of the medium, will have the form,

$$\left.\begin{aligned} a_h u_h + a_h' \dot{u}_h + a_h'' \ddot{u}_h &= u \\ b_h v_h + b_h' \dot{v}_h + b_h'' \ddot{v}_h &= v \\ c_h w_h + c_h' \dot{w}_h + c_h'' \ddot{w}_h &= w \end{aligned}\right\} \quad \ldots\ldots(4),$$

u_h, v_h, w_h being the components of d_h.

In the case of vibrations of frequency n, we then have

$$u_h (a_h + \iota 2\pi n a_h' - 4\pi^2 n^2 a_h'') = u \quad \ldots\ldots(5)$$

and two similar equations: whence U, V, W being the components of D

$$U = u + \Sigma u_h = u \{1 + \Sigma (a_h + \iota 2\pi n a_h' - 4\pi^2 n^2 a_h'')^{-1}\} \quad \ldots\ldots(6),$$

and two similar equations, and introducing these values of u, v, w into equations (1) we obtain

$$\dot{D} = -\operatorname{curl} \varpi, \quad \dot{\varpi} = \operatorname{curl} E \quad \ldots\ldots(7),$$

* Voigt, *Komp. der theor. Physik*, II. 747.

the components of E being

$$(E_1, E_2, E_3) = \tfrac{1}{2}\left(\frac{\partial}{\partial U}, \frac{\partial}{\partial V}, \frac{\partial}{\partial W}\right)(a^2U^2 + b^2V^2 + c^2W^2) \quad \text{......(8)},$$

where

$$\left.\begin{aligned} a^2 &= \Omega^2\left\{1 + \Sigma\frac{1}{a_h + \iota 2\pi n a_h' - 4\pi^2 n^2 a_h''}\right\}^{-1} \\ b^2 &= \Omega^2\left\{1 + \Sigma\frac{1}{b_h + \iota 2\pi n b_h' - 4\pi^2 n^2 b_h''}\right\}^{-1} \\ c^2 &= \Omega^2\left\{1 + \Sigma\frac{1}{c_h + \iota 2\pi n c_h' - 4\pi^2 n^2 c_h''}\right\}^{-1} \end{aligned}\right\} \quad \text{............(9)}.$$

These equations are those that we have adopted to represent the case of absorbing crystals, in which the axes of absorption coincide with those of polarisation.

226. Taking the case of an isotropic medium, we have $a = b = c$ and the complex propagational speed is given by

$$\begin{aligned} \varpi^2 &= \Omega^2\left(1 + \Sigma\frac{1}{a_h + \iota 2\pi n a_h' - 4\pi^2 n^2 a_h''}\right)^{-1} \\ &= \Omega^2\left(1 + \Sigma\frac{1/a_h}{1 + \iota\alpha_h n - n^2/n_h^2}\right)^{-1} \quad \text{......................(10)}, \end{aligned}$$

where α_h is written for $2\pi a_h'/a_h$ and n_h is the frequency of the free undamped vibrations of the vector d_h. Hence if μ be the refractive index and ν the coefficient of absorption

$$\mu^2(1 - \iota\nu)^2 = 1 + \Sigma\frac{1/a_h}{1 + \iota\alpha_h n - n^2/n_h^2} \quad \text{..................(11)}.$$

Let us first suppose that α_h is very small: then provided that the frequencies n_h are well outside the limits of the visible spectrum, we may neglect the term containing α_h, in which case there is no absorption throughout that part of the spectrum and

$$\begin{aligned} \mu^2 &= 1 + \Sigma\frac{1/a_h}{1 - n^2/n_h^2} \\ &= 1 + \Sigma\frac{1/a_v}{1 - n^2/n_v^2} - \Sigma\frac{n_r^2}{n^2}\frac{1/a_r}{1 - n_r^2/n^2} \\ &= 1 + \Sigma a_v^{-1} + n^2\Sigma\frac{a_v^{-1}}{n_v^2} + n^4\Sigma\frac{a_v^{-1}}{n_v^4} + \ldots \\ &\quad - \frac{1}{n^2}\Sigma a_r^{-1}n_r^2 - \frac{1}{n^4}\Sigma a_r^{-1}n_r^4 - \ldots \quad \text{......................(12)}, \end{aligned}$$

where the subscripts v and r refer to the vectors d_h that have their frequencies beyond the violet and the red end of the spectrum respectively,

Retaining only the first term of the second series, the formula connecting the refractive index with the period τ of the light is of the form

$$\mu^2 = -A'\tau^2 + A + \frac{B}{\tau^2} + \frac{C}{\tau^4} + \dots \qquad (13),$$

which is found to represent with considerable accuracy the law of dispersion of transparent bodies.

227. Passing now to the case that we have reserved, we see that there will be absorption when the frequency is near one of the critical frequencies n_h, even when $a_h = 0$; for as n increases through this value μ^2 passes from ∞ to $-\infty$ and when μ^2 is negative the wave ceases to be transmitted and absorption occurs. Also there is an abnormal increase of the refractive index for frequencies less than n_h and an abnormal decrease for frequencies greater than n_h, so that the dispersion is anomalous.

We will however consider the more general case in which α_h though small is not actually zero. Writing

$$n_1 = \iota\frac{\alpha_h n_h^2}{2} + \sqrt{n_h^2 - \alpha_h^2 n_h^4/4},$$

$$n_2 = \iota\frac{\alpha_h n_h^2}{2} - \sqrt{n_h^2 - \alpha_h^2 n_h^4/4},$$

so that n_1, n_2 are the roots of the equation

$$n^2 - \iota\alpha_h n_h^2 n - n_h^2 = 0,$$

we have $$\frac{a_h^{-1}}{1 + \iota\alpha_h n - n^2/n_h^2} = \frac{n_h^2 a_h^{-1}}{(n_1 - n_2)(n_1 - n)} - \frac{n_h^2 a_h^{-1}}{(n_1 - n_2)(n_2 - n)}.$$

But when the frequency n is very nearly equal to n_h, the absolute value of $n_1 - n$ will be very small compared with that of $n_2 - n$ and we may write approximately

$$\frac{a_h^{-1}}{1 + \iota\alpha_h n - n^2/n_h^2} = \frac{n_h^2 a_h^{-1}}{(n_1 - n_2)(n_1 - n)} = -\frac{n_h a_h^{-1}}{2\left(n - n_h - \frac{1}{2}\alpha_h n_h^2 \iota\right)},$$

since $$n_1 - n_2 \doteq 2n_h, \quad n_1 \doteq \tfrac{1}{2}\alpha_h n_h^2 \iota + n_h.$$

This expression attains its maximum value for $n = n_h$ and is relatively small when the frequency differs in a marked degree from this critical value, and we may, when the periods of the free undamped vibrations of the vectors d_h are not too close together, retain only the one term of the summation in (10) and write

$$\varpi^2 = \Omega^2\left\{1 - \frac{n_h a_h^{-1}}{2\left(n - n_h - \frac{1}{2}\alpha_h n_h^2 \iota\right)}\right\}^{-1} \qquad (14).$$

In order to determine the form of the curve of dispersion, let us write

$$\frac{a_h^{-1}}{\alpha_h n_h} = A, \quad \frac{n - n_h}{n_h^2 \alpha_h/2} = N,$$

then
$$\bar{\mu}^2 = \frac{\Omega^2}{\bar{\omega}^2} = 1 - \frac{A}{N - \iota},$$
where N denotes on a scale of frequencies considerably magnified the distance from the point $n = n_h$.

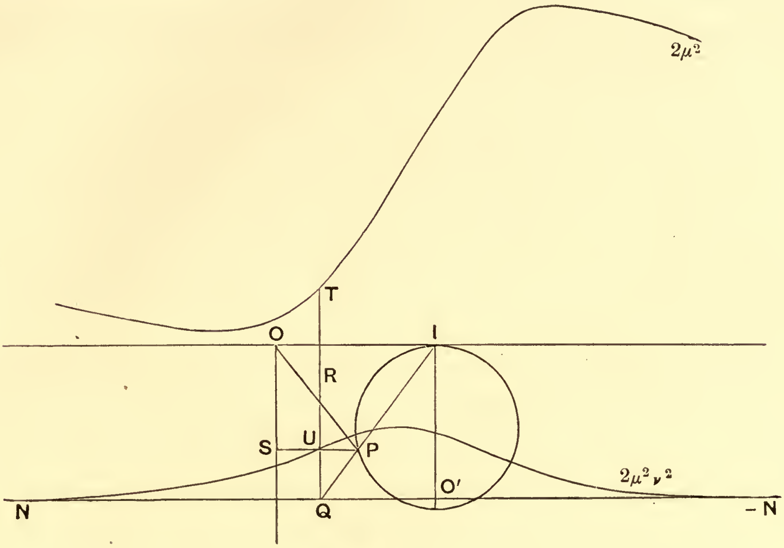


Fig. 51.

With centre $1 - \iota A/2$ and radius equal to $A/2$ describe a circle in the plane representing the complex variable $\xi + \iota\eta$; then the line joining the points 1 and $(1 - N) - \iota$ will cut this circle in the point P that corresponds to the complex number $\bar{\mu}^2$. If then $\mu = x - y\iota$, the absolute value of OP is $x^2 + y^2$ and the abscissa SP of the point P is $x^2 - y^2$, and the values of $2\mu^2$ and $2\mu^2\nu^2$ corresponding to the given value of N are respectively the sum and difference of the tensors of the vectors OP and SP. If then on the ordinate through Q we take $QR = OP$ and $RT = RU = SP$, the lengths QT and QU will represent the values of $2\mu^2$ and $2\mu^2\nu^2$ corresponding to the value of $O'Q$ of N. In this manner the curves representing the values of these quantities are constructed, and we see that the dispersion is anomalous, as the value of μ^2 increases largely on approaching the region of absorption from the side of frequencies less than the critical value and is abnormally decreased on the other side of the absorption-band*.

228. Anomalous dispersion is most usually investigated by Newton's arrangement of crossed prisms, and this was adopted by Kundt in his experiments. A fine thread is stretched across the slit of a spectroscope and the light from the collimator before entering the telescope is made to pass through two prisms with their refracting edges at right angles, that of the

* Kayser, *Handb. der Spectroscopie*, II. p. 652.

first being parallel to the slit. When the dispersion of both prisms is normal, the oblique spectrum thus obtained will be divided into two parts by a dark line forming a continuous curve in the direction of the length of the spectrum: if however the dispersion of the second prism be abnormal, this line will be interrupted by the absorption-bands and on crossing these there occurs a displacement of the line, that indicates an abrupt change of the refractive index of the substance of the prism.

The disadvantage of this method is the great loss of light in traversing the absorbing material, which necessitates the employment of prisms of very small angle, and with liquid prisms capillarity may affect the concentration of the liquid at different distances from the edge. Moreover it is by no means certain that with prisms of very absorbing materials the refraction may not be modified by changes of phase, that vary with the wave-length and are dependent upon the thickness that is traversed.

A second method* is to employ an interferential apparatus, placing a thin film of the substance in the path of one of the interfering streams. If white light be allowed to pass and be subsequently analysed by a spectroscope with its slit perpendicular to the direction of the fringes, the spectrum in the case of normal dispersion will be traversed by dark bands spreading out like a fan from the violet to the red: when however the dispersion is abnormal, these dark lines will be broken by the absorption-bands into portions of distinct curves, and if the absorption be not too vigorous, the separate parts will be joined by rapidly curved pieces passing through the region of absorption. Here again we are met by the difficulty that, even though the incidence be normal, there may be a change of phase dependent upon the wave-length on entering and traversing the film.

A third plan, free from the foregoing objections, is to employ the method of total reflection†. A right-angled glass prism is placed on the plate of a spectroscope and its hypothenuse face is brought into optical contact with the substance to be examined. The slit of the spectroscope is placed at right-angles to the edge of the prism and the light internally reflected within the prism passes through a direct-vision combination of prisms with their edges parallel to the slit and then enters the telescope. When the dispersion of the substance is normal and the pencil from the centre of the slit is incident upon it at the critical angle for rays of mean period, the spectrum is divided into a brighter and a darker region by a line corresponding to the limit of total reflection, that traverses it obliquely from the red to the violet (fig. 52); but in the case of anomalous dispersion this line will consist of distinct branches. Thus in the case of a solution of fuchsine and a flint glass prism,

* Mach and Osnobischin, *Anzeiger Wien Akad.* XII. 51, 82 (1875); *J. de Phys.* V. 34 (1876) *Carl. Rep.* XI. 178 (1875).

† Mach and Arbes, *Wied. Ann.* XXVII. 436 (1886).

the appearance is as represented in fig. 53, the dark band at E corresponding to rays for which the refractive indices of the fuchsine and the prism have nearly the same value. In many cases the high reflecting power of the

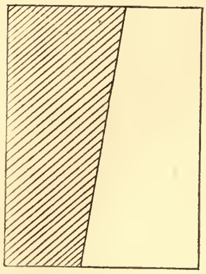


Fig. 52.

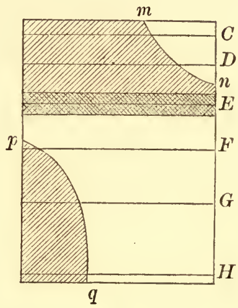


Fig. 53.

substance exhibiting anomalous dispersion masks the division between the partial and the total reflection, but in these cases the dividing line may be rendered sharper by having recourse to multiple reflections.

CHAPTER XVIII.

STRUCTURALLY ACTIVE MEDIA.

229. A PLATE of an uniaxal crystal cut perpendicularly to the optic axis does not as a general rule modify in any way the polarisation of a stream of plane polarised light that passes through it along the normal to its faces, and the emergent light, when analysed by a double image prism, is divided into two colourless beams, one of which can be made to vanish by placing the principal plane of the prism either parallel or perpendicular to the plane of polarisation of the original stream. With a plate of quartz this is no longer the case, and for each position of the prism there are two emergent pencils, exhibiting complementary colours that change in a marked manner as the analysing prism is turned.

This phenomenon was discovered in 1811 by Arago*, who pointed out that it could be explained by the supposition that each monochromatic constituent of the stream remains plane polarised after its passage through the quartz, but that its plane of polarisation has turned through an angle dependent upon the wave-length.

The subject of rotary polarisation was next investigated with remarkable skill and diligence by Biot†, who gave as the results of his experiments the following general laws of the phenomenon:

(1) The rotation of the plane of polarisation produced by a plate of quartz cut perpendicularly to the optic axis is proportional to the thickness of the plate: it is the same for plates cut from different crystals and does not change when the plate is reversed.

(2) Among crystals of quartz there are some that rotate the plane of polarisation from the left to the right of an observer receiving the light, while in the case of others the rotation is in the opposite direction: the former are called right-handed, the latter left-handed crystals. Plates of the two kinds of crystals that have the same thickness produce equal rotations in opposite directions.

(3) The rotation of the plane of polarisation increases with the frequency of the light and varies very nearly as the inverse square of the wave-length.

* *Mém. de la prem. classe de l'Inst.* XII. (1) 93 (1811); *Œuvres complètes*, x. 36.

† *Mém. de la prem. classe de l'Inst.* XIII. (1) 218 (1813); *Mém. de l'Acad. des Sc.* II. 41 (1818).

230. Certain uniaxal crystals, such as cinnabar and the hyposulphates of potassium, calcium, strontium and lead, possess the same property as quartz in the direction of their axes, and a few cubic crystals, such as the chlorate and the bromate of sodium, as well as some liquids and even vapours, impress a rotation on the plane of polarisation of the light that traverses them, whatever may be the direction of the stream.

Such substances are termed active media, and in all cases the rotation of the plane of polarisation is proportional to the distance travelled in the medium. The rotation produced by unit length of the medium is called the rotary power of the substance.

From the fact that active liquids do not lose the rotary property, except in degree, by dilution with inactive substances and retain it even in a state of vapour, it was inferred by Biot that the property is inherent in the ultimate molecules, whence the quotient of the rotary power by the density of the active medium is sometimes called the molecular rotary power. If then R be the rotation produced by a column of length l of a solution of the active substance, the molecular rotary power is

$$[\rho] = R(p+P)/(lp\delta),$$

where p, P are the masses of the active substance and of the inactive solvent and δ is the density of the solution. When there is no chemical action between the substance and the solvent, the molecular rotary power is in most cases constant.

For a mixture of active substances that have no chemical action on one another, the total rotation is the algebraic sum of the separate rotations, so that with a solution of density δ containing a mass P of inactive solvent and masses $p_1, p_2 \ldots$ of active substances with molecular rotary powers $[\rho_1], [\rho_2] \ldots$ the rotation produced by a column of length l is

$$R = \frac{l\delta}{P + p_1 + p_2 + \ldots}\{p_1[\rho_1] + p_2[\rho_2] + \ldots\}.$$

When however there is chemical action between the substances dissolved, the molecular rotary powers are in general altered.

231. Shortly after Arago's discovery, Fresnel* showed that rotary polarisation could be explained kinematically on the principles of the wave-theory by the supposition that a stream of plane polarised light on entering an active medium is divided into two oppositely circularly polarised streams of half the intensity that traverse the medium with unequal speeds.

* *Mém. de l'Acad. des Sc.* XX. 163 (1849), presented in 1818; *Ann. de Ch. et de Phys.* (2) XVII. 172 (1821); *Œuvres complètes*, I. 655.

The incident stream of plane polarised light characterised by the vector

$$\xi = a \cos \frac{2\pi}{\lambda} \omega t,$$

is equivalent to the two circularly polarised streams represented by the vectors

$$\xi_1 = \frac{a}{2} \cos \frac{2\pi}{\lambda} \omega t, \qquad \eta_1 = \frac{a}{2} \sin \frac{2\pi}{\lambda} \omega t,$$

and

$$\xi_2 = \frac{a}{2} \cos \frac{2\pi}{\lambda} \omega t, \qquad \eta_2 = -\frac{a}{2} \sin \frac{2\pi}{\lambda} \omega t,$$

of which the first is left- and the second is right-handed.

If these travel with the different speeds ω_1 and ω_2, their retardations (measured in length in air) will be on emergence

$$\delta_1 = \omega T/\omega_1, \qquad \delta_2 = \omega T/\omega_2,$$

where T is the distance traversed, and the polarisation-vectors of the emergent streams will be

$$\xi_1 = \frac{a}{2} \cos \frac{2\pi}{\lambda} (\omega t - \delta_1), \qquad \eta_1 = \frac{a}{2} \sin \frac{2\pi}{\lambda} (\omega t - \delta_1),$$

and

$$\xi_2 = \frac{a}{2} \cos \frac{2\pi}{\lambda} (\omega t - \delta_2), \qquad \eta_2 = -\frac{a}{2} \sin \frac{2\pi}{\lambda} (\omega t - \delta_2).$$

These are equivalent to a single stream of plane polarised light, for which the components of the polarisation-vector are

$$\xi = a \cos \frac{\pi}{\lambda} (\delta_2 - \delta_1) \cos \frac{2\pi}{\lambda} \left(\omega t - \frac{\delta_1 + \delta_2}{2} \right),$$

$$\eta = a \sin \frac{\pi}{\lambda} (\delta_2 - \delta_1) \cos \frac{2\pi}{\lambda} \left(\omega t - \frac{\delta_1 + \delta_2}{2} \right).$$

Thus the effect of the passage through the plate is to introduce a retardation of phase $\pi(\delta_1 + \delta_2)/\lambda$ and to rotate the plane of polarisation through an angle $R = \pi(\delta_2 - \delta_1)/\lambda$, which is to the right or left according as δ_1 is greater or less than δ_2, that is according as the right- or the left-handed circularly polarised stream travels at the greater rate in the medium.

232. Fresnel* argued that, if this explanation be correct, it must be possible to separate the two coincident circularly polarised streams by limiting the medium, in which they travel, by a face oblique to their direction of propagation; for since the streams have different velocities, they must be differently refracted on emergence.

The amount of their divergence will indeed be exceedingly small, for the difference of the refractive indices is $\mu_2 - \mu_1 = R\lambda/(\pi T)$, where R is the

* *Ann. de Ch. et de Phys.* (2) XXVIII. 147 (1825); *Œuvres complètes*, I. 731.

rotation produced by a length T of the medium, and in the case of quartz, a millimetre length of which gives a rotation of $21\frac{2}{3}°$ with sodium light, the difference of the indices is only about ·00007. Fresnel however succeeded in effecting the separation of the streams by a combination of left- and right-handed prisms of quartz arranged so as to double the deviation.

This experiment has been regarded as a confirmation of Fresnel's views on the cause of rotary polarisation, and in fact v. Fleischl* in 1885 employed a similar arrangement for showing the existence of circular polarisation in active liquids. A little consideration will however show that the result is merely a consequence of the equivalence of a plane polarised stream and two oppositely circularly polarised streams, and is independent of the state of affairs within the active medium, provided this be such as to produce a rotation of the plane of polarisation and a retardation of phase of the transmitted stream, both of which are proportional to the distance traversed†.

To make this clear, let us suppose that a prism of quartz has one of its faces perpendicular to the optic axis, and that a train of plane waves polarised in the normal section of the prism is incident normally on this face. Now the experimental fact of rotary polarisation with which we have to deal is, that on the face of emergence along any line parallel to the edge of the prism the plane of polarisation of the emergent light has been turned through an angle proportional to the distance of the line from the edge of the prism and the retardation of phase of the stream is proportional to the same quantity.

If then the polarisation-vector of the incident light be $\xi = a \cos 2\pi\omega t/\lambda$ the vector of the emergent stream along a line on the face of exit distant x from the edge of the prism will have for its components

$$\xi = a \cos\left(\frac{2\pi}{\lambda} kx \sin A\right) \cos\frac{2\pi}{\lambda}(\omega t - lx \sin A)$$

$$= \frac{a}{2}\cos\frac{2\pi}{\lambda}\{\omega t - (l-k)\, x \sin A\} + \frac{a}{2}\cos\frac{2\pi}{\lambda}\{\omega t - (l+k)\, x \sin A\},$$

$$\eta = a \sin\left(\frac{2\pi}{\lambda} kx \sin A\right) \cos\frac{2\pi}{\lambda}(\omega t - lx \sin A)$$

$$= \frac{a}{2}\sin\frac{2\pi}{\lambda}\{\omega t - (l-k)\, x \sin A\} - \frac{a}{2}\sin\frac{2\pi}{\lambda}\{\omega t - (l+k)\, x \sin A\},$$

where A is the angle of the prism and l, k are constants. That is, the emergent light is equivalent to two trains of oppositely circularly polarised plane waves that are inclined at angles $\sin^{-1}\{(l \mp k) \sin A\}$ to the face of emergence, and these are precisely the directions of the emergent streams on Fresnel's theory.

* *Wied. Ann.* XXIV. 127 (1885).
† Gouy, *C. R.* XC. 992 (1880).

233. Fresnel suggested*, as a second method of verifying his conclusions, an experiment depending upon the interference of the circularly polarised streams emergent from an active substance.

A stream of white light from a slit passes through a polarising prism and a plate of quartz cut perpendicularly to the optic axis and then falls upon Fresnel's mirrors or other interferential apparatus giving two images, real or virtual, of the slit. These images act as proximate sources of light, from each of which according to Fresnel's view emanate two correlated streams of light, that are circularly polarised in opposite directions, the right-hand streams being relatively accelerated or retarded in phase according as the plate of quartz is right- or left-handed.

Now the two pair of similarly polarised streams will give two coincident systems of interference fringes situated at the centre of the field, but the oppositely polarised streams are incapable of interfering unless the light is passed through an analyser. When however an analyser is used, there will appear on either side of the central fringes a system of lateral bands, which are produced by the interference of the left-handed stream from the one image and the right-handed stream from the other image, these streams starting from the sources with an initial difference of phase.

The achromatic bands of these lateral systems will occur at the points, where the retardation of phase is stationary for light of mean wave-length λ_0, that is at the points given by

$$\frac{d}{d\lambda_0}\left\{\frac{2\pi}{\lambda}\frac{cx}{d} \pm 2R\right\} = 0, \quad \text{or} \quad \frac{\pi cx}{\lambda_0^2 d} \mp \frac{dR}{d\lambda} = 0,$$

where c is the distance between the sources, d their distance from the screen of observation and R is the rotation produced by the quartz.

Let x_0 be the distance from the centre of the field of the point at which the difference of phase of the interfering streams is zero for light of mean wave-length λ_0; then

$$\frac{cx_0}{d} = \mp \frac{\lambda_0}{\pi} R, \text{ and } x = \pm \frac{\lambda_0^2}{\pi}\frac{d}{c}\frac{dR}{d\lambda_0} = -x_0 \frac{\lambda_0}{R}\frac{dR}{d\lambda_0}.$$

Assuming that the rotary power is given by the law $R = kT\lambda^{-n}$ we have

$$\frac{\lambda}{R}\frac{dR}{d\lambda} = -n \text{ and } x = nx_0.$$

According to Biot $n = 2$ and the achromatic fringes of the lateral systems are at distances from the centre of the field that are double that of the points at which the interfering streams have the same phase; but the value $n = 2{\cdot}13$ gives results that are more in accord with the observed positions of the bands†.

* *Œuvres*, I. 657.

† Cornu, *C. R.* XCIII. 809 (1881).

It is however possible to explain this experiment of interference without having recourse to the hypothesis of circularly polarised streams within the quartz. When monochromatic light is employed, an extended system of bands is obtained as in all cases of interference and these reach beyond the regions occupied by the lateral systems of bands in Fresnel's experiment; when the light is white and no analyser is used, the visible interference shrinks into a small central system, as the actual interference is quickly masked by the superposition at each point of maxima and minima due to streams of slightly different frequency. In the case of the polarised streams that emerge from the quartz plate, the azimuth of the plane of polarisation is a function of the wave-length, and when the light is passed through an analyser, those constituents are suppressed that have their plane of polarisation nearly parallel to a given direction, so that the interference will again become visible at two determined points where the maxima coincide for the streams that still subsist.

On this explanation the function of the quartz plate and analyser is simply to weed out the constituents of the composite stream that cause the obliteration of the interference phenomenon and as Righi* has shown, the appearance of the lateral systems may be brought about by employing other methods of suppressing these constituents.

234. Fresnel's theory only applies, in the case of quartz and other uniaxal active crystals, to streams propagated in the direction of the axis, but Airy† in 1831 generalised it to include the passage of waves in any direction within the active media.

Starting from the hypothesis that streams travelling along the axis are oppositely circularly polarised and observing that in a direction perpendicular to the axis they are practically plane polarised in and perpendicular to the principal section, he was led by principles of continuity to assume that in intermediate cases the two streams that are propagated in the same direction are oppositely elliptically polarised, their planes of maximum and minimum polarisation being respectively parallel and perpendicular to the principal plane of the waves.

Let us suppose that a stream of light, plane polarised in an azimuth i with respect to the principal section, falls normally on a plate of quartz cut obliquely to the optic axis.

The incident stream may be replaced by the elliptically polarised stream represented by the polarisation-vector

$$\xi_1 = c_1 \cos\beta e^{\iota(nt+\epsilon_1)}, \qquad \eta_1 = -\iota c_1 \sin\beta e^{\iota(nt+\epsilon_1)} \quad \ldots\ldots\ldots\ldots(1),$$

* *Mem. dell' Accad. R. di Bologna* (3) VIII. 87 (1877); *N. Cim.* (3) II. 181 (1877); *J. de Phys.* VII. 25 (1878).

† *Camb. Phil. Trans.* IV. Part 1, 79, 199 (1831).

with its plane of maximum polarisation (β being less than $\pi/4$) in the principal section, together with the oppositely polarised stream represented by the polarisation-vector

$$\xi_2 = -c_2 \sin\beta e^{\iota(nt+\epsilon_2)}, \qquad \eta_2 = -\iota c_2 \cos\beta e^{\iota(nt+\epsilon_2)} \quad \text{.........(2)},$$

where

$$\left.\begin{aligned} c_1 e^{\epsilon_1 \iota} &= r(\cos i \cos\beta + \iota \sin i \sin\beta) \\ c_2 e^{\epsilon_2 \iota} &= r(-\cos i \sin\beta + \iota \sin i \cos\beta) \end{aligned}\right\} \quad \text{...............(3)},$$

the polarisation-vector of the incident stream being $r\exp(\iota nt)$.

According to Airy's generalisation of Fresnel's theory, these oppositely polarised streams will traverse the plate with different speeds, and, emerging with a relative retardation of phase δ, will compound into an elliptically polarised stream. Let

$$\xi = c\cos\gamma e^{\iota(nt+\epsilon)}, \qquad \eta = -\iota c \sin\gamma e^{\iota(nt+\epsilon)},$$

γ being less than $\pi/4$, be the polarisation-vector of the resultant stream and let θ be the angle that its plane of maximum polarisation makes with the principal section of the plate: then we must have

$$c(\cos\gamma\cos\theta + \iota\sin\gamma\sin\theta)e^{\iota\left(\epsilon+\frac{\delta}{2}\right)} = c_1\cos\beta e^{\iota\left(\epsilon_1+\frac{\delta}{2}\right)} - c_2\sin\beta e^{\iota\left(\epsilon_2-\frac{\delta}{2}\right)},$$

$$c(\cos\gamma\sin\theta - \iota\sin\gamma\cos\theta)e^{\iota\left(\epsilon+\frac{\delta}{2}\right)} = -\iota c_1\sin\beta e^{\iota\left(\epsilon_1+\frac{\delta}{2}\right)} - \iota c_2\cos\beta e^{\iota\left(\epsilon_2-\frac{\delta}{2}\right)},$$

when substituting for $c_1 e^{\epsilon_1 \iota}$ and $c_2 e^{\epsilon_2 \iota}$ from (3) and writing

$$\tan R = \sin 2\beta \tan\frac{\delta}{2}, \qquad \tan\frac{\Delta}{2} = \cot 2\beta \sin R \quad \text{............(4)},$$

we obtain

$$ce^{\iota\left(\epsilon+\frac{\delta}{2}\right)}(\cos\gamma\cos\theta + \iota\sin\gamma\sin\theta) = r\left\{\cos(i+R)\cos\frac{\Delta}{2} + \iota\cos i\sin\frac{\Delta}{2}\right\},$$

$$ce^{\iota\left(\epsilon+\frac{\delta}{2}\right)}(\cos\gamma\sin\theta - \iota\sin\gamma\cos\theta) = r\left\{\sin(i+R)\cos\frac{\Delta}{2} - \iota\sin i\sin\frac{\Delta}{2}\right\},$$

or

$$\left.\begin{aligned} &ce^{\iota\left(\epsilon+\frac{\delta}{2}\right)}\left\{\cos\gamma\cos\left(\theta-\frac{R}{2}\right) + \iota\sin\gamma\sin\left(\theta-\frac{R}{2}\right)\right\} \\ &\qquad = r\cos\left(i+\frac{R}{2}\right)\left(\cos\frac{\Delta}{2} + \iota\sin\frac{\Delta}{2}\right) \\ &ce^{\iota\left(\epsilon+\frac{\delta}{2}\right)}\left\{\cos\gamma\sin\left(\theta-\frac{R}{2}\right) - \iota\sin\gamma\cos\left(\theta-\frac{R}{2}\right)\right\} \\ &\qquad = r\sin\left(i+\frac{R}{2}\right)\left(\cos\frac{\Delta}{2} - \iota\sin\frac{\Delta}{2}\right) \end{aligned}\right\} \quad \text{...............(5)}.$$

The sum of the squares of these equations gives

$$c^2 e^{\iota(2\epsilon+\delta)}\cos 2\gamma = r^2\{\cos\Delta + \iota\cos(2i+R)\sin\Delta\},$$

whence

$$\tan(2\epsilon+\delta) = \cos(2i+R)\tan\Delta \quad \text{.........................(6)},$$

and their ratio gives

$$\frac{\cos\gamma\cos\left(\theta-\frac{R}{2}\right)+\iota\sin\gamma\sin\left(\theta-\frac{R}{2}\right)}{\cos\gamma\sin\left(\theta-\frac{R}{2}\right)-\iota\sin\gamma\cos\left(\theta-\frac{R}{2}\right)}=\cot\left(i+\frac{R}{2}\right)(\cos\Delta+\iota\sin\Delta),$$

whence
$$\frac{\cos 2\gamma\sin(2\theta-R)}{1-\cos 2\gamma\cos(2\theta-R)}=\cot\left(i+\frac{R}{2}\right)\cos\Delta,$$
$$\frac{\sin 2\gamma}{1-\cos 2\gamma\cos(2\theta-R)}=\cot\left(i+\frac{R}{2}\right)\sin\Delta,$$

and from these equations we find

$$\left.\begin{aligned}\tan(2\theta-R)&=\tan(2i+R)\cos\Delta\\ \sin 2\gamma&=\sin(2i+R)\sin\Delta\\ \tan^2\gamma&=\tan(i+\theta)\tan(i+R-\theta)\end{aligned}\right\}\ldots\ldots\ldots\ldots(7).$$

An investigation of the emergent elliptically polarised stream gives the angles θ and γ; hence if we know the angle i, we can determine the angles R and Δ and then by equations (4) find the angles β and δ that give the ratio of the axes of the elliptic vibrations of the streams within the crystal and their relative retardation produced by the passage through the plate.

235. Airy assumed that in any direction within an active crystal two streams can be propagated without alteration of their state of polarisation and that these streams are oppositely polarised with their planes of maximum polarisation in and perpendicular to the principal plane of the streams. Gouy* on the other hand has proposed to deduce the existence of these streams of permanent type from the hypothesis that the action of the medium may be represented by a superposition of the effects of ordinary double refraction and of an independent rotary power of the crystal.

Adopting the geometrical representation of the state of polarisation of a stream of light given in § 202, the result of normal passage through unit thickness of a plate of an active crystal is given on Gouy's hypothesis by a rotation through an angle $\kappa(\mu_1-\mu_2)$ round the axis CA corresponding to the principal section of the plate, C being the centre of the sphere, together with a rotation 2ρ round the polar axis CP, where $\kappa=2\pi/\lambda$, μ_1 and μ_2 are the refractive indices of streams polarised in and perpendicularly to the principal section of the plate, supposed devoid of rotary power.

Regarding these rotations as small and neglecting small quantities of the second order, the resultant rotation is represented by the sum of the vectors

* *J. de Phys.* (2) IV. 149 (1885). Lefèbre, *ibid.* (3) I. 121 (1892). Beaulard, *ibid.* (3) II. 393 (1893).

obtained by taking along CA, CP lengths proportional to the rotations round these axes, and is therefore a rotation

$$\sqrt{\kappa^2 (\mu_1 - \mu_2)^2 + 4\rho^2}$$

round the axis CM, where

$$\tan AM = 2\rho / \{\kappa (\mu_1 - \mu_2)\}.$$

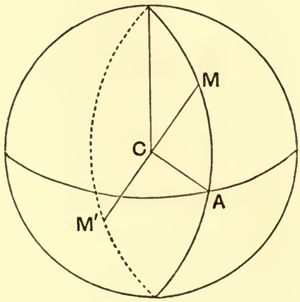


Fig. 54.

Since the point M and the diametrically opposite point M' remain fixed, they represent the polarisations of streams of permanent type, and we see that these streams are oppositely polarised with their planes of maximum polarisation respectively in and perpendicular to the principal section, and that the ratio $\tan \beta$ of the axes of the elliptic vibrations of the ends of their polarisation-vectors is given by

$$\tan 2\beta = 2\rho / \{\kappa (\mu_1 - \mu_2)\} \quad \text{.........................(8)}.$$

In the case of quartz and other positive crystals, $\mu_2 > \mu_1$ and the stream with its plane of maximum polarisation in the principal section of the plate, is left- or right-handed according as ρ is positive or negative, that is according as the crystal is left- or right-handed.

In traversing unit thickness the phase of the vibrations in the right-handed stream is retarded relatively to that of the left-handed vibrations by an amount

$$\kappa (\mu' - \mu'') = \pm \sqrt{\kappa^2 (\mu_1 - \mu_2)^2 + 4\rho^2} \quad \text{.......................(9)},$$

the upper or lower sign being taken, according as ρ is positive or negative.

If we wish to obtain the actual retardations of phase $\kappa\mu'$ and $\kappa\mu''$ of the streams, we require to know the value of $\kappa (\mu' + \mu'')$. In order to determine this*, let a stream of permanent type be replaced by its components polarised in planes parallel and perpendicular to the principal section with the polarisation-vectors

$$\xi = Ae^{\iota nt}, \qquad \eta = Be^{\iota nt},$$

* Poincaré, *Théorie Math. de a Lumière*, II. p. 299.

and suppose that after traversing unit thickness these become

$$\xi' = A'e^{\iota nt}, \qquad \eta' = B'e^{\iota nt};$$

then we have

$$A' = \alpha A + \beta B, \qquad B' = \gamma A + \delta B,$$

where α, β, γ, δ are constants depending upon the nature of the plate.

But the stream being of permanent type, we have

$$A'/A = B'/B = e^{-\iota\kappa x},$$

where κx is the retardation of phase; hence

$$(\alpha - e^{-\iota\kappa x}) A + \beta B = 0, \qquad \gamma A + (\delta - e^{-\iota\kappa x}) B = 0,$$

and

$$\begin{vmatrix} \alpha - e^{-\iota\kappa x}, & \beta \\ \gamma, & \delta - e^{-\iota\kappa x} \end{vmatrix} = 0.$$

The roots of this equation give the values of $e^{-\iota\kappa\mu'}$, $e^{-\iota\kappa\mu''}$, and their product is

$$\alpha\delta - \beta\gamma = e^{-\iota\kappa(\mu'+\mu'')}.$$

But according to Gouy's hypothesis

$$\alpha/\cos\rho = -\beta/\sin\rho = e^{-\iota\kappa\mu_1}, \qquad \gamma/\sin\rho = \delta/\cos\rho = e^{-\iota\kappa\mu_2},$$

whence

$$\alpha\delta - \beta\gamma = e^{-\iota\kappa(\mu_1+\mu_2)}$$

and

$$\mu' + \mu'' = \mu_1 + \mu_2 \qquad\qquad (10).$$

236. The theories hitherto considered are merely kinematical equivalents of the phenomenon of rotary polarisation and give no account of the physical character of the active substance: the case is otherwise with a theory elaborated by Mallard*.

This theory is based on some experiments made by Reusch in 1869† on the optical properties of combinations of thin mica plates, in which it was found that a series of p identical parallel plates, arranged so that each was turned through an angle π/p with respect to the former plate, possessed a rotary power just as a plate of quartz cut perpendicularly to the optic axis.

Such a combination of crystalline plates is called by Mallard a packet, and the packet is said to be symmetrical when all the plates are identical and the angles of combination are the same: a symmetrical packet of p plates is closed, if the common angle of combination be π/p. The superposition of a number of packets constitutes a pile.

* *Ann. des Mines* (7) x. 60 (1876); xix. 256 (1881): *C. R.* xcii. 1155 (1881): *J. de Phys.* x. 479 (1881): *Traité de Crist.* ii. 262. Sohncke, *Pogg. Ann. Ergb.* viii. 16 (1878): *Math. Ann.* ix. 504 (1876): *Zeitschr. f. Kryst.* xiii. 229 (1888). Poincaré, *Théorie Math. de la Lumière*, ii. ch. 12.

† *Pogg. Ann.* cxxxviii. 628 (1869): *Berl. Monatsber.* (1869) 530.

237. We will first take the case of a closed symmetrical packet. Adopting the geometrical representation of the state of polarisation by points on a sphere, the effect of the passage of a stream of polarised light through the packet is given by the resultant of successive rotations about axes CA_1, CA_2, ... in the plane of the equator through an angle δ, where C is the centre of the sphere, δ the relative retardation of phase introduced by each plate, and $A_1A_2 = A_2A_3 = \ldots = 2\pi/p$, p being the number of plates in the packet.

If then Δ_n denote a rotation δ round the axis CA_n and S_p a rotation $2\pi/p$ round the polar axis CP, the combined rotation is

$$\Delta_1 . \Delta_2 . \Delta_3 \ldots \Delta_p = \Delta_1 . S_{-p}\Delta_1 S_p . S_{-2p}\Delta_1 S_{2p} . \ldots S_{-(p-1)p}\Delta_1 S_{(p-1)p},$$

but
$$S_{(p-1)p} . S_p = S_{p.p} = S_{2\pi} = 1 \text{ or } S_{(p-1)p} = S_{-p},$$

whence
$$\Delta_1 . \Delta_2 . \Delta_3 \ldots \Delta_p = (\Delta_1 S_{-p})^p;$$

or the effect of the p successive rotations is the same as p times the resultant of the rotations Δ_1 and S_{-p}.

To determine this resultant, we must draw through A_1 a great circle making an angle $\delta/2$ with A_1P in a direction opposite to the rotation round CA_1, and through P a great circle making with PA_1 an angle π/p in the same direction as the rotation round CP; then if these circles intersect in the point M, the resultant of the two successive rotations is a rotation round CM through an angle equal to $2A_1MP$.

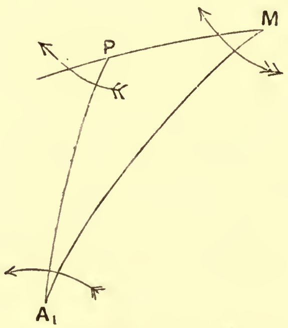


Fig. 55.

Now since $A_1P = \pi/2$, we have

$$\tan PM = \tan\frac{\delta}{2}\operatorname{cosec}\frac{\pi}{p} \text{ or } PM = \frac{\delta}{2}\operatorname{cosec}\frac{\pi}{p},$$

if δ be small: also

$$\cos A_1MP = \cos\frac{\delta}{2}\cos\frac{\pi}{p},$$

whence if $A_1MP = \pi/p + \omega$, we have neglecting ω^2,

$$\omega = \frac{\delta^2}{8}\cot\frac{\pi}{p},$$

and the resultant rotation $(\Delta_1 S_{-p})^p$ is a rotation round CM through an angle

$$\left(\frac{2\pi}{p} + \frac{\delta^2}{4}\cot\frac{\pi}{p}\right) \times p = 2\pi + \frac{p\delta^2}{4}\cot\frac{\pi}{p}.$$

Hence if δ be very small, the effect of the packet is very nearly to move the representative point along a parallel of latitude to a meridian differing from the original meridian by an angle

$$2R = \frac{p\delta^2}{4}\cot\frac{\pi}{p},$$

the rotation being in a direction opposite to that in which the angle of combination of the plates is measured.

Thus in traversing a packet the plane of maximum polarisation of a stream of polarised light is turned through an angle R, the form of the vibrations of the extremity of the polarisation-vector remaining unchanged. The rotation produced by a pile of n packets is nR, and the factor $np\delta^2$ is proportional to the thickness of the pile and very nearly to the inverse square of the wave-length, which represents very approximately the rotary power of quartz.

238. In the general case of an open unsymmetrical packet, if $\delta_1, \delta_2, \ldots$ be the relative retardations of phase introduced by the constituent plates, and $\theta_1, \theta_2, \ldots$ be the successive angles of combination, we want to find the effect of successive rotations $\delta_1, \delta_2, \ldots$ round the equatorial radii $CA_1, CA_2, \ldots$ where

$$A_1A_2 = 2\theta_1, \qquad A_2A_3 = 2\theta_2 \ldots.$$

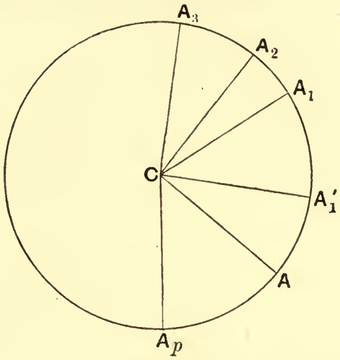


Fig. 56.

These rotations may be replaced by a rotation δ round some axis CA in the plane of the equator, together with a rotation 2ρ round the polar axis CP, and we have to determine these rotations, or what comes to the same thing, the two rotations that will bring the representative point to its primitive position after the successive rotations have been performed.

Let us take a polyhedral angle* $CBB_1 \ldots B_p$, such that the dihedral angle $CB_n = \pi - \delta_n$ and the angle $B_nCB_{n+1} = A_nCA_{n+1} = 2\theta_n$ and let us roll this angle on the equator, starting with the face BCB_1 in this plane and CB_1 coincident with CA_1: then the edges CB_2, CB_3, ... will in turn occupy the positions $CA_2, CA_3, \ldots$ until finally the edges CB_p, CB coincide respectively with CA_p, CA, and in order to bring the pyramid and with it the representative point (supposed rigidly attached to the pyramid) back to its primitive position, we must rotate it round CA through an angle δ, where the dihedral angle CB is $\pi - \delta$, thus bringing CB_1 to CA_1' and then turn it about an axis perpendicular to the plane of the equator through an angle $A_1'CA_1 = 2\rho$ which is the excess of 2π over the sum of the faces of the polyhedral angle.

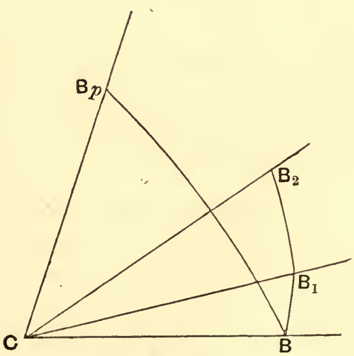


Fig. 57.

* Poincaré, *loc. cit.* p. 296.

The axis CA represents the principal section of the packet, δ is its doubly refracting power and ρ is its rotary power.

If a sphere be described with unit radius round the vertex of the polyhedral angle as centre, the faces of the angle will intersect the sphere in a spherical polygon, and the polar polygon $bb_1 \ldots b_p b$ is such that

$$bb_1 = \delta_1, \quad b_1b_2 = \delta_2, \ldots b_pb = \delta, \quad \angle bb_1b_2 = \pi - 2\theta_1, \quad \angle b_1b_2b_3 = \pi - 2\theta_2, \ldots,$$

and the sum of the angles of the polar polygon is

$$p\pi - 2(\theta_1 + \theta_2 + \ldots) = p\pi - (2\pi - 2\rho),$$

so that 2ρ is the area of the polygon.

When the rotations $\delta_1, \delta_2, \ldots$ are very small, the polar polygon becomes

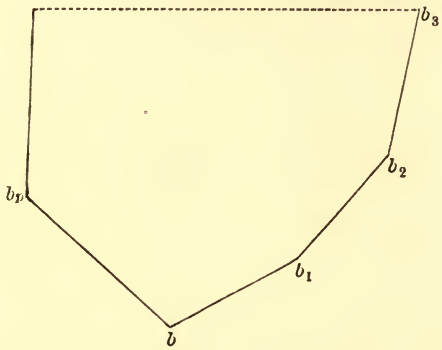


Fig. 58.

practically plane and the properties of the packet are found by drawing the line $bb_1b_2 \ldots b_p$, of which the successive parts $bb_1, b_1b_2, \ldots$ are parallel to the axes $CA_1, CA_2, \ldots$ and represent the rotations round these axes: then bb_p is parallel to the equatorial radius that corresponds to the principal section of the packet and its length gives the relative retardation of phase due to the double refraction and the area of the polygon is on the same scale twice the rotary power of the packet. These quantities being known, the streams of permanent type and the relative retardation of phase produced by their passage through the packet may be determined as in § 235.

To find the actual retardations of phase $\kappa\mu'$, $\kappa\mu''$ of the streams of permanent type, we may notice that (§ 235) $\exp\{-\kappa(\mu' + \mu'')\}$ is equal to the determinant of the linear substitutions that give the coefficients of vibration of the polarisation-vector on emergence in terms of the original coefficients. Now in the case of a packet this determinant is the product of the determinants of the linear substitutions relative to each plate of the packet, and these are respectively

$$\exp\{-\kappa(\mu'_1 + \mu''_1)\}, \qquad \exp\{-\kappa(\mu'_2 + \mu''_2)\} \ldots,$$

where $\kappa\mu'_n$, $\kappa\mu''_n$ are the retardations of phase due to the nth constituent plate. Hence

$$\kappa(\mu' + \mu'') = \sum_1^p \kappa(\mu'_n + \mu''_n).$$

239. In obtaining the differential equations that relate to ponderable media, the assumption was made (Chapter XVII.) that the forced vibrations of the auxiliary vectors d_h at a given point are determined by the value thereat of the polarisation-vector d relating to the pure ether: this is equivalent to neglecting the linear dimensions of the molecules in comparison with the wave-length of light. If however the molecules are of finite extent, the intra-molecular vibrations will depend not only on the value of d at a given point but also on its values in the vicinity and we must introduce on the right-hand side of equations (4) in § 225 the differential coefficients of u, v, w with respect to the coordinates. We shall then have three equations of the form*

$$\begin{aligned} a_h u_h + a'_h \dot{u}_h + a''_h \ddot{u}_h = u &+ \rho_{11}\frac{\partial u}{\partial x} + \rho_{12}\frac{\partial u}{\partial y} + \rho_{13}\frac{\partial u}{\partial z} \\ &+ \rho_{21}\frac{\partial v}{\partial x} + \cdots \\ &+ \rho_{31}\frac{\partial w}{\partial x} + \ldots \quad \text{.........................(11).} \end{aligned}$$

If the active substance be isotropic, the equations must remain unaltered when the coordinate axes are turned as a whole, but must change their form when one axis alone is reversed: the equation connecting d_h and d then becomes

$$a_h d_h + a'_h \dot{d}_h + a''_h \ddot{d}_h = d + \rho_h \operatorname{curl} d \quad \text{.....................(12).}$$

In the case of vibrations of frequency n, writing

$$a_h + \iota 2\pi n a'_h - 4\pi^2 n^2 a''_h = A_h,$$

this gives

$$d_h = A_h^{-1} d + \rho_h A_h^{-1} \operatorname{curl} d,$$

whence

$$\begin{aligned} D &= (1 + \Sigma A_h^{-1})\, d + \Sigma \rho_h A_h^{-1} \operatorname{curl} d \\ &= \alpha d + \rho \operatorname{curl} d \ (\text{say}) \quad \text{..................................(13),} \end{aligned}$$

and since ρ is in all cases very small

$$d = \alpha^{-1} D - \sigma' \operatorname{curl} D \quad \text{..............................(14).}$$

Hence the fundamental equations

$$\dot{D} = -\operatorname{curl} \varpi, \qquad \dot{\varpi} = \operatorname{curl} e \quad \text{.........................(15),}$$

take the form

$$\dot{D} = -\operatorname{curl} \varpi, \qquad \dot{\varpi} = \operatorname{curl} E + \sigma \nabla^2 D \quad \text{...........(16),}$$

* Drude, *Physik des Æthers*, p. 535.

the components of E being given by

$$E_1, E_2, E_3 = \left(\frac{\partial}{\partial U}, \frac{\partial}{\partial V}, \frac{\partial}{\partial W}\right)\Phi \qquad\qquad (17),$$

where
$$2\Phi = \frac{\Omega^2}{\alpha}(U^2 + V^2 + W^2) \qquad\qquad (18).$$

The equations for the case of crystalline active media may be deduced from (11); but in view of the smallness of the rotary power and the weak double refraction of all known crystals, we shall obtain a sufficiently accurate result by assuming that the rotary terms in the differential equations have the same form as in the case of isotropic bodies, and the equations for active crystals will then be (16) and (17) with

$$2\Phi = a_{11}U^2 + a_{22}V^2 + a_{33}W^2 + 2a_{23}VW + 2a_{31}WU + 2a_{12}UV \qquad (19).$$

Taking the plane $x = 0$ as the interface between two media that have different optical properties, the boundary conditions given by (15) are the continuity of ϖ_2, ϖ_3, e_2, e_3 to which may be added the continuity of U and ϖ_1. In terms of the vector D these become the continuity of

$$\varpi_1,\ \varpi_2,\ \varpi_3,\ U,\ \frac{\partial\Phi}{\partial V} - \sigma\left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}\right),\ \ \frac{\partial\Phi}{\partial W} - \sigma\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right).$$

240. Eliminating ϖ between equations (16) we have

$$\ddot{D} = \nabla^2 E - \nabla \operatorname{div} E - \sigma\nabla^2 \operatorname{curl} D,$$

of which the Cartesian equivalents are

$$\begin{aligned}(\ddot{U}, \ddot{V}, \ddot{W}) = \nabla^2&\left(\frac{\partial\Phi}{\partial U}, \frac{\partial\Phi}{\partial V}, \frac{\partial\Phi}{\partial W}\right)\\ &- \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\frac{\partial\Phi}{\partial U} + \frac{\partial}{\partial y}\frac{\partial\Phi}{\partial V} + \frac{\partial}{\partial z}\frac{\partial\Phi}{\partial W}\right)\\ &- \sigma\nabla^2\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},\ \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x},\ \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) \qquad (20).\end{aligned}$$

Let
$$(U, V, W) = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})A \exp\{\iota\kappa(lx + my + nz - \omega t)\} \qquad (21);$$

then these represent the components of the polarisation-vector of a stream of elliptically polarised light, so long as the ratio $\bar{\alpha} : \bar{\beta} : \bar{\gamma}$ is not real, and if we so choose the origin of time that

$$\bar{\alpha}A = \alpha L - \iota\alpha' L', \quad \bar{\beta}A = \beta L - \iota\beta' L', \quad \bar{\gamma}A = \gamma L - \iota\gamma' L' \qquad (22),$$

then α, β, γ and α', β', γ' are the direction-cosines of the axes of the ellipse traced by the extremity of the polarisation-vector and L, L' are the lengths of the axes in these directions.

Taking the axes of symmetry as the coordinate axes

$$2\Phi = a^2U^2 + b^2V^2 + c^2W^2,$$

and substituting the values (21) in equations (20) we obtain

$$\left.\begin{aligned} (a^2-\omega^2)\,\bar{\alpha} + \iota\sigma\kappa n\bar{\beta} - \iota\sigma\kappa m\bar{\gamma} - \bar{F}l &= 0 \\ -\iota\sigma\kappa n\bar{\alpha} + (b^2-\omega^2)\,\bar{\beta} + \iota\sigma\kappa l\bar{\gamma} - \bar{F}m &= 0 \\ \iota\sigma\kappa m\bar{\alpha} - \iota\sigma\kappa l\bar{\beta} + (c^2-\omega^2)\,\bar{\gamma} - \bar{F}n &= 0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(23),$$

where
$$\bar{F} = a^2l\bar{\alpha} + b^2m\bar{\beta} + c^2n\bar{\gamma},$$

and since $\operatorname{div} D = 0$

$$l\bar{\alpha} + m\bar{\beta} + n\bar{\gamma} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

Eliminating $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{F}$ between (23) and (24), we obtain

$$\begin{vmatrix} a^2-\omega^2, & \iota\sigma\kappa n, & -\iota\sigma\kappa m, & -l \\ -\iota\sigma\kappa n, & b^2-\omega^2, & \iota\sigma\kappa l, & -m \\ \iota\sigma\kappa m, & -\iota\sigma\kappa l, & c^2-\omega^2, & -n \\ l, & m, & n, & 0 \end{vmatrix} = 0,$$

or

$$(b^2-\omega^2)(c^2-\omega^2)\,l^2 + (c^2-\omega^2)(a^2-\omega^2)\,m^2 + (a^2-\omega^2)(b^2-\omega^2)\,n^2 = \sigma^2\kappa^2 \ldots(25),$$

which is the polar equation of the surface of wave-quickness.

Let ω_1, ω_2 be the roots of this equation, then

$$(z-\omega_1^2)(z-\omega_2^2) = (b^2-z)(c^2-z)\,l^2 + (c^2-z)(a^2-z)\,m^2 + (a^2-z)(b^2-z)\,n^2 - \sigma^2\kappa^2,$$

is identically true for all values of z. Writing $z = b^2$, we have

$$(\omega_1^2-b^2)(b^2-\omega_2^2) = (a^2-b^2)(b^2-c^2)\,m^2 + \sigma^2\kappa^2,$$

which is always positive and greater than $\sigma^2\kappa^2$: hence the roots of (25) are one greater and one less than b^2 and can never become equal, so that the surface of wave-quickness consists of two distinct sheets one within and one without the sphere of radius b.

Solving (25) we find

$$2\omega^2 = l^2(b^2+c^2) + m^2(c^2+a^2) + n^2(a^2+b^2) \pm \sqrt{A^2+B^2+C^2-2(AB+BC+CA)+4\sigma^2\kappa^2} \ldots(26),$$

where
$$A = l^2(b^2-c^2), \quad B = m^2(c^2-a^2), \quad C = n^2(a^2-b^2);$$

writing the radical in the form

$$(A+B-C)^2 - 4AB + 4\sigma^2\kappa^2,$$

we see that the difference between ω_1 and ω_2 becomes least, when

$$A + B - C = 0 \text{ and } AB = 0,$$

and these equations combined with $l^2 + m^2 + n^2 = 1$ give

$$l = \pm\sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad m = 0, \quad n = \sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

which are the direction-cosines of the optic axes of the medium supposed to be deprived of its activity*.

If χ, χ' be the angles that the wave-normal makes with these directions, equation (26) gives

$$2\omega^2 = a^2 + c^2 + (a^2 - c^2)\cos\chi\cos\chi' \pm \sqrt{(a^2-c^2)^2\sin^2\chi\sin^2\chi' + 4\sigma^2\kappa^2} \quad \ldots(27).$$

241. Now α, β, γ and α', β', γ' and l, m, n being the direction-cosines of three vectors at right-angles to one another we have

$$\alpha' = \gamma m - \beta n, \qquad \beta' = \alpha n - \gamma l, \qquad \gamma' = \beta l - \alpha m,$$

$$\alpha = -(\gamma' m - \beta' n), \quad \beta = -(\alpha' n - \gamma' l), \quad \gamma = -(\beta' l - \alpha' m),$$

whence
$$\iota(m\bar{\gamma} - n\bar{\beta}) A = (\iota L\gamma + L'\gamma') m - (\iota L\beta + L'\beta') n$$
$$= -L'\alpha + \iota L\alpha'.$$

Hence separating real and imaginary parts, equations (23) give the six equations

$$\left.\begin{aligned}\left(a^2 - \omega^2 + \sigma\kappa\frac{L'}{L}\right)\alpha = Fl, \; \left(b^2 - \omega^2 + \sigma\kappa\frac{L'}{L}\right)\beta = Fm, \; \left(c^2 - \omega^2 + \sigma\kappa\frac{L'}{L}\right)\gamma = Fn \\ \left(a^2 - \omega^2 + \sigma\kappa\frac{L}{L'}\right)\alpha' = F'l, \left(b^2 - \omega^2 + \sigma\kappa\frac{L}{L'}\right)\beta' = F'm, \left(c^2 - \omega^2 + \sigma\kappa\frac{L}{L'}\right)\gamma' = F'n\end{aligned}\right\} \quad \ldots\ldots\ldots(28),$$

where
$$F = a^2 l\alpha + b^2 m\beta + c^2 n\gamma, \quad F' = a^2 l\alpha' + b^2 m\beta' + c^2 n\gamma'.$$

Multiplying these equations by α, β, γ and α', β', γ' respectively, and adding, we obtain

$$\begin{aligned}\sigma\kappa\left(\frac{L'}{L} + \frac{L}{L'}\right) &= 2\omega^2 - a^2(\alpha^2 + \alpha'^2) - b^2(\beta^2 + \beta'^2) - c^2(\gamma^2 + \gamma'^2) \\ &= 2\omega^2 - a^2(m^2 + n^2) - b^2(n^2 + l^2) - c^2(l^2 + m^2) \\ &= 2\omega^2 - \omega_1^2 - \omega_2^2 \quad \ldots\ldots\ldots(29),\end{aligned}$$

where ω_1 and ω_2 are the wave-velocities in the direction (l, m, n).

Hence using the subscripts $(_1)$, $(_2)$ to refer to the quicker and the slower wave respectively, we have

$$\sigma\kappa\left(\frac{L_1'}{L_1} + \frac{L_1}{L_1'}\right) = \omega_1^2 - \omega_2^2 = -\sigma\kappa\left(\frac{L_2'}{L_2} + \frac{L_2}{L_2'}\right),$$

or
$$\left(1 + \frac{L_1 L_2}{L_1' L_2'}\right)\left(\frac{L_1'}{L_1} + \frac{L_2'}{L_2}\right) = 0.$$

Now the solution $L_2'/L_2 = -L_1'/L_1$ expresses that the streams are of opposite rotations with their planes of maximum polarisation coincident and must therefore be rejected on account of the continuity between active and

* Cf. Clebsch, *Crelle's J.* LVII. 319 (1860). Weder, *N. Jahrb. f. Min.* Beil.-Bd. XI. 1 (1898).

inactive media. We therefore have $L_2/L_2' = -L_1'/L_1$, or the polarisations of the streams propagated in a given direction are opposite. From the specification of U, V, W it follows that a positive value of L'/L denotes a left-handed stream, and hence the quicker wave is left- or right-handed according as σ is positive or negative.

Introducing the angles χ, χ' that the wave-normal makes with the optic axes, we have

$$2\sigma\kappa\frac{L_1}{L_1'} = -2\sigma\kappa\frac{L_2'}{L_2} = -(a^2-c^2)\sin\chi\sin\chi' + \sqrt{(a^2-c^2)\sin^2\chi\sin^2\chi' + 4\sigma^2\kappa^2} \quad \text{...(30).}$$

242. The interference patterns obtained with plates of quartz cut at right-angles to the optic axis present certain notable characteristics that serve to distinguish them from similar plates of an inactive crystal. These peculiarities were first observed by Airy and were explained by him with the aid of the hypotheses considered in § 234*.

In order to obtain a result that will be useful to us later, we will first obtain an expression for the intensity, when a pencil of elliptically polarised light falls upon a plate of quartz and after traversing the same is transmitted through a plane analyser.

Let the primitive stream be replaced by its components polarised in planes parallel and perpendicular to its plane of maximum polarisation with the polarisation-vectors

$$\xi = \cos\beta' e^{\iota nt}, \quad \eta = -\iota\sin\beta' e^{\iota nt},$$

where β' is numerically less than $\pi/4$ and positive or negative according as the stream is left- or right-handed.

On entering the plate the primitive stream is replaced by two oppositely polarised streams with their planes of maximum polarisation respectively parallel and perpendicular to the principal section and the polarisation-vectors of these streams may be represented by the components

$$\xi_1 = c_1\cos\beta e^{\iota nt}, \quad \eta_1 = -\iota c_1\sin\beta e^{\iota nt},$$

and

$$\xi_2 = c_2\sin\beta e^{\iota nt}, \quad \eta_2 = \iota c_2\cos\beta e^{\iota nt},$$

where β is numerically less than $\pi/4$ and positive or negative according as the plate is left- or right-handed. Whence if α be the angle that the plane of maximum polarisation of the primitive stream makes with the principal section

$$c_1\cos\beta + c_2\sin\beta = \cos\alpha\cos\beta' + \iota\sin\alpha\sin\beta',$$

$$c_1\sin\beta - c_2\cos\beta = \cos\alpha\sin\beta' + \iota\sin\alpha\cos\beta',$$

or

$$c_1 = \cos\alpha\cos(\beta-\beta') + \iota\sin\alpha\sin(\beta+\beta'),$$

$$c_2 = \cos\alpha\sin(\beta-\beta') - \iota\sin\alpha\cos(\beta+\beta').$$

* *loc. cit.*

In passing through the plate, the phase of the vibrations in the second stream is retarded relatively to the phase of the vibrations in the first by an amount δ, where δ is a positive quantity, and if the plane of analysation make an angle γ with the principal section, the polarisation-vector of the stream emergent from the analyser may be represented by

$$\Xi = \{(c_1 \cos\beta + c_2 \sin\beta e^{-\iota\delta})\cos\gamma - \iota(c_1 \sin\beta - c_2\cos\beta e^{-\iota\delta})\sin\gamma\}\, e^{\iota(nt-\epsilon)},$$

and the intensity, obtained by multiplying this by the conjugate expression, is

$$\begin{aligned} I = &\{c_1c_1' \cos^2\beta + c_2c_2' \sin^2\beta + (c_1c_2' e^{\iota\delta} + c_1'c_2 e^{-\iota\delta})\sin\beta\cos\beta\}\cos^2\gamma \\ &+ \{c_1c_1'\sin^2\beta + c_2c_2'\cos^2\beta - (c_1c_2'e^{\iota\delta} + c_1'c_2e^{-\iota\delta})\sin\beta\cos\beta\}\sin^2\gamma \\ &- \iota(c_1c_2'e^{\iota\delta} - c_1'c_2e^{-\iota\delta})\sin\gamma\cos\gamma, \end{aligned}$$

where c_1', c_2' are conjugate to c_1, c_2 respectively.

But

$$\begin{aligned} c_1c_1' &= \cos^2\alpha\cos^2(\beta-\beta') + \sin^2\alpha\sin^2(\beta+\beta'), \\ c_2c_2' &= \cos^2\alpha\sin^2(\beta-\beta') + \sin^2\alpha\cos^2(\beta+\beta'), \\ c_1c_2' &= \cos^2\alpha\sin(\beta-\beta')\cos(\beta-\beta') - \sin^2\alpha\sin(\beta+\beta')\cos(\beta+\beta') \\ &\qquad + \iota\sin\alpha\cos\alpha\cos 2\beta', \\ c_1'c_2 &= \cos^2\alpha\sin(\beta-\beta')\cos(\beta-\beta') - \sin^2\alpha\sin(\beta+\beta')\cos(\beta+\beta') \\ &\qquad - \iota\sin\alpha\cos\alpha\cos 2\beta', \end{aligned}$$

whence

$$\begin{aligned} I = &\Big[\cos^2\alpha\cos^2\beta' + \sin^2\alpha\sin^2\beta' - \sin 2\alpha\cos 2\beta'\sin 2\beta\sin\frac{\delta}{2}\cos\frac{\delta}{2} \\ &- \{\cos^2\alpha\sin 2(\beta-\beta') - \sin^2\alpha\sin 2(\beta+\beta')\}\sin 2\beta\sin^2\frac{\delta}{2}\Big]\cos^2\gamma \\ &+ \Big[\cos^2\alpha\sin^2\beta' + \sin^2\alpha\cos^2\beta' + \sin 2\alpha\cos 2\beta'\sin 2\beta\sin\frac{\delta}{2}\cos\frac{\delta}{2} \\ &+ \{\cos^2\alpha\sin 2(\beta-\beta') - \sin^2\alpha\sin 2(\beta+\beta')\}\sin 2\beta\sin^2\frac{\delta}{2}\Big]\sin^2\gamma \\ &+ \Big[\sin 2\alpha\cos 2\beta' + 2\{\cos^2\alpha\sin 2(\beta-\beta') - \sin^2\alpha\sin 2(\beta+\beta')\}\sin\frac{\delta}{2}\cos\frac{\delta}{2} \\ &- 2\sin 2\alpha\cos 2\beta'\sin^2\frac{\delta}{2}\Big]\sin\gamma\cos\gamma \\ = &\cos^2\beta'\cos^2(\gamma-\alpha) + \sin^2\beta'\sin^2(\gamma-\alpha) \\ &+ \cos 2\beta'\Big\{\sin 2\beta\sin 2(\gamma-\alpha)\sin\frac{\delta}{2}\cos\frac{\delta}{2} \\ &- (\cos 2\alpha\cos 2\gamma\sin^2 2\beta + \sin 2\alpha\sin 2\gamma)\sin^2\frac{\delta}{2}\Big\} \\ &+ \sin 2\beta'\cos 2\beta\Big\{\cos 2\gamma\sin 2\beta\sin^2\frac{\delta}{2} - \sin 2\gamma\sin\frac{\delta}{2}\cos\frac{\delta}{2}\Big\} \\ = &\sin^2\beta' + \cos 2\beta'\Big[\Big\{\cos(\gamma-\alpha)\cos\frac{\delta}{2} + \sin(\gamma-\alpha)\sin 2\beta\sin\frac{\delta}{2}\Big\}^2 \\ &+ \cos^2(\gamma+\alpha)\cos^2 2\beta\sin^2\frac{\delta}{2}\Big] \\ &+ \sin 2\beta'\cos 2\beta\Big(\cos 2\gamma\sin 2\beta\sin^2\frac{\delta}{2} - \sin 2\gamma\sin\frac{\delta}{2}\cos\frac{\delta}{2}\Big) \quad \text{............(31).} \end{aligned}$$

This expression with the same limitations as in Chapter XIV. may be applied to the case in which the pencil of light incident on the plate is conical.

243. Let us first suppose that the incident light is plane polarised, then $\beta' = 0$ and

$$I = \left\{\cos(\gamma - \alpha)\cos\frac{\delta}{2} + \sin(\gamma - \alpha)\sin 2\beta \sin\frac{\delta}{2}\right\}^2 + \cos^2(\gamma + \alpha)\cos^2 2\beta \sin^2\frac{\delta}{2}$$

$$= \left(\cos\psi \cos\frac{\delta}{2} + \sin\psi \sin 2\beta \sin\frac{\delta}{2}\right)^2 + \cos^2(2\gamma - \psi)\cos^2 2\beta \sin^2\frac{\delta}{2} \quad \text{.......(32)},$$

where ψ is the angle between the final and the primitive plane of polarisation of the stream.

When the planes of polarisation and analysation are crossed $\psi = \pi/2$, and

$$I = (\sin^2 2\beta + \cos^2 2\beta \sin^2 2\gamma)\sin^2\frac{\delta}{2} \quad \text{.................(33)}.$$

The intensity is thus a minimum when $\gamma = 0°$ or $90°$ or $180°$ or $270°$ and there are therefore dark brushes parallel and perpendicular to the primitive plane of polarisation, but these will be insensible near the centre of the field, since in the vicinity of the axis β is approximately $\pi/4$.

In addition to these we have the dark curves of constant retardation given by $\delta = 2n\pi$. Now if the crystal were inactive, we should have

$$\delta = \frac{2\pi}{\lambda}\frac{T}{a}\{\sqrt{\omega^2 - c^2\sin^2 i} - \sqrt{\omega^2 - a^2\sin^2 i}\} \doteqdot \frac{\pi}{\lambda}\frac{T(a^2 - c^2)}{a\omega}\sin^2 i,$$

where T is the thickness of the plate and i the angle of incidence, whence if the angle of incidence be small, we may take in the case of an active plate

$$\delta = \frac{\pi T}{\lambda}\left\{\frac{a^2 - c^2}{a\omega}\sin^2 i + \frac{2\rho}{\pi\lambda}\right\},$$

where $\rho T\lambda^{-2}$ is the rotation of the plane of polarisation when the light passes through in the direction of the axis. The dark curves are therefore circles and the difference of the squares of the sines of their angular radii is constant for small angles of incidence.

When the planes of polarisation and analysation are parallel

$$I = 1 - (\sin^2 2\beta + \cos^2 2\beta \sin^2 2\gamma)\sin^2\frac{\delta}{2} \quad \text{..............(34)},$$

and the interference pattern is complementary to the above.

If the planes of polarisation and analysation be neither parallel nor crossed,

$$I = (\cos^2\psi + \sin^2\psi \sin^2 2\beta)\cos^2\left(\frac{\delta}{2} - \chi\right) + \cos^2(2\gamma - \psi)\cos^2 2\beta \sin^2\frac{\delta}{2} \quad \text{...(35)}$$

where $$\tan\chi = \sin 2\beta \tan\psi.$$

Now if we suppose that the different points of a given curve of retardation are not at very different distances from the centre, we may regard β as sensibly constant for all points of this curve, and we obtain the approximate equation of the curves of maximum and of minimum intensity by equating to zero the derivative of I with respect to δ, regarding β as constant. This gives

$$\tan(\delta-\chi)=\sin 2\beta \tan\psi \frac{\cos^2\psi+\sin^2\psi\sin^2 2\beta+\cos^2 2\beta\cos^2(2\gamma-\psi)}{\cos^2\psi+\sin^2\psi\sin^2 2\beta-\cos^2 2\beta\cos^2(2\gamma-\psi)}$$
$$=\tan\omega \text{ (say)} \qquad\qquad (36),$$

and therefore δ exceeds χ or $\chi+\pi$... by the angle ω dependent upon the angle γ. If the angle of incidence be small, so that $\tan\beta$ does not differ greatly from unity and if ψ be less than $\pi/2$, $\tan\omega$ is always positive and attains its maximum value when $\gamma=\psi/2+n\pi/2$ and its minimum value when $\gamma=\psi/2+(2n+1)\pi/4$. Hence to obtain the form of the dark curves, it is necessary to describe a circle of radius OC and to increase the radii of this circle by amounts variable with their direction, that attain their maximum value along the internal and external bisectors of the angle between the primitive and final planes of polarisation and their minimum value along directions inclined at 45° to the former. The result is a kind of square with rounded corners, known as a "quadratic curve." On the other hand, if ψ be greater than $\pi/2$, ω is then negative and the greatest contractions from the circular form occur when $2\gamma=n\pi+\psi$ and the least contractions when

$$2\gamma=(2n+1)\pi/2+\psi.$$

It follows then if the primitive plane of polarisation be vertical, the highest corner of the quadratic curve is to the left of this plane (fig. 59).

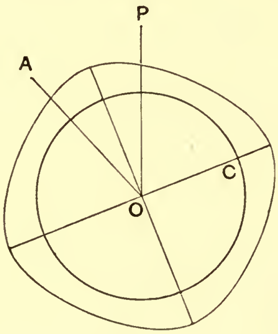


Fig. 59.

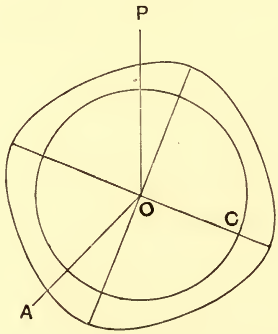


Fig. 60.

With a right-handed plate, β is negative and consequently ω is positive or negative according as ψ is greater or less than $\pi/2$. Consequently in this case the highest corner of the curve is to the right of the vertical plane (fig. 60).

The intensity on one of the dark quadratic curves is

$$I = \tfrac{1}{2}\,[\cos^2\psi + \sin^2\psi \sin^2 2\beta + \cos^2(2\gamma - \psi)\cos^2 2\beta$$
$$- \sqrt{\{\cos^2\psi - \sin^2\psi\sin^2 2\beta - \cos^2(2\gamma-\psi)\cos^2 2\beta\}^2 + \sin^2 2\beta \sin^2 2\psi}\,]$$

which is a maximum or a minimum according as

$$2\gamma = n\pi + \psi \;\text{ or }\; = (2n+1)\,\pi/2 + \psi\,;$$

hence the greatest intensity on the curve occurs at the corners or at the centres of the sides according as ψ is less or greater than $\pi/2$.

At the centre of the field $\beta = \pi/4$ and the intensity is $\cos^2(\delta/2 - \psi)$: this is zero, if $\psi = \pi/2 + \rho$, where ρ is the rotation of the plane of polarisation produced by passage through the plate in the direction of the optic axis, and in this case the first term in (35) is very small for points near the centre and the intensity is approximately given by

$$I = \cos^2(2\gamma - \psi)\cos^2 2\beta \sin^2\frac{\delta}{2},$$

which is a maximum or a minimum according as

$$2\gamma = \psi + n\pi \;\text{ or }\; = \psi + (2n+1)\,\pi/2.$$

The central spot will then be dark and extended in the direction of the diagonals of the quadratic curves, as a kind of rectangular cross.

244. When the primitive light is circularly polarised, $\beta' = \pm\,\pi/4$, the upper or lower sign being taken according as the stream is left- or right-handed. Hence in this case

$$I = \tfrac{1}{2}\left\{1 \pm 2\cos 2\beta\left(\cos 2\gamma \sin 2\beta \sin^2\frac{\delta}{2} - \sin 2\gamma \sin\frac{\delta}{2}\cos\frac{\delta}{2}\right)\right\}$$
$$= \tfrac{1}{2}\,\{1 \pm \cos 2\gamma \sin 2\beta \cos 2\beta$$
$$\mp \cos 2\beta\sqrt{\sin^2 2\gamma + \cos^2 2\gamma \sin^2 2\beta}\,\cos(\delta - \chi)\} \quad \text{.........(37)},$$

where $$\tan\chi = \tan 2\gamma/\sin 2\beta.$$

Taking the upper sign, the dark curves are determined by

$$\delta = 2n\pi + \chi,$$

and writing as a first approximation $\sin 2\beta = 1$, we have for the dark curves

$$\delta = 2n\pi + 2\gamma, \text{ nearly.}$$

Consequently δ, and hence also i, increases continually as γ increases and this shows that the dark curves are two mutually inwrapping spirals, that are right-handed, since γ is measured from the principal section to the plane of analysation in a counter-clockwise direction. At the centre these spirals touch the line OQ that makes an angle R with the plane of analysation on

the right-hand side of it, R being the rotation of the plane of polarisation produced by passage through the plate in the direction of the axis.

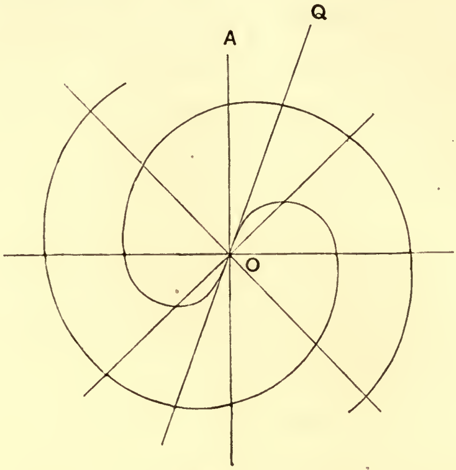


Fig. 61.

The actual curves are however not so simple, for χ is only equal to 2γ, when $\gamma = m\pi/4$, and is in excess or defect of this quantity according as γ is between $m\pi/2$ and $(2m+1)\pi/4$ or between $(2m+1)\pi/4$ and $(m+1)\pi/2$: hence δ and therefore also i is, in the odd octants, counted to the right from the plane of analysation, too great and in the even octants too small for an uniform spiral, and the dark curves consequently have the form of quadratic spirals.

If the primitive stream be right-handed, we must take the lower sign in (37) and the dark curves are given by $\delta = (2n+1)\pi + \chi$; that is we have the same curves as before, but they are turned through a right-angle.

On the other hand if the plate be right-handed, β is negative and the spirals are left-handed.

245. An interesting case treated by Airy is that in which a conical pencil of plane polarised light passes in succession through two plates of quartz perpendicular to the axis, of opposite sign and equal thickness, and is subsequently analysed.

Taking the first plate as left-handed, the light emergent from it consists of the two oppositely polarised streams with the polarisation-vectors

$$\xi_1 = c_1 \cos\beta e^{\iota(nt-\epsilon)}, \qquad \eta_1 = -\iota c_1 \sin\beta e^{\iota(nt-\epsilon)},$$

$$\xi_2 = c_2 \sin\beta e^{\iota(nt-\epsilon-\delta)}, \qquad \eta_2 = \iota c_2 \cos\beta e^{\iota(nt-\epsilon-\delta)},$$

where
$$c_1 = \cos\alpha\cos\beta + \iota\sin\alpha\sin\beta,$$

$$c_2 = \cos\alpha\sin\beta - \iota\sin\alpha\cos\beta,$$

α being the angle that the primitive plane of polarisation makes with the principal section and the first stream having its plane of maximum polarisation parallel to the principal section.

In the second plate on the other hand it is the right-handed stream that has its plane of maximum polarisation in the principal section, and therefore on entry into the plate, the pencil is divided into two streams with the polarisation-vectors

$$\xi_1' = k_1 \cos \beta e^{\iota(nt-\epsilon)}, \qquad \eta_1' = \iota k_1 \sin \beta e^{\iota(nt-\epsilon)},$$

$$\xi_2' = k_2 \sin \beta e^{\iota(nt-\epsilon)}, \qquad \eta_2' = -\iota k_2 \cos \beta e^{\iota(nt-\epsilon)},$$

where
$$k_1 \cos \beta + k_2 \sin \beta = c_1 \cos \beta + c_2 \sin \beta e^{-\iota\delta},$$

$$k_1 \sin \beta - k_2 \cos \beta = - c_1 \sin \beta + c_2 \cos \beta e^{-\iota\delta},$$

or
$$k_1 = c_1 \cos 2\beta + c_2 \sin 2\beta e^{-\iota\delta},$$

$$k_2 = c_1 \sin 2\beta - c_2 \cos 2\beta e^{-\iota\delta}.$$

In traversing the plate the phase of the vibrations in the second stream is relatively retarded by an amount δ, and if γ be the azimuth of the plane of analysation with respect to the principal section, the polarisation-vector of the final plane polarised stream is

$$\Xi = \{(k_1 \cos \beta + k_2 \sin \beta e^{-\iota\delta}) \cos \gamma + \iota (k_1 \sin \beta - k_2 \cos \beta e^{-\iota\delta}) \sin \gamma\} \; e^{\iota(nt-\epsilon)},$$

giving as the intensity

$$\begin{aligned} I = &\{k_1 k_1' \cos^2 \beta + k_2 k_2' \sin^2 \beta + (k_1 k_2' e^{\iota\delta} + k_1' k_2 e^{-\iota\delta}) \sin \beta \cos \beta\} \cos^2 \gamma \\ &+ \{k_1 k_1' \sin^2 \beta + k_2 k_2' \cos^2 \beta - (k_1 k_2' e^{\iota\delta} + k_1' k_2 e^{-\iota\delta}) \sin \beta \cos \beta\} \sin^2 \gamma \\ &+ \iota (k_1 k_2' e^{\iota\delta} - k_1' k_2 e^{-\iota\delta}) \sin \gamma \cos \gamma, \end{aligned}$$

where k_1', k_2' are the expressions conjugate to k_1, k_2 respectively.

Now

$$\begin{aligned} k_1 k_1' &= c_1 c_1' \cos^2 2\beta + c_2 c_2' \sin^2 2\beta + (c_1 c_2' e^{\iota\delta} + c_1' c_2 e^{-\iota\delta}) \sin 2\beta \cos 2\beta \\ &= \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta - \sin 2\alpha \sin 4\beta \sin \frac{\delta}{2} \cos \frac{\delta}{2} \\ &\qquad - \cos 2\alpha \sin 2\beta \sin 4\beta \sin^2 \frac{\delta}{2}, \end{aligned}$$

$$\begin{aligned} k_2 k_2' &= c_1 c_1' \sin^2 2\beta + c_2 c_2' \cos^2 2\beta - (c_1 c_2' e^{\iota\delta} + c_1' c_2 e^{-\iota\delta}) \sin 2\beta \cos 2\beta \\ &= \cos^2 \alpha \sin^2 \beta + \sin^2 \alpha \cos^2 \beta + \sin 2\alpha \sin 4\beta \sin \frac{\delta}{2} \cos \frac{\delta}{2} \\ &\qquad + \cos 2\alpha \sin 2\beta \sin 4\beta \sin^2 \frac{\delta}{2}, \end{aligned}$$

$$k_1k_2'e^{\iota\delta}+k_1'k_2e^{-\iota\delta}$$

$$=2\,(c_1c_1'-c_2c_2')\sin 2\beta\cos 2\beta\cos\delta-(c_1c_2'e^{\iota 2\delta}+c_1'c_2e^{-\iota 2\delta})\cos^2 2\beta$$
$$+(c_1c_2'+c_1'c_2)\sin^2 2\beta$$

$$=\cos 2\alpha\sin 2\beta+4\sin 2\alpha\cos^2 2\beta\left(\sin\frac{\delta}{2}\cos\frac{\delta}{2}-2\sin^3\frac{\delta}{2}\cos\frac{\delta}{2}\right)$$
$$+2\cos 2\alpha\cos 2\beta\sin 4\beta\left(\sin^2\frac{\delta}{2}-2\sin^4\frac{\delta}{2}\right),$$

$$k_1k_2'e^{\iota\delta}-k_1'k_2e^{-\iota\delta}$$

$$=2\iota\,(c_1c_1'-c_2c_2')\sin 2\beta\cos 2\beta\sin\delta-(c_1c_2'e^{\iota 2\delta}-c_1'c_2e^{-\iota 2\delta})\cos^2 2\beta$$
$$-(c_1c_2'-c_1'c_2)\sin^2 2\beta$$

$$=-\iota\left(\sin 2\alpha-8\sin 2\alpha\cos^2 2\beta\sin^2\frac{\delta}{2}\cos^2\frac{\delta}{2}\right.$$
$$\left.-4\cos 2\alpha\cos 2\beta\sin 4\beta\sin^3\frac{\delta}{2}\cos\frac{\delta}{2}\right);$$

hence

$$I=\left\{\cos^2\alpha-2\sin 2\alpha\cos 2\beta\sin 4\beta\sin^3\frac{\delta}{2}\cos\frac{\delta}{2}\right.$$
$$\left.-\cos 2\alpha\sin^2 4\beta\sin^4\frac{\delta}{2}\right\}\cos^2\gamma$$
$$+\left\{\sin^2\alpha+2\sin 2\alpha\cos 2\beta\sin 4\beta\sin^3\frac{\delta}{2}\cos\frac{\delta}{2}\right.$$
$$\left.+\cos 2\alpha\sin^2 4\beta\sin^4\frac{\delta}{2}\right\}\sin^2\gamma$$
$$+2\left\{\sin\alpha\cos\alpha-4\sin 2\alpha\cos^2 2\beta\sin^2\frac{\delta}{2}\cos^2\frac{\delta}{2}\right.$$
$$\left.-2\cos 2\alpha\cos 2\beta\sin 4\beta\sin^3\frac{\delta}{2}\cos\frac{\delta}{2}\right\}\sin\gamma\cos\gamma$$
$$=\cos^2(\gamma-\alpha)-4\cos^2 2\beta\sin^2\frac{\delta}{2}\left\{\sin 2\alpha\sin 2\gamma\cos^2\frac{\delta}{2}\right.$$
$$\left.+\sin 2(\gamma+\alpha)\sin 2\beta\sin\frac{\delta}{2}\cos\frac{\delta}{2}+\cos 2\alpha\cos 2\gamma\sin^2 2\beta\sin^2\frac{\delta}{2}\right\}$$
$$\text{.........(38).}$$

If the planes of polarisation and analysation be crossed, so that

$$\gamma=\alpha+\pi/2,$$

$$I=4\cos^2 2\beta\sin^2\frac{\delta}{2}\left\{\sin 2\alpha\cos\frac{\delta}{2}+\cos 2\alpha\sin 2\beta\sin\frac{\delta}{2}\right\}^2$$
$$=4\,(\sin^2 2\alpha+\cos^2 2\alpha\sin^2 2\beta)\cos^2 2\beta\sin^2\frac{\delta}{2}\sin^2\left(\frac{\delta}{2}+\chi\right)\quad\text{......(39),}$$

where $$\tan\chi=\tan 2\alpha/\sin 2\beta.$$

In this case the expression for the intensity vanishes, when $\delta = 2n\pi$, which gives a series of dark circles as in the case of a single plate: it also vanishes when $\delta = 2n\pi - 2\chi$, and writing as a first approximation $\sin 2\beta = 1$, this corresponds to a second system of dark curves given by

$$\delta = 2n\pi - 4\alpha, \text{ nearly.}$$

Now α is measured from the primitive plane of polarisation to the principal section in a clockwise direction, and since from the above δ and consequently i increases as α diminishes, it follows that this second system of dark curves consists of four similar left-handed spirals, each of which is turned through 90° from the position of that adjacent to it. At the centre these spirals touch the lines COC, $C'OC'$ inclined at an angle $R/2$ to the planes of polarisation and analysation on the left-hand side, where R is the rotation of the plane of polarisation produced by normal passage through either of the plates.

Since $\sin^2 \delta/2$ and $\sin^2(\delta/2 + \chi)$ have the same value when $\alpha = \pm n\pi/2$, the points in which the spirals intersect the circles lie in directions parallel and perpendicular to the primitive plane of polarisation, but as χ is in excess or defect of 2α, according as α lies between $n\pi/2$ and $(2n+1)\pi/4$ or between $(2n+1)\pi/4$ and $(n+1)\pi/2$, the spirals intersect the circles at angles somewhat greater than those at which an uniform spiral would cut them.

When β is very small, the intensity is a minimum when $\alpha = m\pi/2$; hence at a distance from the centre there will be faint brushes parallel and perpendicular to the primitive plane of polarisation.

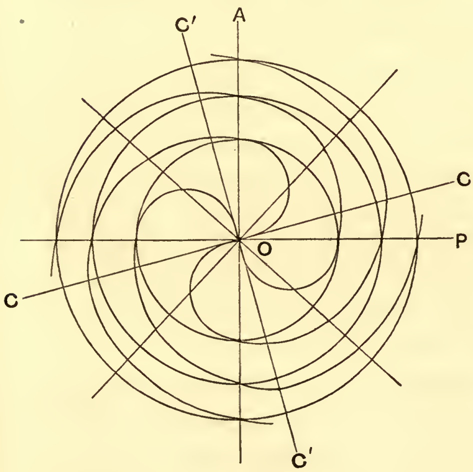


Fig. 62.

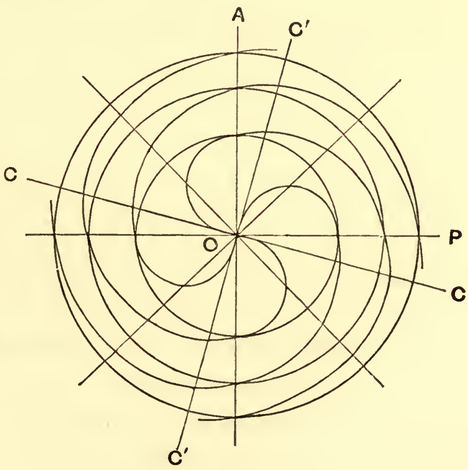


Fig. 63.

When the plate that receives the incident light is right-handed, β is negative: this changes the sign of χ and the spirals will be right-handed.

CHAPTER XIX.

MAGNETICALLY ACTIVE MEDIA.

246. In the course of a series of experiments instituted with the view of establishing a relation between electric and magnetic phenomena and those of light Faraday* in 1845 discovered, that a transparent body in itself inactive acquires the property of rotating the plane of polarisation of a stream of light, when it is placed in a magnetic field, the effect being a maximum when the stream traverses the medium in the direction of the lines of force of the field and nil when the magnetic force is at right-angles to the pencil of rays. The list of magnetically active media is now known to include a large number of solids and liquids†, in fact most diamagnetic substances and even gases and vapours‡.

The distinguishing features of the magnetic rotation of the plane of polarisation are:

(1) that the activity is temporary, appearing and disappearing simultaneously with the magnetic force that produces the field§;

(2) that the direction of rotation is changed with respect to the observer, when the direction of propagation is reversed: in other words, the direction of rotation in space is the same whether the light travels along or in a direction opposite to the lines of force, whereas in structurally active media reversal of the stream involves a change in the direction of rotation in space. It follows then that if a pencil of polarised light, after passing through the substance, be reflected by a mirror so as to be sent through it again in the opposite direction, the rotation is doubled when it results from magnetic action and is annulled if the medium be structurally active.

The direction of rotation is thus determined by that of the magnetic force and in the case of most diamagnetic substances is the same as that in which

* *Phil. Trans.* CXXXVI. 1 (1846): *Exp. Res.* XIXth series, § 26, Arts. 2146—2242.

† Matthiessen, *C. R.* XXIV. 969; XXV. 20, 173 (1847). Bertin, *Ann. de Ch. et de Phys.* (3) XXIII. 5 (1848). Ed. Becquerel, *ibid.* (3) XXVIII. 334 (1850).

‡ H. Becquerel, *J. de Phys.* VIII. 198 (1879); IX. 265 (1880). Bichat, *ibid.* VIII. 204 (1879); IX. 275 (1880).

§ Bichat and Blondlot, *J. de Phys.* (2) I. 364 (1882).

a positive current must circulate round the stream, in order to produce the magnetic field that actually exists. There are however notable exceptions to this rule that the rotation is positive for diamagnetic and negative for ferromagnetic media; for oxide of manganese and the salts of nickel and cobalt, though magnetic, give a positive rotation, while the diamagnetic chromate of potash and bichloride of titanium produce a negative rotation*.

The angle through which the plane of polarisation is turned in passing through a magnetically active medium is found to be proportional to:

(1) the distance within the medium over which the light travels;

(2) the intensity of the resolved part of the magnetic force in the direction of propagation of the stream†.

These laws may clearly be replaced by the single general statement that the angular rotation between two points is proportional to the difference of their magnetic potentials.

The coefficient of proportionality is a physical constant as characteristic of the substance as its refractive index and is known as "Verdet's constant": in C.G.S. units it is the rotation of the plane of polarisation between two points, one centimetre apart, the magnetic potentials of which differ by a C.G.S. unit.

In the case of thin transparent plates of iron, nickel and cobalt the above statement has to be modified, for as the intensity of the magnetic force is increased, the rotation produced by these metals rises to a maximum and then remains sensibly constant. The law of magnetic rotation, as given by Du Bois, is in the case of these metals that the angular rotation between two points is proportional to the difference of their potentials of magnetisation. In all three metals the rotation is positive and extraordinarily great: the theoretical value of the maximum rotation of red light produced by a plate of the thickness of 1 cm. is in the case of nickel 89,000°, of cobalt 198,000° and of iron 200,000°‡.

In a transparent medium the rotations of the plane of polarisation of streams of different frequencies vary approximately as the inverse square of the period: but this law is not exact, as the product of the rotation by the square of the period increases with the frequency of the light, the substances for which this increase is most marked being those that have the greater dispersive power§. Absorbing media however form exceptions to these rules,

* Verdet, *C. R.* XLIV. 1209 (1857): *Œuvres*, I. 168: *Ann. de Ch. et de Phys.* (3) LII. 129 (1858): *Œuvres*, I. 176.

† Wiedemann, *Pogg. Ann.* LXXXII. 215 (1851). Verdet, *C. R.* XXXVIII. 613; XXXIX. 548 (1854): *Ann. de Ch. et de Phys.* (3) XLI. 370 (1854); XLIII. 37 (1855): *Œuvres*, I. 107, 112, 152, 155. Cornu and Potier, *C. R.* CII. 385 (1886).

‡ Kundt, *Wied. Ann.* XXIII. 228 (1884); XXVII. 191 (1886). Du Bois, *ibid.* XXXI. 941 (1887).

§ Ed. Becquerel, *Ann. de Ch. et de Phys.* (3) XVII. 437 (1846). Wiedemann, *Pogg. Ann.* LXXXII. 215 (1851). Verdet, *C. R.* LVI. 630; LVII. 670 (1863): *Ann. de Ch. et de Phys.* (3) LXIX. 415 (1863): *Œuvres*, I. 205, 209, 214.

as indeed might be expected from the fact that near and within the region of absorption the refractive index experiences abnormal variations.

H. Becquerel* has from theoretical considerations stated as the law of magnetic rotary dispersion that the magnetic rotary power varies as $\lambda\,(d\mu/d\lambda)$ and has shown that this formula holds in the case of creosote and carbon bisulphide, while it gives in general a good approximation to the observed order of magnitude; but there is at present no experimental proof that this relation is true in the case of absorbing media†.

Attempts have been made without much success to find a relation between the magnetic rotary power of a substance and its refractive index. De la Rive‡ advanced the statement, to a great extent confirmed by Bertin's experiments§, that the magnetic rotation ought to increase with the refractive index of the medium, but Verdet's experiments|| have conclusively shown that, though substances with a high refractive index have in general a large rotary power when placed in a magnetic field, there is no constant relation between these quantities.

247. Another action of magnetism on light has been discovered by Kerr¶, who found that when plane polarised light is reflected from the polished pole of an electromagnet, the plane of polarisation of the reflected light is in certain cases altered when the magnet is excited. The following results have been obtained by Kerr and have been confirmed by later investigations**.

When the mirror is magnetised normally, light polarised in one of the principal azimuths gives at normal or at oblique incidence a reflected stream that is slightly elliptically polarised with the plane of maximum polarisation rotated from the primitive plane of polarisation in a direction opposite to that of the current exciting the pole. For light polarised in a plane perpendicular to the plane of incidence the rotation is a maximum for an angle of incidence between 44° and 68°: in the case of light polarised in the plane of incidence the rotation decreases continuously as the angle of incidence increases.

In the case of tangential magnetisation of the mirror, no change is produced either at normal incidence or when the plane of incidence is perpendicular to the lines of magnetic force: but when the lines of force are in the plane of incidence and the incidence is oblique, the reflected light is elliptically

* *C. R.* cxxv. 679 (1897).

† Cf. Cotton, *Le Phénomène de Zeeman, Scientia*, No. 5, p. 81.

‡ *Traité de l'Électricité*, I. 505 (1854).

§ *loc. cit.*

|| *C. R.* XLIII. 529 (1856): *Ann. de Ch. et de Phys.* (3) LII. 129 (1858): *Œuvres*, I. 163, 176.

¶ *Phil. Mag.* (5) III. 321 (1877); v. 161 (1878).

** Righi, *Ann. de Ch. et de Phys.* (6) IV. 433 (1885); IX. 65 (1886); X. 200 (1887): *Mém. R. Accad. Lincei* (4ª) I. 367; III. 14, 562 (1885–6). Kundt, *Wied. Ann.* XXIII. 228 (1884); XXVII. 191 (1886). Du Bois, *ibid.* XXXIX. 25 (1890). Sissingh, *ibid.* XLII. 115 (1891); *Arch. Néerl.* XXVII. 173 (1894). Zeeman, *ibid.* XXVII. 252 (1894); (2) I. 354, 376 (1897).

polarised, and if the primitive plane of polarisation coincide with the plane of incidence, the plane of maximum polarisation of the reflected light is rotated from the primitive plane in all cases in a direction opposite to that of the current that would produce a field of the same sign as the magnet; if the light be polarised at right-angles to the plane of incidence, the rotation is in the same direction as the current for angles of incidence between 0° and from 75° to 80°, and in the opposite direction for larger angles of incidence.

248. In 1896 Zeeman* discovered the remarkable effect produced on the character of the radiations by placing the source of light in an uniform magnetic field.

Earlier investigators† had indeed observed that the insertion of a Geissler's tube between the poles of an electromagnet influenced the colour and the spectrum of the light issuing from it and Fievez‡ found that the aspect of the sodium lines was modified, when the source emitting these radiations was placed in a magnetic field. These earlier discoveries however have characteristics that distinguish them from the phenomenon known as "the Zeeman effect" and are most probably to be attributed in the one case to chemical action within the tube and in the second case to a change in the form of the flame produced by a lack of uniformity in the magnetic field.

It has already been stated that a rotation of the molecules of a source as wholes will affect the character of the radiations emitted therefrom, and in this manner it is possible to give an account of the main features of the Zeeman effect. Whatever be the nature of the vibrations within a radiating molecule, they may in the case of strictly monochromatic light be represented by a vector, the extremity of which executes in general elliptic vibrations: this vector may be replaced by a component vibrating in the direction along which the radiation is considered, together with a component with elliptic vibrations in the perpendicular plane, this latter component giving rise to an elliptically polarised stream of definite frequency. If now we impress upon the vector an uniform rotation round the direction of propagation as axis, this elliptically polarised stream will be replaced by two oppositely circularly polarised streams with frequencies in excess and defect respectively of the primitive frequency by an amount equal to the number of revolutions per

* *Zittingsversl. Kon. Akad. v. Wet. Amsterdam*, v. 181, 242 (1896); vi. 13, 99, 260, 408 (1897); vii. 122 (1898); viii. 328 (1899): *Arch. Néerl.* (2) i. 44, 217, 383 (1897); v. 237 (1900): *Phil. Mag.* (5) xliii. 226; xliv. 55, 255 (1897); xlv. 197 (1898): *Astrophys. J.* v. 332 (1897); ix. 47 (1899).

† Plücker, *Pogg. Ann.* civ. 113 (1858). Trève, *C. R.* lxx. 36 (1870). Ångström, *Pogg. Ann.* cxliv. 300 (1872): *C. R.* lxxiii. 369 (1871): *Phil. Mag.* (4) xlii. 398 (1871). Daniel, *C. R.* lxx. 183 (1870). Secchi, *ibid.* lxx. 431 (1870). Chautard, *ibid.* lxxix. 1123 (1874); lxxx. 1161; lxxxi. 75 (1875); lxxxii. 272 (1876). Van Aubel, *J. de Phys.* (3) vii. 408 (1898). Thénard, *C. R.* lxxxix. 298 (1879); xci. 387 (1880).

‡ *Bull. Acad. Bruxelles* (3) ix. 327, 381 (1885); xii. 30

second of the impressed rotation. When a single molecule is considered, the intensities of the two streams will be unequal, but since the elliptic vibrations of the vectors corresponding to the different molecules of the source must be supposed to have all possible orientations, the two circularly polarised streams will in the aggregate be of the same intensity, provided we suppose the rotation to be the same for all molecules. Again in a direction perpendicular to the axis of rotation, we shall have two streams polarised in the same plane with the same frequencies as the two circularly polarised streams, and a stream polarised in the perpendicular plane of the primitive frequency.

These are the characteristics of the simplest form of the Zeeman effect. When a luminous source is placed in a strong uniform magnetic field and the radiation in the direction of the lines of force is examined with a spectroscope of considerable resolving power, it is found that a spectral line is replaced by a doublet, the constituents of which have equal intensities and on a scale of frequencies are symmetrically placed on either side of the primitive line with a distance between them proportional to the strength of the field: these two components are circularly polarised in opposite directions, that with the higher frequency having the direction of the current that produces the field. The constituents of the doublet are in general complex, which may be accounted for by the fact that the original radiation is itself complex and that the magnetic field may not exercise the same influence on all the molecules of the source.

In a direction perpendicular to the lines of force, the radiations may be divided into two groups, that are polarised respectively in planes parallel and perpendicular to the magnetic force. The constituents of the group with polarisation parallel to the lines of force agree in all cases with the doublet observed in the direction of the field: but in the case of the second group several variations are observed. The simplest phenomenon is that of a single line coinciding in position with the primitive line, but sometimes there is a doublet with constituents symmetrically placed with respect to the original line, and a triplet and even a more complicated system has been obtained*.

* For the above and other results, and for modifications of the normal type of the Zeeman effect, see: Cotton, *Le Phénomène de Zeeman*, *Scientia*, No. 5 (1899). Zeeman, *loc. cit.* Cornu, *C. R.* CXXV. 555 (1897); CXXVI. 181, 300 (1898): *J. de Phys.* (3) VI. 673 (1897). Preston, *Dublin Trans.* (2) VI. 385 (1897); VII. 7 (1898): *Proc. R. S.* LXIII. 26 (1898): *Phil. Mag.* (5) XLV. 325 (1899); XLVII. 165 (1899): *Nature*, LVII. 173 (1897); LIX. 224, 248, 485, 605 (1899); LX. 175 (1899); LXI. 11 (1899). König, *Wied. Ann.* LXII. 240; LXIII. 268 (1897). Becquerel, *C. R.* CXXV. 679 (1897): *J. de Phys.* (3) VI. 681 (1897). Becquerel and Deslandres, *C. R.* CXXVI. 997; CXXVII. 18 (1898). Lodge and Davies, *Proc. R. S.* LX. 513 (1897); LXI. 413 (1897). Ames, Echart and Reese, *Astrophys. J.* VIII. 48 (1898). Reese, *ibid.* XII. 120 (1900): *Phil. Mag.* (5) XLVIII. 317 (1899). Michelson, *Phil. Mag.* (5) XLIV. 109 (1897); XLV. 348 (1898): *Astrophys. J.* VI. 48 (1897); VII. 131 (1898): *Nature*, LIX. 440 (1899). Righi, *Rend. Lincei* (5) VII. [1] 295 (1898): *N. Cim.* (4) X. 20 (1899); XI. 177 (1900): *Phys. Zeitschr.* I. 329 (1900): *Mem. R. Accad. Bologna* (5) VIII. 263 (1900). Blythswood and Marchant, *Phil. Mag.* (5) XLIX. 384 (1900). Blythswood and Allen, *Nature*, LXV. 79 (1901). Kent, *Astrophys. J.* XIII. 289 (1901). Runge, *Phys. Zeitschr.* III. 441

These more complicated phenomena may possibly indicate a longitudinal effect on the vibrating molecules produced by the action of the magnetic field.

Egoroff and Géorgiewsky*, observing without a spectroscope, found that in a direction perpendicular to the lines of force the light from a sodium flame placed in a magnetic field is partially polarised in a plane parallel to the lines of force. This appears to imply that the sum of the intensities of the two components polarised in this direction exceeds that of the stream polarised in the perpendicular plane, a fact that is unaccounted for by the elementary explanation given above. This result may however be attributed either to an orientation of the molecules of the source by the field† or to an inequality in the absorption of light polarised in the two azimuths in traversing the outer mantle of the flame‡.

The phenomenon of absorption can, by Kirchhoff's principle, be employed for exhibiting and studying the influence of the magnetic field on the radiations from a luminous source§; but Macaluso and Corbino|| have shown that, when a sodium flame is placed in a magnetic field and a stream of polarised light passes through it along the lines of force, it is necessary to take account not only of the fact that absorption of light of a given period has been replaced by absorption of two streams of circularly polarised light of opposite signs and of periods in excess and defect respectively of the natural period, but also of the fact that the plane of polarisation of the light is rotated during the passage through the flame. This rotation, though generally very small in the case of a gas or a vapour, becomes of primary importance when the period of the light is near that corresponding to an absorption-band¶.

(1902). Runge and Paschen, *ibid.* I. 480 (1900): *Astrophys. J.* XV. 235, 333 (1902). Shedd, *Phys. Zeitschr.* I. 270 (1900); II. 278 (1901). Gray and Stewart, *Nature*, LXV. 54 (1901). For Lorentz' theory of the Zeeman effect see Lorentz, *Wied. Ann.* LXIII. 278 (1897): *Arch. Néerl.* (2) II. 1. 412 (1899); VII. 299 (1902): *Phys. Zeitschr.* I. 39, 498, 514 (1900): *Rapp. prés. au Congr. Intern. de Phys.* III. 1 (1900). Cotton, *Écl. Élect.* XIV. 311 (1898). Voigt, *Phys. Zeitschr.* I. 116, 128, 138 (1899).

* *C. R.* CXXIV. 748, 949; CXXV. 16 (1897).

† Cf. Voigt, *Gött. Nachr.* (1901) 169.

‡ Lorentz, *Zittingsversl. Kon. Akad. v. Wet. Amsterdam*, VI. 193 (1897): *Arch. Néerl.* (2) II. 1. 412 (1899): *Rapp. prés. au Congr. Intern. de Phys.* III. 28 (1900). Voigt, *Wied. Ann.* LXIX. 290 (1899).

§ Cotton, *C. R.* CXXV. 865 (1897). König, *Wied. Ann.* LXII. 240 (1897); LXIII. 268 (1897). Righi, *Rend. Lincei* (5) VII. [2] 41 (1898): *N. Cim.* (4) VIII. 102 (1898): *C. R.* CXXVII. 216 (1898); CXXVIII. 45 (1899).

|| *C. R.* CXXVII. 548, 951 (1898): *Rend. Lincei* (5) VII. [2] 293 (1898); VIII. [1] 38, 116, 250 (1899).

¶ H. Becquerel, *C. R.* CXXVII. 647, 899, 953 (1898); CXXVIII. 145 (1899). For the anomalous dispersion of sodium vapour see Kundt, *Wied. Ann.* X. 321 (1880); Winkelmann, *ibid.* XXXII. 439 (1887). Rotation of the plane of polarisation within the absorption-band has been observed by Schmauss, *Drude's Ann.* II. 280 (1900); VIII. 842 (1902) and Corbino, *Rend. Lincei* (5) X. [2] 137 (1901): *N. Cim.* (5) III. 121 (1902). Zeeman, *Astrophys. J.* XVI. 106 (1902): *Arch. Néerl.* (2 VII 465 (1902); *Rend. Lincei* (5) XI. [1] 470 (1902).

Again when the light traverses the flame in a direction perpendicular to the magnetic field, the phenomenon of rotation does not intervene but the sodium vapour becomes doubly-refracting and this fact has to be considered as well as the change in the absorption produced by the field*.

249. We have seen that the facts of dispersion and absorption can be represented by taking as the differential equations applicable to ponderable media

$$\dot{D} = -\operatorname{Curl} \varpi, \qquad \dot{\varpi} = \operatorname{Curl} e \quad \text{......................(1)},$$

$$D = d + \Sigma d_h \quad \text{...(2)},$$

where d is the polarisation-vector of the pure ether, d_h a vector representative of the intra-molecular vibrations and the components of e are given by

$$(e_1, e_2, e_3) = \frac{1}{2}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)(\Omega^2 d^2),$$

d and d_h being connected by the relations

$$\left.\begin{aligned} a_h u_h + a_h' \dot{u}_h + a_h'' \ddot{u}_h &= u \\ b_h v_h + b_h' \dot{v}_h + b_h'' \ddot{v}_h &= v \\ c_h w_h + c_h' \dot{w}_h + c_h'' \ddot{w}_h &= w \end{aligned}\right\} \quad \text{........................(3)}.$$

It is now necessary to extend these equations, so as to render an account of the phenomena that have just been described†.

If we limit ourselves to linear functions of A, B, C the components of the intensity of the magnetic field H, the symmetry of equations (3) leads us to represent the action of the magnetic force by the addition of a vector at right-angles to the vectors H and $\dot{d}_h$ and proportional to their vector product. We then have in place of (3)

$$\left.\begin{aligned} a_h u_h + a_h' \dot{u}_h + a_h'' \ddot{u}_h + e_h'(C\dot{v}_h - B\dot{w}_h) &= u \\ b_h v_h + b_h' \dot{v}_h + b_h'' \ddot{v}_h + e_h'(A\dot{w}_h - C\dot{u}_h) &= v \\ c_h w_h + c_h' \dot{w}_h + c_h'' \ddot{w}_h + e_h'(B\dot{u}_h - A\dot{v}_h) &= w \end{aligned}\right\} \quad \text{..................(3')},$$

where e_h' is the constant of the magneto-optic effect‡; or in the case of an isotropic medium, taking the z-axis in the direction of the lines of force

$$\left.\begin{aligned} a_h u_h + a_h' \dot{u}_h + a_h'' \ddot{u}_h + e_h' H \dot{v}_h &= u \\ a_h v_h + a_h' \dot{v}_h + a_h'' \ddot{v}_h - e_h' H \dot{u}_h &= v \\ a_h w_h + a_h' \dot{w}_h + a_h'' \ddot{w}_h \qquad\qquad &= w \end{aligned}\right\} \quad \text{..................(3'')}.$$

* Cotton, *C. R.* CXXVII. 953, 1256 (1898); CXXVIII. 294 (1899). Voigt, *Gött. Nachr.* (1898) 329, 355: *Wied. Ann.* LXVII. 345 (1899): *Drude's Ann.* VIII. 872 (1902): *Rend. Lincei* (5) XI. [1] 459 (1902). H. Becquerel, *C. R.* CXXVIII. 145 (1899).

† Voigt, *Gött. Nachr.* (1898), 329, 349, 355: *Wied. Ann.* LXVII. 345; LXVIII. 352; LXIX. 290 (1899): *Drude's Ann.* VI. 784 (1901).

‡ These equations assume that the medium, though crystalline, is magnetically isotropic; cf. Larmor, *Matter and Æther*, p. 198.

250. Let us first consider the propagation of plane homogeneous waves along the lines of force of the field: then all the vectors are dependent upon t and z alone and may be taken proportional to Exp $\{2\pi n(t - z/\overline{\omega})\iota\}$, where $\overline{\omega} = \omega/(1 - \iota\nu)$, ω being the real propagational speed of the waves and ν the index of absorption.

The equations then give

$$\left.\begin{array}{c} U = -\varpi_2/\overline{\omega}, \quad V = \varpi_1/\overline{\omega}, \quad W = 0 \\ \varpi_1 = \Omega^2 v/\overline{\omega}, \quad \varpi_2 = -\Omega^2 u/\overline{\omega}, \quad \varpi_3 = 0 \\ (a_h + \iota\,.\,2\pi n a_h' - 4\pi^2 n^2 a_h'')\,u_h + \iota\,.\,e_h' H 2\pi n v_h = u \\ (a_h + \iota\,.\,2\pi n a_h' - 4\pi^2 n^2 a_h'')\,v_h - \iota\,.\,e_h' H 2\pi n u_h = v \end{array}\right\} \quad \ldots\ldots\ldots(4),$$

and we have

$$\left.\begin{array}{c} U = (\Omega/\overline{\omega})^2 u, \quad V = (\Omega/\overline{\omega})^2 v \\ (u_h \pm \iota v_h)\,\{a_h + 2\pi n(\iota a_h' \pm e_h' H) - 4\pi^2 n^2 a_h''\} = u \pm \iota v \end{array}\right\} \ldots\ldots\ldots(5),$$

whence, writing

$$2\pi n_h = \sqrt{a_h/a_h''}, \quad \alpha_h = 2\pi n_h^2 a_h'/a_h, \quad e_h = 2\pi n_h^2 e_h'/a_h, \quad \epsilon_h = n_h/a_h,$$

we obtain

$$(\Omega/\overline{\omega})^2 (u \pm \iota v) = U \pm \iota V$$
$$= [1 - \Sigma\epsilon_h n_h \{n^2 - n_h^2 - n(\iota\alpha_h \pm e_h H)\}^{-1}](u \pm \iota v) \quad \ldots\ldots\ldots(6).$$

Hence, either

$$u - \iota v = 0 \text{ and } (\Omega/\overline{\omega})^2 = 1 - \Sigma\epsilon_h n_h \{n^2 - n_h^2 - n(\iota\alpha_h + e_h H)\}^{-1} \ldots\ldots(7),$$

or

$$u + \iota v = 0 \text{ and } (\Omega/\overline{\omega})^2 = 1 - \Sigma\epsilon_h n_h \{n^2 - n_h^2 - n(\iota\alpha_h - e_h H)\}^{-1} \ldots\ldots(8).$$

In the first case we have a circularly polarised stream, of the same sign as the current that would produce the field, having a complex velocity $\overline{\omega}_+$ given by (7); and in the second case the circularly polarised stream has the opposite sign and its complex velocity $\overline{\omega}_-$ is determined from (8). The rotation of the plane of polarisation of a stream of plane polarised light produced by traversing a length l of the medium in the direction of the lines of force is

$$R = \pi n l(\omega_-^{-1} - \omega_+^{-1}) = \frac{\pi n l \omega_0}{2}\left(\frac{1}{\omega_-^2} - \frac{1}{\omega_+^2}\right) \quad \ldots\ldots\ldots\ldots\ldots(9),$$

where ω_0 is a mean velocity given by $2\omega_0^{-1} = \omega_-^{-1} + \omega_+^{-1}$.

If the medium be transparent, $\alpha_h = 0$ and

$$(\Omega/\omega_\pm)^2 = 1 - \Sigma\epsilon_h n_h \{n^2 - n_h^2 \mp e_h H\,.\,n\}^{-1},$$

whence

$$R = \frac{\pi n^2 l \omega_0}{\Omega^2} \Sigma \frac{n_h \epsilon_h e_h H}{(n^2 - n_h^2)^2 - e_h^2 H^2 n^2}$$
$$\doteqdot \frac{\pi n^2 l \omega_0 H}{\Omega^2} \Sigma \frac{n_h \epsilon_h e_h}{(n^2 - n_h^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

if $e_h H$ be small compared with $n - n_h$, that is, when the frequency of the light is not near one of the critical frequencies.

To the same approximation, ω_0 is the propagational speed before the field is established and is given by

$$(\Omega/\omega_0)^2 = \mu_0^2 = 1 - \Sigma n_h \epsilon_h (n^2 - n_h^2)^{-1},$$

whence $$\mu_0 (d\mu_0/dn) = n\Sigma n_h \epsilon_h (n^2 - n_h^2)^{-2}.$$

Hence if the intra-molecular vibrations have a single period

$$R = klHn (d\mu_0/dn),$$

in accordance with Becquerel's law, where k is a constant for the given medium.

When the frequency of the light is nearly the same as one of the critical frequencies n_h, we may simplify the expressions for the complex wave-velocities as in Chapter XVII, and we obtain

$$(\Omega/\overline{\omega}_\pm)^2 = 1 - \epsilon_h \{2(n - n_h) \mp e_h H - \iota\alpha_h\}^{-1} \quad \text{..............(11).}$$

This equation only differs from that obtained when the medium is removed from the magnetic field by having $n \mp e_h H/2$ written in place of n: it therefore follows that the curves of dispersion and absorption for each of the circularly polarised streams have the same form as those of the medium when uninfluenced by the magnetic force: the curves for the positive stream being obtained by a displacement parallel to the axis of abscissæ by an amount $e_h H/2$, and those for the negative stream by an equal displacement in the opposite direction. Thus the effect of the field is to resolve a single absorption-band into two symmetrically placed with respect to the original band*.

Writing

$$2(n - n_h)/\alpha_h = \Delta, \qquad e_h H/\alpha_h = P, \qquad \epsilon_h/\alpha_h = A,$$

and separating real and imaginary parts we have

$$\left.\begin{aligned} \left(\frac{\Omega}{\omega_\pm}\right)^2 (1 - \nu_\pm^2) &= 1 - \frac{A(\Delta \mp P)}{(\Delta \mp P)^2 + 1} \\ 2\left(\frac{\Omega}{\omega_\pm}\right)^2 \nu_\pm &= \frac{A}{(\Delta \mp P)^2 + 1} \end{aligned}\right\} \quad \text{..................(12),}$$

whence if ν be very small

$$R = \frac{\pi n l \omega_0 A P}{\Omega^2} \frac{\Delta^2 - P^2 - 1}{(\Delta^2 - P^2 - 1)^2 + 4\Delta^2} \quad \text{..................(13),}$$

and at a moderate distance from the region of absorption

$$R = \frac{\pi n l \omega_0 A P}{\Omega^2 \Delta^2} = \frac{\pi n l \omega_0}{4\Omega^2} \frac{\epsilon_h e_h H}{(n - n_h)^2}.$$

* If there be two vectors d_h for which the frequencies are the same, while the magneto-optic parameters are different, the application of the magnetic force will resolve a single band into four; and so on.

Comparing this expression with (10) which to the same degree of approximation may be taken as the rotation for frequencies differing widely from a critical frequency, we see that the rotation, though imperceptible in other parts of the spectrum, may attain measurable dimensions as an absorption-band is approached.

Within the region of absorption, we may write

$$\mu_0 R = KP \frac{\Delta^2 - P^2 - 1}{(\Delta^2 - P^2 - 1)^2 + 4\Delta^2},$$

where μ_0, K may be regarded as constant, μ_0 being a mean refractive index and $K = \pi n l A/\Omega$. The rotation vanishes when $\Delta = \pm\sqrt{P^2+1}$, and is a maximum or a minimum when $\Delta = 0$ and when

$$\Delta^2 = P^2 + 1 \pm 2\sqrt{P^2+1} \quad \text{.........................(14)},$$

its value then being given by

$$\mu_0 R = -KP/(P^2+1),$$

and $$\mu_0 R = \pm K\{\sqrt{P^2+1} \mp 1\}/(4P),$$

respectively.

If P be less than $\sqrt{3}$, only the upper sign in (14) gives a real value of Δ and then

$$(\mu_0 R)_1 = -KP/(P^2+1) \text{ is a minimum value,}$$

$$(\mu_0 R)_2 = K\{\sqrt{P^2+1} - 1\}/(4P) \text{ is a maximum value;}$$

on the other hand when P exceeds $\sqrt{3}$, both signs correspond to real values of Δ and

$$(\mu_0 R)_1 = -KP/(P^2+1) \text{ is a maximum value,}$$

$$(\mu_0 R)_2 = -K\{\sqrt{P^2+1} + 1\}/(4P) \text{ is a minimum value,}$$

$$(\mu_0 R)_3 = K\{\sqrt{P^2+1} - 1\}/(4P) \text{ is a maximum value.}$$

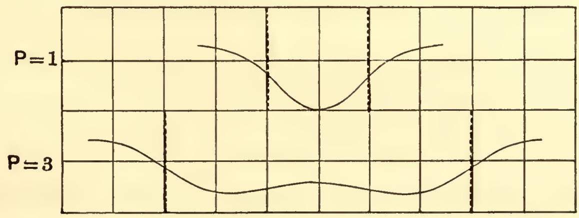


Fig. 64.

The curves in figure (64) represent the values of $\mu_0 R$ when $P = 1$ and $P = 3$, the unit for Δ being $\frac{3}{4}$ cm., that for $\mu_0 R$ $1\frac{1}{2}$ cm.: the vertical dotted lines give the approximate places of the maximum of absorption.

251. When the light travels in a direction perpendicular to that of the field, or parallel, say, to the axis of x, the vectors are proportional to

$$\text{Exp}\,\{2\pi n\,(t - x/\overline{\omega})\,\iota\},$$

and we then have

$$\left.\begin{aligned} U=0,\quad V=-\varpi_3/\overline{\omega},\quad W=\varpi_2/\overline{\omega} \\ \varpi_1=0,\quad \varpi_2=\Omega^2 w/\overline{\omega},\quad \varpi_3=-\,\Omega^2 v/\overline{\omega} \end{aligned}\right\}\quad\text{.................(15)},$$

with

$$\left.\begin{aligned} (n_h^2+\iota\alpha_h n-n^2)\,u_h+\iota e_h H n v_h &= n_h\epsilon_h u \\ (n_h^2+\iota\alpha_h n-n^2)\,v_h-\iota e_h H n u_h &= n_h\epsilon_h v \\ (n_h^2+\iota\alpha_h n-n^2)\,w_h &= n_h\epsilon_h w \end{aligned}\right\}\quad\text{..............(16)};$$

whence $\qquad U=0,\qquad V=(\Omega/\overline{\omega})^2\,v,\qquad W=(\Omega/\overline{\omega})^2\,w,$

$$(u_h\pm\iota v_h)\,(n_h^2+\iota\alpha_h n-n^2\pm e_h H n)=n_h\epsilon_h\,(u\pm\iota v),$$

and

$$\left.\begin{aligned} \pm(\Omega/\overline{\omega})^2\,v\iota &= \{1-\Sigma n_h\epsilon_h\,(n^2-n_h^2\mp e_h H n-\iota\alpha_h n)^{-1}\}\,(u\pm\iota v) \\ (\Omega/\overline{\omega})^2\,w &= \{1-\Sigma n_h\epsilon_h\,(n^2-n_h^2-\iota\alpha_h n)^{-1}\}\,w \end{aligned}\right\}\quad\text{...(17)}.$$

Hence, if $w\neq 0$, we have

$$(\Omega/\overline{\omega})^2=1-\Sigma n_h\epsilon_h\,(n^2-n_h^2-\iota\alpha_h n)^{-1}\quad\text{..................(18)},$$

and with this value of $\overline{\omega}$, the first of equations (17) is only satisfied by $u=v=0$. This then is the case of plane polarised light with its polarisation-vector parallel to the lines of force, and we see that the absorption and the propagational-speed are unaffected by the magnetic force*.

If on the other hand $w=0$, then

$$u=\left\{-1+\frac{1}{p_-}\left(\frac{\Omega}{\overline{\omega}}\right)^2\right\}v\iota=\left\{1-\frac{1}{p_+}\left(\frac{\Omega}{\overline{\omega}}\right)^2\right\}v\iota,$$

where $\qquad p_\pm=1-\Sigma n_h\epsilon_h\,(n^2-n_h^2\pm e_h\,Hn-\iota\alpha_h n)^{-1},$

whence $\qquad \left(\dfrac{1}{p_+}+\dfrac{1}{p_-}\right)\left(\dfrac{\Omega}{\overline{\omega}}\right)^2=2,$

or

$$\left(\frac{\Omega}{\overline{\omega}}\right)^2=1-\Sigma\frac{n_h\epsilon_h\,(n^2-n_h^2-\iota\alpha_h n)}{(n^2-n_h^2-\iota\alpha_h n)^2-e_h^2H^2n^2}$$

$$-\frac{\left\{\Sigma\dfrac{n_h\epsilon_h\,e_h\,Hn}{(n^2-n_h^2-\iota\alpha_h n)^2-e_h^2H^2n^2}\right\}^2}{1-\Sigma\dfrac{n_h\epsilon_h\,(n^2-n_h^2-\iota\alpha_h n)}{(n^2-n_h^2-\iota\alpha_h n)^2-e_h^2H^2n^2}}\quad\text{.........................(19)}.$$

In this case the polarisation-vector is perpendicular to the direction of the

* In order to explain the resolution of this absorption-band by the action of the magnetic force, Voigt introduces on the left-hand side of the third of equations (3″) a term $f_h w_h'$, where

$$f_h w_h+a_h w_h'+\alpha_h'\frac{\partial w_h'}{\partial t}+\beta_h''\frac{\partial^2 w_h'}{\partial t^2}=0,$$

and f_h is a function of H, that vanishes when $H=0$.

field, and the application of the magnetic force has the effect of altering the speed of the waves and also of resolving each absorption-band into two components*.

Hence in a direction perpendicular to the lines of force there are two streams of permanent type propagated with different velocities. The medium thus becomes doubly refracting under the influence of the magnetic force, but in general the difference in the speeds of the two waves will only become marked when their frequency is near one of the critical values.

As the formulæ in their general form are too complicated to be of any special interest, we will consider only the case in which there is a single vector d_h dependent upon the intra-molecular vibrations and consequently only a single absorption-band, when the medium is removed from the magnetic field.

Using the subscripts $(_y)$ and $(_z)$ to distinguish between the streams that have their polarisation-vectors perpendicular and parallel respectively to the lines of force, we have

$$\left(\frac{\Omega}{\omega_y}\right)^2 = 1 - \frac{n_h\epsilon_h(n^2 - n_h^2 - n_h\epsilon_h - \iota\alpha_h n)}{\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2 - \alpha_h^2n^2\} - \iota\alpha_h n\{2(n^2-n_h^2) - n_h\epsilon_h\}},$$

whence

$$\left.\begin{aligned} &\left(\frac{\Omega}{\omega_y}\right)^2(1-\nu_y^2) \\ &\quad = 1 - \frac{n_h\epsilon_h[(n^2-n_h^2-n_h\epsilon_h)\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2\} + \alpha_h^2n^2(n^2-n_h^2)]}{\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2 - \alpha_h^2n^2\}^2 + \alpha_h^2n^2\{2(n^2-n_h^2) - n_h\epsilon_h\}^2} \\ &2\left(\frac{\Omega}{\omega_y}\right)^2\nu_y = \frac{n_h\epsilon_h\alpha_h n\{(n^2-n_h^2-n_h\epsilon_h)^2 + e_h^2H^2n^2 + \alpha_h^2n^2\}}{\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2 - \alpha_h^2n^2\}^2 + \alpha_h^2n^2\{2(n^2-n_h^2) - n_h\epsilon_h\}^2} \end{aligned}\right\} \quad \text{.........(20);}$$

also

$$\left.\begin{aligned} \left(\frac{\Omega}{\omega_z}\right)^2(1-\nu_z^2) &= 1 - \frac{n_h\epsilon_h(n^2-n_h^2)}{(n^2-n_h^2)^2 + \alpha_h^2n^2} \\ 2\left(\frac{\Omega}{\omega_z}\right)^2\nu_z &= \frac{n_h\epsilon_h\alpha_h n}{(n^2-n_h^2)^2 + \alpha_h^2n^2} \end{aligned}\right\} \quad \text{..................(21).}$$

The difference of phase between these two streams produced by a passage through a length l of the medium is

$$\delta = 2\pi nl(\omega_z^{-1} - \omega_y^{-1}) = \pi nl\omega_0(\omega_z^{-2} - \omega_y^{-2}),$$

where ω_0 is a mean velocity given by $2\omega_0^{-1} = \omega_z^{-1} + \omega_y^{-1}$. Hence if the indices of absorption be very small, we obtain from the above formulæ

$$\delta = \frac{\pi nl\omega_0 n_h\epsilon_h e_h^2H^2n^2}{\Omega^2}$$

$$\times \frac{(n^2-n_h^2)\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2 - \alpha_h^2n^2\} - \alpha_h^2n^2\{2(n^2-n_h^2) - n_h\epsilon_h\}}{\{(n^2-n_h^2)^2 + \alpha_h^2n^2\}[\{(n^2-n_h^2)^2 - n_h\epsilon_h(n^2-n_h^2) - e_h^2H^2n^2 - \alpha_h^2n^2\}^2 + \alpha_h^2n^2\{2(n^2-n_h^2) - n_h\epsilon_h\}^2]}$$

.........(22);

* See note on page 378.

when the frequency of the light is near the critical value n_h, this reduces to

$$\delta_0 = \frac{\pi l \omega_0 n_h \epsilon_h e_h^2 H^2}{\Omega^2}$$
$$\times \frac{2(n-n_h)\{4(n-n_h)^2 - 2\epsilon_h(n-n_h) - e_h^2 H^2 - \alpha_h^2\} - \alpha_h^2\{4(n-n_h) - \epsilon_h\}}{\{4(n-n_h)^2+\alpha_h^2\}\left[\{4(n-n_h)^2 - 2\epsilon_h(n-n_h) - e_h^2H^2 - \alpha_h^2\}^2 + \alpha_h^2\{4(n-n_h)-\epsilon_h\}^2\right]} \quad \text{.........(23).}$$

A further simplification is introduced, when α_h is very small, as (22) and (23) then become

$$\delta = \frac{\pi n l \omega_0}{\Omega^2} \frac{n_h \epsilon_h e_h^2 H^2 n^2}{(n^2 - n_h^2)\{(n^2 - n_h^2)^2 - n_h \epsilon_h (n^2 - n_h^2) - e_h^2 H^2 n^2\}} \quad \text{.......(24),}$$

$$\delta_0 = \frac{\pi l \omega_0}{\Omega^2} \frac{n_h \epsilon_h e_h^2 H^2}{2(n-n_h)\{4(n-n_h)^2 - 2\epsilon_h(n-n_h) - e_h^2 H^2\}} \quad \text{...........(25).}$$

Now (24) may be written

$$\delta = \frac{\pi n l \omega_0}{\Omega^2}\left[\frac{n_h \epsilon_h e_h^2 H^2 n^2}{(n^2-n_h^2)\{(n^2-n_h^2)^2 - e_h^2 H^2 n^2\}} + \frac{n_h^2 \epsilon_h^2 e_h^2 H^2 n^2}{\{(n^2-n_h^2)^2 - e_h^2 H^2 n^2\}^2}\left\{1 - \frac{n_h \epsilon_h (n^2 - n_h^2)}{(n^2-n_h^2)^2 - e_h^2 H^2 n^2}\right\}^{-1}\right] \text{...(26);}$$

but in the case of a glowing vapour that has, when there is no magnetic field, a refractive index nearly equal to unity for frequencies not too close to that of the light absorbed, the first of equations (21) shows that $n_h \epsilon_h$ is very small compared with $n^2 - n_h^2$, whence if $e_h H$ be not nearly equal to $n - n_h$, we have

$$\delta = \frac{\pi n l \omega_0}{\Omega^2} \frac{n_h \epsilon_h e_h^2 H^2 n^2}{(n^2 - n_h^2)\{(n^2-n_h^2)^2 - e_h^2 H^2 n^2\}},$$

holding for parts of the spectrum at a considerable distance from the absorption-band: on the other hand at a moderate distance from the region of absorption

$$\delta_0 = \frac{\pi l \omega_0}{\Omega^2} \frac{n_h \epsilon_h e_h^2 H^2}{2(n-n_h)\{4(n-n_h)^2 - e_h^2 H^2\}}.$$

Thus we see that a vapour, such as that of a soda-flame, when placed in a magnetic field, may become strongly doubly refracting for light of periods near that which it emits.

252. Returning to the fundamental equations (1), (2), (3′) we have in the case of periodic vibrations

$$\frac{\partial u_h}{\partial t} = \iota 2\pi n u_h, \quad \frac{\partial^2 u_h}{\partial t^2} = -4\pi^2 n^2 u_h,$$

and so on.. Hence equations (3′) take the form

$$\left.\begin{aligned} A_h u_h + e_h' (C\dot{v}_h - B\dot{w}_h) &= u \\ B_h v_h + e_h' (A\dot{w}_h - C\dot{u}_h) &= v \\ C_h w_h + e_h' (B\dot{u}_h - A\dot{v}_h) &= w \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

whence

$$A_h u_h = u - e_h' (C\dot{v}_h - B\dot{w}_h) \fallingdotseq u - e_h' \left(\frac{C}{B_h}\dot{v} - \frac{B}{C_h}\dot{w}\right) \ldots\ldots\ldots(28),$$

and two similar equations, if we neglect the squares and products of the small quantities $e_h'A$, $e_h'B$, $e_h'C$.

Hence $$U = \left(1 + \Sigma \frac{1}{A_h}\right) u - C\Sigma \frac{e_h}{A_h B_h} \dot{v} + B\Sigma \frac{e_h}{C_h A_h} \dot{w} \ldots\ldots\ldots(29),$$

and to the same degree of approximation, we obtain for u, v, w expressions of the form

$$\left.\begin{aligned} \Omega^2 u &= \alpha^{-1} U + b_3 \dot{V} - b_2 \dot{W} \\ \Omega^2 v &= \beta^{-1} V + b_1 \dot{W} - b_3 \dot{U} \\ \Omega^2 w &= \gamma^{-1} W + b_2 \dot{U} - b_1 \dot{V} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

Hence if b_1, b_2, b_3 be the components of a vector B, the differential equations take the form

$$\dot{D} = -\operatorname{Curl} \varpi, \quad \dot{\varpi} = \operatorname{Curl} E + B\nabla\dot{D} \ldots\ldots\ldots\ldots\ldots(31),$$

where the components of E are given by

$$(E_1, E_2, E_3) = \left(\frac{\partial}{\partial U}, \frac{\partial}{\partial V}, \frac{\partial}{\partial W}\right)\Phi \ldots\ldots\ldots\ldots\ldots\ldots(32),$$

with $$2\Phi = (\alpha^{-1} U^2 + \beta^{-1} V^2 + \gamma^{-1} W^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(33);$$

more generally when the coordinate axes are not coincident with the axes of optical symmetry, we have

$$2\Phi = a_{11} U^2 + a_{22} V^2 + a_{33} W^2 + 2a_{23} VW + 2a_{31} WU + 2a_{12} UV \ldots(34).$$

In this form the equations admit of the following simple geometrical interpretation*.

In any direction within a magnetically active crystal two oppositely polarised streams can be propagated that have their planes of maximum polarisation parallel respectively to the axes of the central section of the ellipsoid of polarisation parallel to the plane of the waves: the propagational speeds of these waves are respectively in excess or defect of the speed represented by the reciprocal of the length of either axis of the section by an amount that is inversely proportional to the period of the vibrations and directly proportional to the length of the axis, to the ratio of the axes of the elliptic vibration perpendicular and parallel to the axis, and to the component

* *Proc. R. S.* LXX. 40 (1902).

perpendicular to the section of a vector dependent upon the intensity of the magnetic field.

From equations (31)—(33), the characteristics of wave-propagation in magnetically active media may be determined as in §§ 240, 241, but only one point of special interest need be mentioned.

When the medium is isotropic, the wave-velocity ω in a direction making an angle θ with that of the vector B is given by

$$\omega^2 = \Omega^2 \pm 2\pi n B \cos\theta,$$

where n is the frequency of the waves and Ω is the propagational speed when the field is suppressed. On account of the smallness of the last term, this may be written

$$\omega^2 = \Omega^2 \pm \frac{2\pi n B \cos\theta}{\Omega}\,\omega,$$

and we see that on the application of the magnetic force the original spherical surface of wave-quickness becomes a surface of two sheets, that are approximately spheres of the same radius, the centres of which are displaced in opposite directions parallel to the magnetic field*.

The boundary conditions obtained at once from (1) are, the interface being the plane $x = 0$, the continuity of ϖ_2, ϖ_3, e_2, e_3; the two latter giving in terms of U, V, W the continuity of

$$\frac{\partial\Phi}{\partial V} + b_1\dot{W} - b_3\dot{U}, \quad \text{and} \quad \frac{\partial\Phi}{\partial W} + b_2\dot{U} - b_1\dot{V}.$$

To these may be added the continuity of ϖ_1 and U, since div $\varpi = 0$, div $D = 0$, but these are clearly involved in the four preceding conditions.

These equations suffice for the solution of the problem of reflection at the surface of a magnetised medium†.

* Cornu, *C. R.* XCIX. 1045 (1884).

† Cf. Goldhammer, *Wied. Ann.* XLVI. 71 (1892); XLVIII. 740 (1893); LXV. 111 (1898). Drude, *ibid.* XLVI. 353 (1892); LII. 496 (1894); LXII. 687 (1897). Basset, *Phil. Trans.* CLXXXII. A, 371 (1891); *Amer. J. of Math.* XIX. 60 (1897). Larmor, *B. A. Report* (1893), p. 335; *Proc. Lond. Math. Soc.* XXIV. 280 (1893). Leathem, *Phil. Trans.* CXC. A, 89 (1897). Micheli, *Drude's Ann.* I. 542 (1900).

APPENDIX I.

PROPERTIES OF BESSEL'S AND STRUVE'S FUNCTIONS.

1. BESSEL'S Function of the order n may be defined by the equations

$$J_n(x) = \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \int_0^\pi \sin^{2n} \omega \cos(x \cos \omega)\, d\omega$$

$$= \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \int_0^\pi \sin^{2n} \omega\, e^{\iota x \cos \omega}\, d\omega$$

$$= \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \int_{-1}^1 (1-u^2)^{n-\frac{1}{2}} e^{\iota x u}\, du \quad \text{..................(1)},$$

where n is real and positive and the argument x may be any real or complex quantity.

Taking the first form and writing

$$\cos(x \cos \omega) = \sum_0^\infty (-1)^s \frac{x^{2s} \cos^{2s} \omega}{\lfloor 2s},$$

we obtain

$$J_n(x) = \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \sum_0^\infty (-1)^s \frac{x^{2s}}{\lfloor 2s} \int_0^\pi \sin^{2n} \omega \cos^{2s} \omega\, d\omega$$

$$= \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \sum_0^\infty (-1)^s \frac{x^{2s}}{\lfloor 2s} \frac{\Gamma(n+\frac{1}{2}) \Gamma(s+\frac{1}{2})}{\Gamma(n+s+1)}$$

$$= \sum_0^\infty (-1)^s \frac{x^{n+2s}}{2^{n+2s} \Gamma(s+1) \Gamma(n+s+1)} \quad \text{........................(2)}.$$

When n is not a positive integer, $J_{-n}(x)$ is defined by

$$J_{-n}(x) = \sum_0^\infty (-1)^s \frac{x^{-n+2s}}{2^{-n+2s} \Gamma(s+1) \Gamma(-n+s+1)} \quad \text{..............(3)},$$

and if n be a positive integer,

$$J_{-n}(x) = \mathrm{Lt}_{\epsilon=0} J_{-(n+\epsilon)} = (-1)^n J_n(x) \quad \text{..................(4)}.$$

2. We have from (2)

$$\frac{2n}{x}J_n(x) = \sum_0^\infty (-1)^s \frac{2nx^{n+2s-1}}{2^{n+2s}\Gamma(s+1)\Gamma(n+s+1)}$$

$$= \sum_0^\infty (-1)^s \frac{x^{n+2s-1}}{2^{n+2s-1}\Gamma(s+1)\Gamma(n+s)} - \sum_1^\infty (-1)^s \frac{x^{n+2s-1}}{2^{n+2s-1}\Gamma(s)\Gamma(n+s+1)}$$

$$= J_{n-1} + \sum_0^\infty (-1)^s \frac{x^{n+2s+1}}{2^{n+2s+1}\Gamma(s+1)\Gamma(n+s+2)}$$

$$= J_{n-1} + J_{n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Writing $n+1$ for n, this relation gives

$$xJ_n(x) = 2(n+1)J_{n+1}(x) - xJ_{n+2}(x)$$

$$= 2(n+1)J_{n+1}(x) - 2(n+3)J_{n+3}(x) + xJ_{n+4}(x)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 2\sum_0^\infty (-1)^r (n+2r+1) J_{n+2r+1}(x) \quad \ldots\ldots\ldots\ldots (6).$$

From this it follows that

$$\begin{aligned}\frac{x^2}{4}J_n(x) = (n+1)&\{(n+2)J_{n+2}(x) - (n+4)J_{n+4}(x) + \ldots\} \\ &- (n+3)\{(n+4)J_{n+4}(x) - (n+6)J_{n+6}(x) + \ldots\} \\ &+ (n+5)\{(n+6)J_{n+6}(x) - \ldots\} \\ &- \ldots\ldots\ldots\ldots\end{aligned}$$

$$\begin{aligned}&= (n+1)(n+2)J_{n+2}(x) - 2(n+2)(n+4)J_{n+4}(x) \\ &\quad + 3(n+3)(n+6)J_{n+6}(x) - 4(n+4)(n+8)J_{n+8}(x) \\ &\quad + \ldots\ldots\ldots\ldots\end{aligned}$$

Similarly

$$\begin{aligned}\left(\frac{x}{2}\right)^3 J_n(x) = (n+1)(n+2)&\{(n+3)J_{n+3} - (n+5)J_{n+5} + (n+7)J_{n+7} - \ldots\} \\ &- 2(n+2)(n+4)\{(n+5)J_{n+5} - (n+7)J_{n+7} + \ldots\} \\ &+ 3(n+3)(n+6)\{(n+7)J_{n+7} - \ldots\} \\ &- \ldots\ldots\ldots\ldots\end{aligned}$$

$$\begin{aligned}&= (n+1)(n+2)(n+3)J_{n+3} - 3(n+2)(n+3)(n+5)J_{n+5} \\ &\quad + \frac{3.4}{1.2}(n+3)(n+4)(n+7)J_{n+7} - \frac{3.4.5}{1.2.3}(n+4)(n+5)(n+9)J_{n+9} \\ &\quad + \ldots\ldots\ldots\ldots\end{aligned}$$

These special results suggest the general theorem

$$\begin{aligned}\left(\frac{x}{2}\right)^r J_n(x) = &\frac{\Gamma(n+r)}{\Gamma(n+1)}(n+r)J_{n+r} - r\frac{\Gamma(n+r+1)}{\Gamma(n+2)}(n+r+2)J_{n+r+2} \\ &+ \frac{r(r+1)}{1.2}\frac{\Gamma(n+r+2)}{\Gamma(n+3)}(n+r+4)J_{n+r+4} \\ &- \frac{r(r+1)(r+2)}{1.2.3}\frac{\Gamma(n+r+3)}{\Gamma(n+4)}(n+r+6)J_{n+r+6} \\ &+ \ldots\ldots\end{aligned}$$

Suppose that this is true for a particular value of r, then for the next greater value

$$\begin{aligned}
\left(\frac{x}{2}\right)^{r+1} J_n(x) = {} & \frac{\Gamma(n+r)}{\Gamma(n+1)}(n+r)\left\{(n+r+1)J_{n+r+1} - (n+r+3)J_{n+r+3} + \dots\right\} \\
& - r\frac{\Gamma(n+r+1)}{\Gamma(n+2)}(n+r+2)\left\{(n+r+3)J_{n+r+3}\right. \\
& \qquad \left. - (n+r+5)J_{n+r+5} + \dots\right\} + \dots \\
= {} & \frac{\Gamma(n+r+1)}{\Gamma(n+1)}(n+r+1)J_{n+r+1} \\
& - (r+1)\frac{\Gamma(n+r+2)}{\Gamma(n+2)}(n+r+3)J_{n+r+3} \\
& + \left\{(r+1)\frac{\Gamma(n+r+2)}{\Gamma(n+2)} + \frac{r(r+1)}{1.2}\right. \\
& \qquad \left. \times \frac{\Gamma(n+r+2)}{\Gamma(n+3)}(n+r+4)\right\}(n+r+5)J_{n+r+5} \\
& - \left\{(r+1)\frac{\Gamma(n+r+2)}{\Gamma(n+2)} + \frac{r(r+1)}{1.2}\frac{\Gamma(n+r+2)}{\Gamma(n+3)}(n+r+4)\right. \\
& \left. + \frac{r(r+1)(r+2)}{1.2.3}\frac{\Gamma(n+r+3)}{\Gamma(n+4)}(n+r+6)\right\}(n+r+7)J_{n+r+7} \\
= {} & \frac{\Gamma(n+r+1)}{\Gamma(n+1)}(n+r+1)J_{n+r+1} \\
& - (r+1)\frac{\Gamma(n+r+2)}{\Gamma(n+2)}(n+r+3)J_{n+r+3} \\
& + \frac{(r+1)(r+2)}{1.2}\frac{\Gamma(n+r+3)}{\Gamma(n+3)}(n+r+5)J_{n+r+5} \\
& - \left\{\frac{(r+1)(r+2)}{1.2}\frac{\Gamma(n+r+3)}{\Gamma(n+3)} + \frac{r(r+1)(r+2)}{1.2.3}\right. \\
& \qquad \left. \times \frac{\Gamma(n+r+3)}{\Gamma(n+4)}(n+r+6)\right\}(n+r+7)J_{n+r+7} \\
& + \dots\dots \\
= {} & \frac{\Gamma(n+r+1)}{\Gamma(n+1)}(n+r+1)J_{n+r+1} \\
& - (r+1)\frac{\Gamma(n+r+2)}{\Gamma(n+2)}(n+r+3)J_{n+r+3} \\
& + \frac{(r+1)(r+2)}{1.2}\frac{\Gamma(n+r+3)}{\Gamma(n+3)}(n+r+5)J_{n+r+5} \\
& - \frac{(r+1)(r+2)(r+3)}{1.2.3}\frac{\Gamma(n+r+4)}{\Gamma(n+4)}(n+r+7)J_{n+r+7} \\
& + \dots\dots,
\end{aligned}$$

the form of which shows that the theorem remains true for the next greater value of r, therefore for the value of r next succeeding and so on. But it is true when $r=2$, and therefore for all positive integral values of r,

$$\left(\frac{x}{2}\right)^r J_n(x) = \frac{\Gamma(n+r)}{\Gamma(n+1)}(n+r)\,J_{n+r}(x)$$
$$+\sum_1^\infty (-1)^s \frac{r(r+1)\ldots(r+s-1)}{1.2\ldots s}\frac{\Gamma(n+r+s)}{\Gamma(n+s+1)}(n+r+2s)\,J_{n+r+2s}(x) \ldots (7).$$

3. From the general expression for $J_n(x)$, we find that

$$\frac{\partial}{\partial x}(x^n J_n) = \sum_0^\infty (-1)^s \frac{(2n+2s)\,x^{2n+2s-1}}{2^{n+2s}\,\Gamma(s+1)\,\Gamma(n+s+1)}$$
$$= \sum_0^\infty (-1)^s \frac{x^{2n+2s-1}}{2^{n+2s-1}\,\Gamma(s+1)\,\Gamma(n+s)} = x^n J_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and

$$\frac{\partial}{\partial x}(x^{-n} J_n) = \sum_0^\infty (-1)^s \frac{2s\,.\,x^{2s-1}}{2^{n+2s}\,\Gamma(s+1)\,\Gamma(n+s+1)}$$
$$= x^{-n}\sum_1^\infty (-1)^s \frac{x^{n+2s-1}}{2^{n+2s-1}\,\Gamma(s)\,\Gamma(n+s+1)}$$
$$= -x^{-n}\sum_0^\infty (-1)^s \frac{x^{n+2s+1}}{2^{n+2s+1}\,\Gamma(s+1)\,\Gamma(n+s+2)} = -x^{-n} J_{n+1} \quad \ldots (9).$$

By performing the differentiations on the left-hand sides of these equations, we find

$$\frac{\partial J_n}{\partial x} = J_{n-1} - nx^{-1}J_n,$$

and

$$\frac{\partial J_n}{\partial x} = -J_{n+1} + nx^{-1}J_n,$$

whence by addition

$$\frac{\partial J_n}{\partial x} = \tfrac{1}{2}(J_{n-1} - J_{n+1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

4. From the expressions in the last section, we obtain

$$\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$\int_0^x x^{-n} J_{n+1}(x)\,dx = \left[-x^{-n}J_n(x)\right]_0^x$$
$$= -x^{-n}J_n(x) + \frac{1}{2^n\,\Gamma(n+1)} \ldots\ldots\ldots (12).$$

Again

$$\int_0^x J_n(x)\,dx = \int_0^x \frac{x^{n+1}}{x^{n+1}} J_n(x)\,dx$$
$$= J_{n+1}(x) + (n+1)\int_0^x \frac{J_{n+1}(x)}{x}\,dx, \text{ from (11)}$$
$$= J_{n+1}(x) + \tfrac{1}{2}\int_0^x J_n(x)\,dx + \tfrac{1}{2}\int_0^x J_{n+2}(x)\,dx, \text{ from (5)},$$

$$\therefore \quad \int_0^x J_n(x)\,dx = 2J_{n+1}(x) + \int_0^x J_{n+2}(x)\,dx$$

$$= 2J_{n+1}(x) + 2J_{n+3}(x) + \int_0^x J_{n+4}(x)\,dx$$

$$= 2\sum_0^\infty J_{n+2s+1}(x) \quad \text{..........(13).}$$

Hence

$$\int_0^x \int_0^x J_n(x)\,dx^2 = 2\sum_0^\infty \int_0^x J_{n+2s+1}\,dx$$

$$\begin{aligned} = 2^2 \{&J_{n+2} + J_{n+4} + J_{n+6} + \ldots \\ &+ J_{n+4} + J_{n+6} + \ldots \\ &+ J_{n+6} + \ldots \\ &+ \ldots\} \end{aligned}$$

$$= 2^2 \sum_0^\infty (s+1)\, J_{n+2s+2}(x) \quad \text{..........(14).}$$

By partial integration we obtain

$$\int x J_n dx = x\int J_n dx - \iint J_n dx^2$$

$$= 2\left[x\,(J_{n+1} + J_{n+3} + J_{n+5} + \ldots)\right.$$

$$\left. - x\left(\frac{2}{x}J_{n+2} + \frac{4}{x}J_{n+4} + \frac{6}{x}J_{n+6} + \ldots\right)\right];$$

but

$$\frac{2}{x}J_{n+2s} = \frac{1}{n+2s}(J_{n+2s-1} + J_{n+2s+1}),$$

$$\therefore \quad \int_0^x xJ_n(x)\,dx = 2nx\sum_0^\infty \frac{n+2s+1}{(n+2s)(n+2s+2)} J_{n+2s+1}(x) \quad \text{......(15).}$$

Multiplying (5) by J_n, we have

$$\frac{2n}{x}J_n^2 = J_n(J_{n-1} + J_{n+1}),$$

and this may be written

$$\frac{2n}{x}J_n^2 = J_n(J_{n-1} - J_{n+1}) + 2\sum_1^\infty J_{n+s}(J_{n+s-1} - J_{n+s+1})$$

$$= 2J_n\frac{\partial J_n}{\partial x} + 2\sum_1^\infty 2J_{n+s}\frac{\partial J_{n+s}}{\partial x}$$

$$= \frac{\partial J_n^2}{\partial x} + 2\sum_1^\infty \frac{\partial J^2_{n+s}}{\partial x};$$

whence

$$\int_0^x \frac{J_n^2(x)}{x}\,dx = \frac{1}{2n}\left\{J_n^2(x) + 2\sum_1^\infty J^2_{n+s}(x)\right\} \quad \text{..........(16).}$$

Using the differential properties in § 3, we obtain

$$\frac{\partial}{\partial x}\{xJ_m J_{n+1}\} = \frac{\partial}{\partial x}\{x^{m-n}\,.\,x^{-m}J_m\,.\,x^{n+1}J_{n+1}\}$$

$$= x\,(J_m J_n - J_{m+1}J_{n+1}) + (m-n)\,J_m J_{n+1},$$

and interchanging m and n,

$$\frac{\partial}{\partial x}\{xJ_{m+1}J_n\} = x\,(J_m J_n - J_{m+1}J_{n+1}) - (m-n)\,J_{m+1}J_n;$$

hence by subtraction

$$\frac{\partial}{\partial x}\{x\,(J_m J_{n+1} - J_{m+1}J_n)\} = (m-n)\,(J_m J_{n+1} + J_{m+1}J_n).$$

Again

$$\frac{\partial}{\partial x}(J_m J_n) = \frac{\partial}{\partial x}\{x^{m+n}\,.\,x^{-m}J_m\,.\,x^{-n}J_n\}$$

$$= -\,(J_m J_{n+1} + J_{m+1}J_n) + \frac{m+n}{x}J_m J_n,$$

and therefore

$$\frac{\partial}{\partial x}\{x\,(J_m J_{n+1} - J_{m+1}J_n) + (m-n)\,J_m J_n\} = (m^2-n^2)\,\frac{J_m J_n}{x},$$

which on integration gives that, if $m \neq n$,

$$\int_0^x \frac{J_m J_n}{x}\,dx = \frac{x}{m^2-n^2}(J_m J_{n+1} - J_{m+1}J_n) + \frac{1}{m+n}J_m J_n \quad \text{......(17).}$$

Consider now the special integral

$$\int_0^x xJ_0^2\,dx;$$

integrating by parts, taking x as the integrand, we have

$$\int_0^x xJ_0^2\,dx = \frac{x^2}{2}J_0^2 + \int x^2 J_0 J_1\,dx$$

$$= \frac{x^2}{2}J_0^2 + \int xJ_1\,.\,\frac{\partial}{\partial x}(xJ_1)\,dx$$

$$= \frac{x^2}{2}(J_0^2 + J_1^2) \quad \text{..................................(18).}$$

Writing $n=1$ in (16), we have

$$\int_0^x \frac{J_1^2}{x}\,dx = \tfrac{1}{2}\{J_1^2 + 2J_2^2 + 2J_3^2 + \dots\},$$

and since, as will be shown later § 6,

$$1 = J_0^2 + 2J_1^2 + 2J_2^2 + \dots,$$

this becomes

$$\int_0^x \frac{J_1^2}{x}\,dx = \tfrac{1}{2}(1 - J_0^2 - J_1^2) \quad \text{........................(19).}$$

This result may be obtained directly as follows: differentiating the product on the left-hand side of (8), we have

$$\frac{n}{x} J_n = J_{n-1} - \frac{\partial}{\partial x} J_n,$$

$$\therefore \; n x^{-2n+1} J_n{}^2 = x^{-n+1} J_{n-1} \,.\, x^{-n+1} J_n - x^{-2n+2} J_n \frac{\partial}{\partial x} J_n$$

$$= - x^{-n+1} J_{n-1} \frac{\partial}{\partial x} (x^{-n+1} J_{n-1}) - x^{-2n+2} J_n \frac{\partial}{\partial x} J_n$$

$$= -\tfrac{1}{2} \left[\frac{\partial}{\partial x} \{x^{-n+1} J_{n-1}\}^2 + x^{-2n+2} \frac{\partial}{\partial x} J_n{}^2 \right];$$

whence writing $n = 1$, and integrating

$$\int_0^x \frac{J_1{}^2}{x} dx = \tfrac{1}{2} (1 - J_0{}^2 - J_1{}^2),$$

since $$J_0(0) = 1, \quad J_1(0) = 0.$$

5. Since

$$e^{\frac{x}{2}(y - y^{-1})} = e^{\frac{xy}{2}} \,.\, e^{-\frac{xy^{-1}}{2}} = \sum_0^\infty \frac{x^r y^r}{2^r \lfloor r} \sum_0^\infty (-1)^s \frac{x^s y^{-s}}{2^s \lfloor s}$$

$$= \Sigma \Sigma (-1)^s \frac{x^{r+s}}{2^{r+s} \lfloor r \lfloor s} y^{r-s},$$

and when n is a positive integer

$$J_n(x) = \Sigma (-1)^s \frac{x^{n+2s}}{2^{n+2s} \lfloor n+s \lfloor s},$$

we see that the coefficient of y^n in the above expression is $J_n(x)$ and similarly the coefficient of y^{-n} is $(-1)^n J_n(x)$ or $J_{-n}(x)$. Hence

$$e^{\frac{x}{2}(y - y^{-1})} = J_0 + J_1 \,.\, (y - y^{-1}) + J_2 \,.\, (y^2 + y^{-2}) + J_3 \,.\, (y^3 - y^{-3}) + \dots.$$

Let $y = \cos\omega + \iota \sin\omega$, then

$$e^{\iota x \sin\omega} = J_0 + 2\iota \sin\omega \,.\, J_1 + 2\cos 2\omega \,.\, J_2 + 2\iota \sin 3\omega \,.\, J_3 + \dots$$

and
$$\cos(x \sin\omega) = J_0 + 2\cos 2\omega \,.\, J_2 + 2\cos 4\omega \,.\, J_4 + \dots \quad \text{......(20)},$$

$$\sin(x \sin\omega) = 2\sin\omega \,.\, J_1 + 2\sin 3\omega \,.\, J_3 + \dots \quad \text{...............(21)},$$

and writing $\pi/2 - \omega$ for ω,

$$\cos(x \cos\omega) = J_0 - 2\cos 2\omega \,.\, J_2 + 2\cos 4\omega \,.\, J_4 - \dots \quad \text{......(22)},$$

$$\sin(x \cos\omega) = 2\cos\omega \,.\, J_1 - 2\cos 3\omega \,.\, J_3 + \dots \quad \text{...............(23)}.$$

Writing $\omega = 0$ and $\omega = \pi/2$ the last two equations give

$$\cos x = J_0 - 2J_2 + 2J_4 - \dots \quad \text{......................(24)},$$

$$\sin x = 2J_1 - 2J_3 + \dots \quad \text{............................(25)},$$

$$1 = J_0 + 2J_2 + 2J_4 + \dots \quad \text{......................(26)}.$$

6. Again since

$$e^{\frac{x}{2}(y-y^{-1})} = J_0 + J_1 . (y - y^{-1}) + J_2 . (y^2 + y^{-2}) + \ldots,$$

we have
$$e^{-\frac{x}{2}(y-y^{-1})} = J_0 - J_1 . (y - y^{-1}) + J_2 . (y^2 + y^{-2}) + \ldots,$$

and
$$1 = \{J_0 + J_1 . (y - y^{-1}) + J_2 (y^2 + y^{-2}) + \ldots\}$$
$$\times \{J_0 - J_1 . (y - y^{-1}) + J_2 (y^2 - y^{-2}) - \ldots\}.$$

Since this must hold for all values of y, we have

$$1 = J_0^2 + 2J_1^2 + 2J_2^2 + \ldots \qquad \ldots\ldots(27).$$

Also the coefficient of y^{2m+2} in the product is

$$2\sum_0^\infty J_s J_{2m+s+2} + (-1)^{m+1} J^2_{m+1} + 2\sum_0^{m-1} (-1)^{s+1} J_{s+1} J_{2m-s+1},$$

and since this must be zero,

$$2\sum_0^\infty J_s J_{2m+s+2} = (-1)^m J^2_{m+1} + 2\sum_0^{m-1} (-1)^s J_{s+1} J_{2m-s+1} \qquad \ldots\ldots(28).$$

7. Struve's function of the order n is defined by

$$H_n(x) = \frac{2}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \int_0^{\frac{\pi}{2}} \cos^{2n} \omega . \sin(x \sin \omega)\, d\omega \qquad \ldots\ldots(29).$$

Expanding $\sin(x \sin \omega)$ in a series of powers of $\sin \omega$, we have

$$H_n(x) = \frac{2}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \sum_0^\infty (-1)^s \frac{x^{2s+1}}{\underline{|2s+1}} \int_0^{\frac{\pi}{2}} \sin^{2s+1} \omega . \cos^{2n} \omega\, d\omega$$
$$= \frac{2}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma(n+\frac{1}{2})} \sum_0^\infty (-1)^s \frac{\Gamma(s+1)\Gamma(n+\frac{1}{2})}{2 . \Gamma(2s+2)\Gamma(s+n+1+\frac{1}{2})} x^{2s+1}$$
$$= \sum_0^\infty (-1)^s \frac{x^{n+2s+1}}{2^{n+2s+1} . \Gamma(s+1+\frac{1}{2})\Gamma(n+s+1+\frac{1}{2})} \qquad \ldots\ldots(30).$$

8. From this expression for $H_n(x)$ we obtain

$$\frac{2n}{x} H_n(x) = \sum_0^\infty (-1)^s \frac{(n+s+\frac{1}{2})\, x^{n+2s}}{2^{n+2s}\Gamma(s+1+\frac{1}{2})\Gamma(n+s+1+\frac{1}{2})}$$
$$- \sum_0^\infty (-1)^s \frac{(s+\frac{1}{2})\, x^{n+2s}}{2^{n+2s}\Gamma(s+1+\frac{1}{2})\Gamma(n+s+1+\frac{1}{2})}$$
$$= \sum_0^\infty (-1)^s \frac{x^{n+2s}}{2^{n+2s}\Gamma(s+1+\frac{1}{2})\Gamma(n+s+\frac{1}{2})}$$
$$- \sum_1^\infty (-1)^s \frac{x^{n+2s}}{2^{n+2s}\Gamma(s+\frac{1}{2})\Gamma(n+s+1+\frac{1}{2})} - \frac{x^n}{2^n\sqrt{\pi} . \Gamma(n+1+\frac{1}{2})}$$
$$= H_{n-1} + \sum_0^\infty (-1)^s \frac{x^{n+2s+2}}{2^{n+2s+2}\Gamma(s+1+\frac{1}{2})\Gamma(n+s+2+\frac{1}{2})}$$
$$- \frac{x^n}{2^n\sqrt{\pi} . \Gamma(n+1+\frac{1}{2})}$$
$$= H_{n-1} + H_{n+1} - \frac{x^n}{2^n\sqrt{\pi} . \Gamma(n+1+\frac{1}{2})} \qquad \ldots\ldots(31).$$

9. Multiplying (30) by x^n and differentiating with respect to x, we obtain

$$\frac{\partial}{\partial x}(x^n H_n) = \sum_0^\infty (-1)^s \frac{(2n+2s+1)\,x^{2n+2s}}{2^{n+2s+1}\,\Gamma(s+1+\frac{1}{2})\,\Gamma(n+s+1+\frac{1}{2})}$$

$$= \sum_0^\infty (-1)^s \frac{x^{2n+2s}}{2^{n+2s}\,\Gamma(s+1+\frac{1}{2})\,\Gamma(n+s+\frac{1}{2})}$$

$$= x^n H_{n-1} \quad\text{.....................(32)};$$

and in the same way

$$\frac{\partial}{\partial x}(x^{-n} H_n) = \sum_0^\infty (-1)^s \frac{(2s+1)\,x^{2s}}{2^{n+2s+1}\,\Gamma(s+1+\frac{1}{2})\,\Gamma(n+s+1+\frac{1}{2})}$$

$$= \sum_0^\infty (-1)^s \frac{x^{2s}}{2^{n+2s}\,\Gamma(s+\frac{1}{2})\,\Gamma(n+s+1+\frac{1}{2})}$$

$$= \frac{1}{2^n\sqrt{\pi}.\Gamma(n+1+\frac{1}{2})} + \sum_1^\infty (-1)^s \frac{x^{2s}}{2^{n+2s}\,\Gamma(s+\frac{1}{2})\,\Gamma(n+s+1+\frac{1}{2})}$$

$$= \frac{1}{2^n\sqrt{\pi}.\Gamma(n+1+\frac{1}{2})} - x^{-n}\sum_0^\infty (-1)^s \frac{x^{n+2s+2}}{2^{n+2s+2}\Gamma(s+1+\frac{1}{2})\Gamma(n+s+2+\frac{1}{2})}$$

$$= \frac{1}{2^n\sqrt{\pi}.\Gamma(n+1+\frac{1}{2})} - x^{-n}H_{n+1} \quad\text{.....................(33)}.$$

Differentiating the product on the left-hand side of (32), we obtain

$$\frac{\partial}{\partial x}H_n = H_{n-1} - \frac{n}{x}H_n$$

$$= H_{n-1} - \tfrac{1}{2}\left\{H_{n-1} + H_{n+1} - \frac{x^n}{2^n\sqrt{\pi}\,.\,\Gamma(n+1+\frac{1}{2})}\right\} \text{ from (31)}$$

$$= \tfrac{1}{2}\left\{H_{n-1} - H_{n+1} + \frac{x^n}{2^n\sqrt{\pi}\,.\,\Gamma(n+1+\frac{1}{2})}\right\} \quad\text{.....................(34)}.$$

As special cases of these equations we have

$$\frac{\partial}{\partial x}(xH_1) = xH_0, \qquad \frac{\partial}{\partial x}H_0 = \frac{2}{\pi} - H_1.$$

10. By integration of the expressions (32) and (33) we find

$$\int_0^x x^{n+1}H_n\,.\,dx = x^{n+1}H_{n+1} \quad\text{.....................(35)},$$

$$\int_0^x \frac{H_n}{x^{n-1}} = -\frac{H_{n-1}}{x^{n-1}} + \frac{x}{2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})} \quad\text{.....................(36)}.$$

Thus $$\int_0^x xH_0\,.\,dx = xH_1, \qquad \int_0^x H_1 dx = -H_0 + \frac{2}{\pi}x.$$

11. We may write

$$J_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\int_0^1 \cos xv\,(1-v^2)^{n-\frac{1}{2}}\,dv,$$

and

$$H_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\int_0^1 \sin xv\,(1-v^2)^{n-\frac{1}{2}}\,dv,$$

whence

$$J_n(x) - \iota H_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\int_0^1 e^{-\iota xv}(1-v^2)^{n-\frac{1}{2}}\,dv.$$

Consider the integral $\int e^{-xw}(1+w^2)^{n-\frac{1}{2}}\,dw$, where w is a complex variable of the form $u+\iota v$*. Representing $u+\iota v$ by a point in a plane, the rectangular coordinates of which are u and v, we see that the integral in question has the value zero, if the path of integration be the sides of the rectangle, of which the angular points are $0, h, h+\iota, \iota$, where h is any real positive quantity. Thus

$$\int_0^h e^{-xu}(1+u^2)^{n-\frac{1}{2}}\,du + \int_0^\iota e^{-x(h+\iota v)}\{1+(h+\iota v)^2\}^{n-\frac{1}{2}}\,d(\iota v)$$
$$+\int_h^0 e^{-x(u+\iota)}\{1+(u+\iota)^2\}^{n-\frac{1}{2}}\,du + \int_\iota^0 e^{-\iota xv}(1-v^2)^{n-\frac{1}{2}}\,d(\iota v) = 0.$$

If now we suppose $h=\infty$, the second integral vanishes, and

$$\int_0^1 e^{-\iota xv}(1-v^2)^{n-\frac{1}{2}}\,dv = -\iota\int_0^\infty e^{-xu}(1+u^2)^{n-\frac{1}{2}}\,du$$
$$+\iota\int_0^\infty e^{-x(u+\iota)}\{1+(u+\iota)^2\}^{n-\frac{1}{2}}\,du.$$

The integral on the left-hand side is $2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})\,x^{-n}(J_n-\iota H_n)$, and replacing ux by β on the right-hand side, we obtain

$$2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\tfrac{1}{2})\,x^{-n}(J_n-\iota H_n)$$
$$= -\iota x^{-1}\int_0^\infty e^{-\beta}\left(1+\frac{\beta^2}{x^2}\right)^{n-\frac{1}{2}}d\beta$$
$$+ 2^{n-\frac{1}{2}}x^{-n-\frac{1}{2}}e^{-\iota\left(x-\frac{2n+1}{4}\pi\right)}\int_0^\infty e^{-\beta}\beta^{n-\frac{1}{2}}\left(1-\frac{\iota\beta}{2x}\right)^{n-\frac{1}{2}}d\beta.$$

Expanding by the binomial theorem under the integral signs and then integrating by the formulæ

$$\int_0^\infty e^{-\beta}\beta^{2s}\,d\beta = \Gamma(2s+1),$$

$$\int_0^\infty e^{-\beta}\beta^{n+s-\frac{1}{2}}\,d\beta = \Gamma(n+s+\tfrac{1}{2})$$
$$=(n+s-\tfrac{1}{2})(n+s-\tfrac{3}{2})\dots(n+\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2}),$$

* Lipschitz, *Crelle's J.* LVI. 189 (1859). Lord Rayleigh, *Sound*, II. 153.

we have on equating real and imaginary parts

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \left\{1 - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{1.2(2x)^2} + \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})(n^2 - \frac{25}{4})(n^2 - \frac{49}{4})}{1.2.3.4(2x)^4} - \dots\right\}$$
$$\times \cos\left(x - \frac{2n+1}{4}\pi\right)$$
$$-\sqrt{\frac{2}{\pi x}} \left\{\frac{n^2 - \frac{1}{4}}{2x} - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})(n^2 - \frac{25}{4})}{1.2.3(2x)^3} + \dots\right\} \sin\left(x - \frac{2n+1}{4}\pi\right) \quad \dots(37),$$

and

$$H_n(x) = \frac{x^{n-1}}{2^{n-1}\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})} \left\{1 + (n - \tfrac{1}{2})\frac{\lfloor 2}{x^2} + \frac{(n-\frac{1}{2})(n-\frac{3}{2})}{1.2}\frac{\lfloor 4}{x^2} + \dots\right\}$$
$$+\sqrt{\frac{2}{\pi x}} \left\{1 - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{1.2(2x)^2} + \dots\right\} \sin\left(x - \frac{2n+1}{4}\pi\right)$$
$$+\sqrt{\frac{2}{\pi x}} \left\{\frac{n^2 - \frac{1}{4}}{2x} - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})(n^2 - \frac{25}{4})}{1.2.3(2x)^3} + \dots\right\} \cos\left(x - \frac{2n+1}{4}\pi\right) \quad \dots(38).$$

The following are special cases of interest:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right), \qquad J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(-\sin x - \frac{\cos x}{x}\right),$$

$$H_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} (1 - \cos x),$$

$$H_{\frac{3}{2}}(x) = \sqrt{\frac{x}{2\pi}} \left(1 + \frac{2}{x^2}\right) + \sqrt{\frac{2}{\pi x}} \left(-\sin x - \frac{\cos x}{x}\right).$$

APPENDIX II.

LOMMEL'S FUNCTIONS.

1. THE infinite series

$$U_\nu(y, z) = \sum_0 (-1)^s (y/z)^{\nu+2s} J_{\nu+2s}(z) \quad \text{.................(1)}$$

is convergent under all circumstances, whatever y, z and ν may be, for the quotient of the $(s+1)$th term by the sth is $-(y/z)^2 J_{\nu+2s}/J_{\nu+2s-2}$, which vanishes for $s=\infty$. If $y/z<1$, it converges from the term for which $\nu+2s$ is positive more rapidly than the geometric series $\Sigma (y/z)^{\nu+2s}$.

The infinite series

$$V_\nu(y, z) = \sum_0 (-1)^s (y/z)^{-\nu-2s} J_{-\nu-2s}(z) \quad \text{.................(2)}$$

on the other hand is convergent if ν be a positive or a negative integer (n) or zero, because in that case

$$J_{-\nu-2s} = (-1)^n J_{\nu+2s},$$

and the ratio of two consecutive terms is the same as in the former case; but for fractional values of ν, it is divergent.

From equations (12) and (13) of Chapter VIII., we have for values of ν less than unity and not less than 1/2,

$$U_\nu(y, z) - V_{-\nu+2}(y, z) + \iota\{U_{\nu+1}(y, z) - V_{-\nu+1}(y, z)\}$$
$$= e^{\left(\frac{y}{2}+\frac{z^2}{2y}-\nu\frac{\pi}{2}\right)\iota},$$

which gives $\quad U_\nu(y, z) - V_{-\nu+2}(y, z) = \cos\left(\frac{y}{2}+\frac{z^2}{2y}-\nu\frac{\pi}{2}\right),$

$$U_{\nu+1}(y, z) - V_{-\nu+1}(y, z) = \sin\left(\frac{y}{2}+\frac{z^2}{2y}-\nu\frac{\pi}{2}\right),$$

or since the second equation is obtained from the first by writing $\nu+1$ for ν, and the first from the second by writing $\nu-1$ for ν, the equation

$$U_\nu(y, z) - V_{-\nu+2}(y, z) = \cos\left(\frac{y}{2}+\frac{z^2}{2y}-\nu\frac{\pi}{2}\right) \text{..............(3)},$$

which holds between the limits $-1/2$ and $3/2$ for ν, serves to express V in terms of U.

Since the series $U_\nu(y, z)$ is convergent whatever ν may be, we may go further and define $V_\nu(y, z)$ for any value of ν by the equation

$$V_{-\nu+2}(y, z) = U_\nu(y, z) - \cos\left(\frac{y}{2} + \frac{z^2}{2y} - \nu\frac{\pi}{2}\right)$$

or

$$V_\nu(y, z) = U_{-\nu+2}(y, z) + \cos\left(\frac{y}{2} + \frac{z^2}{2y} + \nu\frac{\pi}{2}\right) \quad\ldots\ldots\ldots\ldots(4).$$

Writing z^2/y for y we obtain

$$U_\nu\left(\frac{z^2}{y}, z\right) - V_{-\nu+2}\left(\frac{z^2}{y}, z\right) = \cos\left(\frac{z^2}{2y} + \frac{y}{2} - \nu\frac{\pi}{2}\right)$$
$$= U_\nu(y, z) - V_{-\nu+2}(y, z) \ldots\ldots\ldots\ldots(5).$$

If ν be an integer (n) or zero, we can express V_n by the convergent series

$$V_n(y, z) = \Sigma(-1)^s\left(\frac{y}{z}\right)^{-n-2s} J_{-n-2s}(z)$$
$$= (-1)^n\Sigma(-1)^s\left(\frac{z}{y}\right)^{n+2s} J_{n+2s}(z) = (-1)^n U_n\left(\frac{z^2}{y}, z\right) \ldots\ldots(6);$$

also

$$U_n(y, z) = (-1)^n V_n\left(\frac{z^2}{y}, z\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From the series for U_ν we have

$$U_\nu(y, z) + U_{\nu+2}(y, z) = \left(\frac{y}{z}\right)^\nu J_\nu(z),$$

but

$$U_\nu(y, z) - V_{-\nu+2}(y, z) = \cos\left(\frac{y}{2} + \frac{z^2}{2y} - \nu\frac{\pi}{2}\right)$$
$$= -U_{\nu+2}(y, z) + V_{-\nu}(y, z),$$
$$\therefore\ U_\nu(y, z) + U_{\nu+2}(y, z) = V_{-\nu}(y, z) + V_{-\nu+2}(y, z)$$
$$= \left(\frac{y}{z}\right)^\nu J_\nu(z) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

2. Since

$$\frac{\partial}{\partial z}\left\{\frac{J_{\nu+2s}(z)}{z^{\nu+2s}}\right\} = -\frac{J_{\nu+2s+1}(z)}{z^{\nu+2s}},$$

we obtain from (1)

$$\frac{\partial U_\nu(y, z)}{\partial z} = -\Sigma(-1)^s\left(\frac{y}{z}\right)^{\nu+2s} J_{\nu+2s+1}(z) = -\frac{z}{y}U_{\nu+1}(y, z) \ldots\ldots\ldots(9),$$

and from (4)

$$\frac{\partial V_\nu(y, z)}{\partial z} = -\frac{z}{y}U_{-\nu+3}(y, z) - \frac{z}{y}\sin\left(\frac{y}{2} + \frac{z^2}{2y} - \nu\frac{\pi}{2}\right)$$
$$= -\frac{z}{y}V_{\nu-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Hence also

$$\frac{\partial^m U_\nu(y, z)}{\partial z^m} = -\frac{z}{y}\frac{\partial^{m-1} U_{\nu+1}(y, z)}{\partial z^{m-1}} - \frac{m-1}{y}\frac{\partial^{m-2} U_{\nu+1}(y, z)}{\partial z^{m-2}} \quad \text{......(11)},$$

$$\frac{\partial^m V_\nu(y, z)}{\partial z^m} = -\frac{z}{y}\frac{\partial^{m-1} V_{\nu-1}(y, z)}{\partial z^{m-1}} - \frac{m-1}{y}\frac{\partial^{m-2} V_{\nu-1}(y, z)}{\partial z^{m-2}} \quad \text{......(12)}.$$

Now by Taylor's theorem

$$U_\nu(y, z+\epsilon) = U_\nu(y, z) + \epsilon\frac{\partial U_\nu(y, z)}{\partial z} + \frac{\epsilon^2}{\lfloor 2}\frac{\partial^2 U_\nu(y, z)}{\partial z^2} + \dots;$$

calculating the successive differential coefficients by (11) and arranging the terms we find

$$U_\nu(y, z+\epsilon) = \Sigma(-1)^s \frac{1}{\lfloor s}\left(\frac{h}{2y}\right)^s U_{\nu+s}(y, z) \quad \text{..............(13)},$$

if $h = 2\epsilon z + \epsilon^2$; in the same way

$$V_\nu(y, z+\epsilon) = \Sigma(-1)^s \frac{1}{\lfloor s}\left(\frac{h}{2y}\right)^s V_{\nu-s}(y, z) \quad \text{..............(14)}.$$

By these formulæ we can interpolate between the tabulated values of the functions U_ν and V_ν. The functions $U_{\nu+1}$, $U_{\nu+2}$, ..., $V_{\nu-1}$, $V_{\nu-2}$, ..., may be found by calculating $U_{\nu+1}$, $V_{\nu-1}$ by the aid of (9) and (10) and then deducing the others by the successive application of (8). Since the series (13), (14) converge very rapidly, only a few terms are required.

If it be required to find the roots of $U_\nu(y, z) = 0$, $V_\nu(y, z) = 0$, the tabular values of these functions nearest to zero are taken, and the equations

$$U_\nu(y, z+\epsilon) = 0, \quad V_\nu(y, z+\epsilon) = 0$$

are solved for $h/(2y)$, and from the value of h thus determined, ϵ is calculated by means of

$$\epsilon^2 + 2z\epsilon - h = 0.$$

3. Differentiating the series (1) with respect to y, we have

$$\begin{aligned}\frac{\partial U_\nu(y,z)}{\partial y} &= \frac{1}{y}\Sigma(-1)^s(\nu+2s)\left(\frac{y}{z}\right)^{\nu+2s} J_{\nu+2s}(z)\\ &= \tfrac{1}{2}\Sigma(-1)^s\left(\frac{y}{z}\right)^{\nu+2s-1} J_{\nu+2s-1}(z) + \tfrac{1}{2}\left(\frac{z}{y}\right)^2\Sigma(-1)^s\left(\frac{y}{z}\right)^{\nu+2s+1} J_{\nu+2s+1}(z)\\ &= \tfrac{1}{2}U_{\nu-1}(y, z) + \tfrac{1}{2}\left(\frac{z}{y}\right)^2 U_{\nu+1}(y, z) \quad \text{..............................(15)},\end{aligned}$$

whence we deduce

$$\frac{\partial V_\nu(y, z)}{\partial y} = \tfrac{1}{2}V_{\nu+1}(y, z) + \tfrac{1}{2}\left(\frac{z}{y}\right)^2 V_{\nu-1}(y, z) \quad \text{...........(16)}.$$

If y be a function of z, then

$$\frac{\partial U_\nu(y, z)}{\partial z} = \frac{\partial U_\nu(y, z)}{\partial z} + \frac{\partial U_\nu(y, z)}{\partial y}\frac{\partial y}{\partial z}$$

$$= -\frac{z}{y} U_{\nu+1}(y, z) + \tfrac{1}{2}\left\{U_{\nu-1}(y, z) + \left(\frac{z}{y}\right)^2 U_{\nu+1}(y, z)\right\}\frac{\partial y}{\partial z};$$

let $y = z^2/c$, then $\partial y/\partial z = 2z/c$, and

$$\frac{\partial}{\partial z} U_\nu\left(\frac{z^2}{c}, z\right) = \frac{z}{c} U_{\nu-1}\left(\frac{z^2}{c}, z\right),$$

or writing y for c,

$$\frac{\partial}{\partial z} U_\nu\left(\frac{z^2}{y}, z\right) = \frac{z}{y} U_{\nu-1}\left(\frac{z^2}{y}, z\right) \qquad \text{......(17)}.$$

In the same way

$$\frac{\partial}{\partial z} V_\nu\left(\frac{z^2}{y}, z\right) = \frac{z}{y} V_{\nu+1}\left(\frac{z^2}{y}, z\right) \qquad \text{......(18)}.$$

4. An important case of Lommel's functions is that in which $z = 0$: then by the definition of U_ν we have

$$U_\nu(y, 0) = \Sigma(-1)^s y^{\nu+2s}\left[z^{-\nu-2s} J_{\nu+2s}(z)\right]_{z=0}$$

$$= \Sigma(-1)^s \frac{(y/2)^{\nu+2s}}{\Gamma(\nu+2s+1)}, \text{ if } \nu > -1 \qquad \text{......(19)}.$$

Differentiating this equation m times with respect to y, we obtain from (15)

$$U_{\nu-m}(y, 0) = \Sigma(-1)^s \frac{(y/2)^{\nu+2s-m}}{\Gamma(\nu+2s-m+1)},$$

and since $\nu - m$, where $\nu > -1$ and m is zero or any positive integer, can have any real value, we may regard (19) as defining $U_\nu(y, 0)$ for any index. We may then define $V_\nu(y, 0)$ by the equation

$$V_\nu(y, 0) = U_{-\nu+2}(y, 0) + \cos\left(\frac{y}{2} + \nu\frac{\pi}{2}\right) \qquad \text{......(20)},$$

and we also have

$$U_\nu(y, 0) + U_{\nu+2}(y, 0) = V_{-\nu}(y, 0) + V_{-\nu+2}(y, 0)$$

$$= \frac{y^\nu}{2^\nu \Gamma(\nu+1)} \qquad \text{......(21)}.$$

In the special case of integral indices, we have

$$U_0(y, 0) = \cos\tfrac{1}{2}y, \quad U_1(y, 0) = \sin\tfrac{1}{2}y, \quad U_2(y, 0) = 1 - \cos\tfrac{1}{2}y,$$

$$U_{2n}(y, 0) = (-1)^n\left\{\cos\tfrac{1}{2}y - \sum_0^{n-1}(-1)^s \frac{(y/2)^{2s}}{\underline{|2s}}\right\},$$

$$U_{2n+1}(y, 0) = (-1)^n\left\{\sin\tfrac{1}{2}y - \sum_0^{n-1}(-1)^s \frac{(y/2)^{2s+1}}{\underline{|2s+1}}\right\},$$

$$U_{-2n}(y, 0) = (-1)^n\cos\tfrac{1}{2}y, \quad U_{-2n-1}(y, 0) = (-1)^{n+1}\sin\tfrac{1}{2}y,$$

or

$$U_{-m}(y, 0) = \cos\left(\tfrac{1}{2}y + m\frac{\pi}{2}\right).$$

Also $$V_0(y, 0) = 1, \quad V_{m+1}(y, 0) = 0 \quad (m = 0, 1, 2, \ldots),$$

$$V_{-2n}(y, 0) = (-1)^n \sum_0^n (-1)^s \frac{(y/2)^{2s}}{\lfloor 2s},$$

$$V_{-2n-1}(y, 0) = (-1)^n \sum_0^n (-1)^s \frac{(y/2)^{2s+1}}{\lfloor 2s+1}.$$

By Taylor's theorem, we obtain

$$U_\nu(y+h, 0) = \Sigma \frac{h^s}{\lfloor s} \frac{\partial^s U_\nu(y, 0)}{\partial y^s} = \Sigma \frac{(h/2)^s}{\lfloor s} U_{\nu-s}(y, 0) \quad \text{.........(22)},$$

$$V_\nu(y+h, 0) = \Sigma \frac{h^s}{\lfloor s} \frac{\partial^s V_\nu(y, 0)}{\partial y^s} = \Sigma \frac{(h/2)^s}{\lfloor s} V_{\nu+s}(y, 0) \quad \text{.........(23)},$$

wherein $U_{\nu-s}$, $V_{\nu+s}$ are determined from (21) when two of these functions with indices differing by unity are known.

5. Writing $x = ru$, we have from equation (10), Chapter VIII., for ν not less than $\frac{1}{2}$,

$$U_\nu(y, z) + \iota U_{\nu+1}(y, z) = \frac{y^\nu}{z^{2\nu-1}} \int_0^1 (zu)^\nu J_{\nu-1}(zu)\, e^{\frac{1}{2}y(1-u^2)\iota}\, du \quad \text{......(24)},$$

and since $$[(zu)^{-\nu+1} J_{\nu-1}(zu)]_{z=0} = \frac{1}{2^{\nu-1}\Gamma(\nu)},$$

we have

$$U_\nu(y, 0) + \iota U_{\nu+1}(y, 0) = \frac{y^\nu}{2^{\nu-1}\Gamma(\nu)} \int_0^1 u^{2\nu-1} e^{\frac{1}{2}y(1-u^2)\iota}\, du \quad \text{......(25)}.$$

Also for $\nu < 1$, equation (ii), Chapter VIII., gives

$$V_{-\nu+2}(y, z) + \iota V_{-\nu+1}(y, z) = -\frac{y^\nu}{z^{2\nu-1}} \int_1^\infty (zu)^\nu J_{\nu-1}(zu)\, e^{\frac{1}{2}y(1-u^2)\iota}\, du \quad \text{...(26)},$$

or writing $-\nu+1$ for ν,

$$V_{\nu+1}(y, z) + \iota V_\nu(y, z) = -\frac{y^{1-\nu}}{z^{1-2\nu}} \int_1^\infty (zu)^{1-\nu} J_{-\nu}(zu)\, e^{\frac{1}{2}y(1-u^2)\iota}\, du$$

$$= -y^{1-\nu} \int_1^\infty (zu)^\nu J_{-\nu}(zu)\, u^{1-2\nu} e^{\frac{1}{2}y(1-u^2)\iota}\, du \quad \text{......(27)},$$

where ν is now greater than zero. But

$$[(zu)^\nu J_{-\nu}(zu)]_{z=0} = 2^\nu / \Gamma(1-\nu),$$

$$\therefore \quad V_{\nu+1}(y, 0) + \iota V_\nu(y, 0) = -\frac{2(y/2)^{1-\nu}}{\Gamma(1-\nu)} \int_1^\infty u^{1-2\nu} e^{\frac{1}{2}y(1-u^2)\iota}\, du$$

$$= -\frac{1}{\Gamma(1-\nu)} \int_0^\infty \frac{e^{-v\iota}}{(y/2+v)^\nu}\, dv \quad \text{......(28)},$$

if v be written for $y(u^2-1)/2$.

Now $$\int_0^\infty e^{-(y/2+v)u}\, u^{\nu-1}\,du = \frac{\Gamma(\nu)}{(y/2+v)^\nu},$$

hence

$$V_{\nu+1}(y,\,0)+\iota V_\nu(y,\,0) = -\frac{1}{\Gamma(\nu)\,\Gamma(1-\nu)}\int_0^\infty e^{-yu/2}u^{\nu-1}du\int_0^\infty e^{-(u+\iota)v}dv$$

$$= -\frac{1}{\Gamma(\nu)\,\Gamma(1-\nu)}\int_0^\infty \frac{u-\iota}{1+u^2}e^{-yu/2}u^{\nu-1}du \quad\text{.....(29)},$$

and equating real and imaginary parts

$$\left.\begin{aligned} V_{\nu+1}(2y,\,0) &= -\frac{1}{\Gamma(\nu)\,\Gamma(1-\nu)}\int_0^\infty \frac{u^\nu e^{-yu}}{1+u^2}du \\ V_\nu(2y,\,0) &= \frac{1}{\Gamma(\nu)\,\Gamma(1-\nu)}\int_0^\infty \frac{u^{\nu-1}e^{-yu}}{1+u^2}du \end{aligned}\right\} \quad\text{.........(30).}$$

In particular when $\nu=\frac{1}{2}$,

$$\left.\begin{aligned} V_{\frac{1}{2}}(2y,\,0) &= \frac{1}{\pi}\int_0^\infty \frac{u^{-\frac{1}{2}}e^{-yu}}{1+u^2}du \\ V_{\frac{3}{2}}(2y,\,0) &= -\frac{1}{\pi}\int_0^\infty \frac{u^{\frac{1}{2}}e^{-yu}}{1+u^2}du \end{aligned}\right\} \quad\text{..................(31),}$$

which are Gilbert's integrals*.

Hence $V_{\frac{1}{2}}(2y, 0)$ is always positive, and since its first differential coefficient $V_{\frac{3}{2}}(2y,\,0)$ is always negative, it decreases continuously as y increases from the value $1/\sqrt{2}$ when $y=0$ to the value 0 for $y=\infty$. $V_{\frac{3}{2}}(2y,\,0)$ is always negative, and since its first differential coefficient, which is the second differential coefficient of $V_{\frac{1}{2}}(2y,\,0)$, is always positive, it increases continuously with y from the value $-1/\sqrt{2}$ for $y=0$ to the value 0 for $y=\infty$.

6. Since $$(zu)^{\frac{1}{2}}J_{-\frac{1}{2}}(zu) = \frac{1}{\sqrt{2\pi}}(e^{zu\iota}+e^{-zu\iota}),$$

we have from (24) by writing $\nu=\frac{1}{2}$,

$$U_{\frac{1}{2}}(y,\,z)+\iota U_{\frac{3}{2}}(y,\,z) = \sqrt{\frac{y}{2\pi}}\left\{\int_0^1 e^{\{\frac{1}{2}y(1-u^2)-zu\}\iota}du + \int_0^1 e^{\{\frac{1}{2}y(1-u^2)+zu\}\iota}du\right\}$$

$$= e^{-\iota z}\sqrt{\frac{y}{2\pi}}\int_0^1 e^{\left\{\sigma-\frac{y}{2}\left(\frac{z}{y}+u\right)^2\right\}\iota}du + e^{\iota z}\sqrt{\frac{y}{2\pi}}\int_0^1 e^{\left\{\delta-\frac{y}{2}\left(\frac{z}{y}-u\right)^2\right\}\iota}du,$$

where $$\sigma=(y+z)^2/(2y),\quad \delta=(y-z)^2/(2y).$$

Writing in the first integral $\frac{y}{2}\left(\frac{z}{y}+u\right)^2=\sigma\zeta^2$, it becomes

$$e^{-\iota z}\sqrt{\frac{\sigma}{\pi}}\int_{\sqrt{z^2/(2y\sigma)}}^1 e^{\sigma(1-\zeta^2)\iota}d\zeta = e^{-\iota z}\sqrt{\frac{\sigma}{\pi}}\left\{\int_0^1 e^{\sigma(1-\zeta^2)\iota}d\zeta - \int_0^{\sqrt{\frac{z^2}{2y\sigma}}} e^{\sigma(1-\zeta^2)\iota}d\zeta\right\}$$

$$= \frac{e^{-\iota z}}{2}\left\{U_{\frac{1}{2}}(2\sigma,\,0)+\iota U_{\frac{3}{2}}(2\sigma,\,0)\right\} - \frac{e^{\frac{y^2+z^2}{2y}\iota}}{\sqrt{\pi}}\int_0^{\sqrt{\frac{z^2}{2y}}} e^{-\zeta^2\iota}d\zeta.$$

* *Mém. couron. de l'Acad. de Brux.* XXXI. 1 (1862).

In the case of the second integral, when $y < z$, we have as in the integral just considered

$$e^{\iota z}\sqrt{\frac{y}{2\pi}}\int_0^1 e^{\left\{\delta-\frac{y}{2}\left(\frac{z}{y}-u\right)^2\right\}\iota}\,du = -\frac{e^{\iota z}}{2}\{U_{\frac{1}{2}}(2\delta, 0) + \iota U_{\frac{3}{2}}(2\delta, 0)\} + \frac{e^{\frac{y^2+z^2}{2y}\iota}}{\sqrt{\pi}}\int_0^{\sqrt{\frac{z^2}{2y}}} e^{-\zeta^2\iota}\,d\zeta;$$

on the other hand if $y > z$,

$$\begin{aligned} e^{\iota z}\sqrt{\frac{y}{2\pi}}\int_0^1 & e^{\left\{\delta-\frac{y}{2}\left(\frac{z}{y}-u\right)^2\right\}\iota}\,du \\ &= e^{\iota z}\sqrt{\frac{y}{2\pi}}\left\{\int_0^{z/y} e^{\left\{\delta-\frac{y}{2}\left(\frac{z}{y}-u\right)^2\right\}\iota}\,du + \int_{z/y}^1 e^{\left\{\delta-\frac{y}{2}\left(u-\frac{z}{y}\right)^2\right\}\iota}\,du\right\} \\ &= e^{\iota z}\sqrt{\frac{\delta}{\pi}}\left\{-\int_{\sqrt{\frac{z^2}{2y\delta}}}^0 e^{\delta(1-\zeta^2)\iota}\,d\zeta + \int_0^1 e^{\delta(1-\zeta^2)\iota}\,d\zeta\right\} \\ &= \frac{e^{\frac{y^2+z^2}{2y}\iota}}{\sqrt{\pi}}\int_0^{\sqrt{\frac{z^2}{2y}}} e^{-\iota\zeta^2}\,d\zeta + \frac{e^{\iota z}}{2}\{U_{\frac{1}{2}}(2\delta, 0) + \iota U_{\frac{3}{2}}(2\delta, 0)\}. \end{aligned}$$

Hence

$$\begin{aligned} U_{\frac{1}{2}}(y, z) + \iota U_{\frac{3}{2}}(y, z) = &\mp \tfrac{1}{2}\{U_{\frac{1}{2}}(2\delta, 0) + \iota U_{\frac{3}{2}}(2\delta, 0)\}\,e^{\iota z} \\ &+ \tfrac{1}{2}\{U_{\frac{1}{2}}(2\sigma, 0) + \iota U_{\frac{3}{2}}(2\sigma, 0)\}\,e^{-\iota z} \quad \text{......(32)}, \end{aligned}$$

the upper or lower sign being taken according as $y <$ or $> z$; and since

$$U_{\frac{1}{2}}(y, z) + \iota U_{\frac{3}{2}}(y, z) = V_{\frac{3}{2}}(y, z) + \iota V_{\frac{1}{2}}(y, z) - \iota e^{\left(\frac{y^2+z^2}{2y}+\frac{\pi}{4}\right)\iota},$$

we obtain

$$\begin{aligned} U_{\frac{1}{2}}(y, z) + \iota U_{\frac{3}{2}}(y, z) = &-\tfrac{1}{2}\{V_{\frac{3}{2}}(2\delta, 0) + \iota V_{\frac{1}{2}}(2\delta, 0)\}\,e^{\iota z} \\ &+ \tfrac{1}{2}\{V_{\frac{3}{2}}(2\sigma, 0) + \iota V_{\frac{1}{2}}(2\sigma, 0)\}\,e^{-\iota z}, \text{ for } y < z \text{ ......(33)}, \end{aligned}$$

and

$$\begin{aligned} V_{\frac{3}{2}}(y, z) + \iota V_{\frac{1}{2}}(y, z) = &-\tfrac{1}{2}\{V_{\frac{3}{2}}(2\delta, 0) + \iota V_{\frac{1}{2}}(2\delta, 0)\}\,e^{\iota z} \\ &+ \tfrac{1}{2}\{V_{\frac{3}{2}}(2\sigma, 0) + \iota V_{\frac{1}{2}}(2\sigma, 0)\}\,e^{-\iota z}, \text{ for } y > z \text{ ......(34)}. \end{aligned}$$

Also

$$\begin{aligned} U_{\frac{1}{2}}(2\sigma, 0) + \iota U_{\frac{3}{2}}(2\sigma, 0) &= 2\sqrt{\frac{\sigma}{\pi}}\int_0^1 e^{\sigma(1-\zeta^2)\iota}\,d\zeta \\ &= e^{\iota\sigma}\,.\sqrt{2}\int_0^{\sqrt{\frac{\sigma}{\pi}}} e^{-\frac{1}{2}\pi v^2\iota}\,dv, \end{aligned}$$

and hence $U_{\frac{1}{2}}(y, z)$ and $U_{\frac{3}{2}}(y, z)$ may be expressed in terms of Fresnel's integrals

$$\int \cos \tfrac{1}{2}\pi v^2 . dv \quad \text{and} \quad \int \sin \tfrac{1}{2}\pi v^2 . dv,$$

while $V_{\frac{1}{2}}(y, z)$ and $V_{\frac{3}{2}}(y, z)$ may be similarly expressed by means of the formulæ

$$V_{\frac{1}{2}}(y, z) = U_{\frac{3}{2}}(y, z) + \cos\left(\frac{y}{2} + \frac{z^2}{2y} + \frac{\pi}{4}\right),$$

$$V_{\frac{3}{2}}(y, z) = U_{\frac{1}{2}}(y, z) - \sin\left(\frac{y}{2} + \frac{z^2}{2y} + \frac{\pi}{4}\right).$$

APPENDIX III.

RADII OF CURVATURE OF FRESNEL'S WAVE-SURFACE.

1. LET σ_1 and σ_2 be the radii-vectores from the centre to the wave-surface in the direction, of which the direction-cosines are λ, μ, ν referred to the principal axes of the surface, then from the equation of the surface we have

$$a^2(b^2-\sigma_1^2)(c^2-\sigma_1^2)\lambda^2+b^2(c^2-\sigma_1^2)(a^2-\sigma_1^2)\mu^2+c^2(a^2-\sigma_1^2)(b^2-\sigma_1^2)\nu^2=0,$$
$$a^2(b^2-\sigma_2^2)(c^2-\sigma_2^2)\lambda^2+b^2(c^2-\sigma_2^2)(a^2-\sigma_2^2)\mu^2+c^2(a^2-\sigma_2^2)(b^2-\sigma_2^2)\nu^2=0,$$

whence since
$$\lambda^2+\mu^2+\nu^2=1,$$

we find
$$Q\sigma_1^2\sigma_2^2\lambda^2=-b^2c^2(b^2-c^2)(a^2-\sigma_1^2)(a^2-\sigma_2^2),$$
$$Q\sigma_1^2\sigma_2^2\mu^2=-c^2a^2(c^2-a^2)(b^2-\sigma_1^2)(b^2-\sigma_2^2),$$
$$Q\sigma_1^2\sigma_2^2\nu^2=-a^2b^2(a^2-b^2)(c^2-\sigma_1^2)(c^2-\sigma_2^2),$$

where
$$Q=(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Hence if x, y, z be the coordinates of the extremity of the radius-vector σ_1, so that $x=\lambda\sigma_1$, $y=\mu\sigma_1$, $z=\nu\sigma_1$, we have

$$\left.\begin{aligned} x&=\frac{1}{\sigma_2}\sqrt{-\frac{b^2c^2(b^2-c^2)}{Q}(a^2-\sigma_1^2)(a^2-\sigma_2^2)} \\ y&=\frac{1}{\sigma_2}\sqrt{-\frac{c^2a^2(c^2-a^2)}{Q}(b^2-\sigma_1^2)(b^2-\sigma_2^2)} \\ z&=\frac{1}{\sigma_2}\sqrt{-\frac{a^2b^2(a^2-b^2)}{Q}(c^2-\sigma_1^2)(c^2-\sigma_2^2)} \end{aligned}\right\}\text{............}(1).$$

2. If now we write

$$dx=a_1d\sigma_1+a_2d\sigma_2,\quad dy=b_1d\sigma_1+b_2d\sigma_2,\quad dz=c_1d\sigma_1+c_2d\sigma_2,$$
$$d^2x=\alpha d\sigma_1^2+\alpha' d\sigma_1 d\sigma_2+\alpha'' d\sigma_2^2,\quad d^2y=\beta d\sigma_1^2+\beta' d\sigma_1 d\sigma_2+\beta'' d\sigma_2^2,$$
$$d^2z=\gamma d\sigma_1^2+\gamma' d\sigma_1 d\sigma_2+\gamma'' d\sigma_2^2,$$

and
$$b_1c_2 - b_2c_1 = A, \quad c_1a_2 - c_2a_1 = B, \quad a_1b_2 - a_2b_1 = C,$$
$$a_1^2 + b_1^2 + c_1^2 = E, \quad a_1a_2 + b_1b_2 + c_1c_2 = F, \quad a_2^2 + b_2^2 + c_2^2 = G,$$
$$A\alpha + B\beta + C\gamma = E', \quad A\alpha' + B\beta' + C\gamma' = F', \quad A\alpha'' + B\beta'' + C\gamma'' = G',$$
$$V^2 = EG - F^2,$$
then the radii of curvature of the surface at the point (x, y, z) are given by
$$\begin{vmatrix} E'\rho - EV, & F'\rho - FV \\ F'\rho - FV, & G'\rho - GV \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

3. Calculating the values of these functions for the case of the wave-surface and writing
$$D = a^2b^2c^2\,(\sigma_1^2/\sigma_2^2 - 1),$$
$$P_1 = (a^2 - \sigma_1^2)\,(b^2 - \sigma_1^2)\,(c^2 - \sigma_1^2),$$
$$P_2 = (a^2 - \sigma_2^2)\,(b^2 - \sigma_2^2)\,(c^2 - \sigma_2^2),$$
we find on reduction
$$E = (D + P_1)/P_1 = e/P_1, \text{ (say)}, \quad F = 0, \quad G = -D/P_2 = g/(-P_2), \text{ (say)},$$
$$E' = \frac{1}{\sigma_1 P_1\sqrt{-P_1P_2}}\{D^2 + D\,(P_1 + a^2b^2c^2 - \sigma_1^4\Sigma a^2 + 2\sigma_1^6) + a^2b^2c^2P_1\}$$
$$= \frac{e'}{P_1\sqrt{-P_1P_2}}, \text{ (say)},$$
$$F' = -\frac{a^2b^2c^2}{\sqrt{-P_1P_2}}\frac{\sigma_1^2}{\sigma_2^3} = -\frac{f'}{(-P_1P_2)}, \text{ (say)},$$
$$G' = \frac{1}{\sigma_1 P_2\sqrt{-P_1P_2}}\{a^2b^2c^2P_1 - D^2\} = \frac{g'}{-P_2\sqrt{-P_1P_2}}, \text{ (say)}.$$

Then the radii of curvature are given by
$$(e'g' - f'^2)\,\rho^2 - \sqrt{eg}\,(eg' + e'g)\,\rho + e^2g^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(3),$$
and
$$e'g' - f'^2 = \frac{D}{\sigma_1^2}\{D + a^2(b^2 - \sigma_1^2)(c^2 - \sigma_1^2)\}\{D + b^2(c^2 - \sigma_1^2)(a^2 - \sigma_1^2)\}$$
$$\times \{D + c^2(a^2 - \sigma_1^2)(b^2 - \sigma_1^2)\},$$
$$eg' + e'g = \frac{D}{\sigma_1}\{D^2 + a^2(b^2 - \sigma_1^2)(c^2 - \sigma_1^2)\}\{D + b^2(c^2 - \sigma_1^2)(a^2 - \sigma_1^2)\}$$
$$\times \{D + c^2(a^2 - \sigma_1^2)(b^2 - \sigma_1^2)\}$$
$$\times \left\{\frac{1}{D + a^2(b^2 - \sigma_1^2)(c^2 - \sigma_1^2)} + \frac{1}{D + b^2(c^2 - \sigma_1^2)(a^2 - \sigma_1^2)} + \frac{1}{D + c^2(a^2 - \sigma_1^2)(b^2 - \sigma_1^2)} - \frac{1}{D}\right\},$$
$$eg = D\,(D + P_1).$$

Hence

$$\rho^2 - \sigma_1 \sqrt{D(D+P_1)} \left\{ \frac{1}{D + a^2(b^2-\sigma_1^2)(c^2-\sigma_1^2)} + \frac{1}{D + b^2(c^2-\sigma_1^2)(a^2-\sigma_1^2)} + \frac{1}{D + c^2(a^2-\sigma_1^2)(b^2-\sigma_1^2)} - \frac{1}{D} \right\} \rho$$

$$+ \frac{D(D+P_1)^2 \sigma_1^2}{\{D + a^2(b^2-\sigma_1^2)(c^2-\sigma_1^2)\}\{D + b^2(c^2-\sigma_1^2)(a^2-\sigma_1^2)\}\{D + c^2(a^2-\sigma_1^2)(b^2-\sigma_1^2)\}} = 0 \quad \text{..........(4).}$$

4. Since D and P_1 always have the same sign, it is clear that D and $D+P_1$ can only vanish when $\sigma_1^2 = \sigma_2^2 = b^2$, that is at the conical points of the wave-surface.

To determine the values of ρ at these points, let $\sigma_1^2 = b^2$, then

$$\rho^2 - bD \left\{ \frac{1}{D - b^2(a^2-b^2)(b^2-c^2)} + \frac{1}{D} \right\} \rho + \frac{Db^2}{D - b^2(a^2-b^2)(b^2-c^2)} = 0,$$

$$\therefore \quad \rho = b \quad \text{or} \quad \frac{bD}{D - b^2(a^2-b^2)(b^2-c^2)} = b \frac{a^2c^2(b^2-\sigma_2^2)}{a^2c^2(b^2-\sigma_2^2) - (a^2-b^2)(b^2-c^2)\sigma_2^2},$$

the second value being zero, if $\sigma_2^2 = b^2$.

Again let $\sigma_2^2 = b^2$, then $D = a^2c^2(\sigma_1^2 - b^2)$, and

$$\rho^2 - \left\{ \frac{(a^2+c^2-\sigma_1^2)^{\frac{3}{2}}}{ac} + \frac{ac(\sigma_1^2-b^2)\sqrt{a^2+c^2-\sigma_1^2}}{a^2c^2 - a^2b^2 - b^2c^2 + b^2\sigma_1^2} \right\} \rho + \frac{(\sigma_1^2-b^2)(a^2+c^2-\sigma_1^2)^2}{a^2c^2 - a^2b^2 - b^2c^2 + b^2\sigma_1^2} = 0,$$

$$\therefore \quad \rho = \frac{(a^2+c^2-\sigma_1^2)^{\frac{3}{2}}}{ac} \quad \text{or} \quad \frac{ac(\sigma_1^2-b^2)\sqrt{a^2+c^2-\sigma_1^2}}{a^2c^2 - a^2b^2 - b^2c^2 + b^2\sigma_1^2},$$

the second value vanishing when $\sigma_1^2 = b^2$.

5. The radius of curvature can only become infinite, if

$$D + a^2(b^2-\sigma_1^2)(c^2-\sigma_1^2) = 0, \quad \text{or} \quad D + b^2(c^2-\sigma_1^2)(a^2-\sigma_1^2) = 0,$$

or

$$D + c^2(a^2-\sigma_1^2)(b^2-\sigma_1^2) = 0.$$

Now

$$D + a^2(b^2-\sigma_1^2)(c^2-\sigma_1^2) = \frac{a^2}{\sigma_2^2}\{b^2c^2(\sigma_1^2-\sigma_2^2) + (b^2-\sigma_1^2)(c^2-\sigma_1^2)\sigma_2^2\}$$

$$= \frac{a^2\sigma_1^2}{\sigma_2^2}\{c^2(b^2-\sigma_2^2) - \sigma_2^2(b^2-\sigma_1^2)\},$$

and since $b^2 - \sigma_2^2$ and $b^2 - \sigma_1^2$ have opposite signs, this can only be zero if $\sigma_1^2 = \sigma_2^2 = b^2$, the case just considered.

Similarly $D + c^2(a^2-\sigma_1^2)(b^2-\sigma_1^2) = 0$, only if $\sigma_1^2 = \sigma_2^2 = b^2$.

But if $D + b^2(c^2-\sigma_1^2)(a^2-\sigma_1^2) = 0$, we have

$$\frac{b^2\sigma_1^2}{\sigma_2^2}\{c^2(a^2-\sigma_2^2) - \sigma_2^2(a^2-\sigma_1^2)\} = 0,$$

which is possible, provided σ_1^2 is greater than b^2.

Hence since D and $D+P_1$ are zero, only if $\sigma_1^2=\sigma_2^2=b^2$, we see that one radius of curvature is infinite at points on the outer sheet of the wave-surface, for which

$$\sigma_2^2=\frac{a^2c^2}{a^2+c^2-\sigma_1^2}.$$

Now from the equation of the wave-surface we have

$$\sigma_1^2\sigma_2^2=\frac{a^2b^2c^2}{a^2\lambda^2+b^2\mu^2+c^2\nu^2},$$

whence substituting for σ_2^2 and remembering that

$$x=\lambda\sigma_1,\quad y=\mu\sigma_1,\quad z=\nu\sigma_1$$

are the coordinates of the extremity of the radius vector σ_1, we find that the points at which one radius of curvature becomes infinite are the intersections of the outer sheet of the wave-surface and the ellipsoid

$$(a^2+b^2)\,x^2+2b^2y^2+(b^2+c^2)\,z^2=b^2\,(a^2+c^2),$$

and these are the circles along which the singular tangent planes touch the surface.

The other radius of curvature at these points is

$$\rho=\frac{D\,(D+P_1)^2\,\sigma_1^2}{\sigma_1\sqrt{D\,(D+P_1)}\,\{D+a^2\,(b^2-\sigma_1^2)\,(c^2-\sigma_1^2)\}\,\{D+c^2\,(a^2-\sigma_1^2)\,(b^2-\sigma_1^2)\}}$$
$$=\frac{(a^2-\sigma_1^2)\,(c^2-\sigma_1^2)}{(a^2-b^2)\,(c^2-b^2)}\,b.$$

6. To find the umbilics we have

$$(e'g'-f'^2)^2\,(\rho-\rho')^2=eg\,\{(eg'-e'g)^2+4egf'^2\},$$

which gives

$$D\,\{D+a^2\,(b^2-\sigma_1^2)\,(c^2-\sigma_1^2)\}^2\,\{D+b^2\,(c^2-\sigma_1^2)\,(a^2-\sigma_1^2)\}^2$$
$$\times\,\{D+c^2\,(a^2-\sigma_1^2)\,(b^2-\sigma_1^2)\}^2\,(\rho-\rho')^2$$
$$=\sigma_1^2\,(D+P_1)\Big[\{a^2b^2c^2\,(D+P_1)^2-D^2\,(\sigma_1^4\Sigma a^2-2\sigma_1^6)\}^2$$
$$-\,4D\,(D+P_1)\,a^4b^4c^4\,\frac{\sigma_1^6}{\sigma_2^6}\,P_1P_2\Big]\quad\text{............(5).}$$

Now P_1P_2 is never positive, and D, $D+P_1$ only vanish when $\sigma_1^2=\sigma_2^2=b^2$, a case already discussed: hence the conditions for an umbilic are

$$P_1P_2=0\quad\text{and}\quad a^2b^2c^2\,(D+P_1)^2=D^2\,\{(a^2+b^2+c^2)\,\sigma_1^4-2\sigma_1^6\}.$$

But if $P_1=0$, the second condition becomes

$$D^2\,\{a^2b^2c^2-(a^2+b^2+c^2)\,\sigma_1^4+2\sigma_1^6\}=0,$$

and $$a^2b^2c^2-(a^2+b^2+c^2)\,\sigma_1^4+2\sigma_1^6$$
$$=P_1+(b^2c^2+c^2a^2+a^2b^2)\,\sigma_1^2-2\,(a^2+b^2+c^2)\,\sigma_1^4+3\sigma_1^6$$
$$=P_1-\sigma_1^2\frac{dP_1}{d\sigma_1^2}.$$

The second condition therefore gives that $dP_1/d\sigma_1^2=0$ or that $P_1=0$ has equal roots, which is impossible.

If $P_2=0$, we have

$$D+P_1=D+P_1-P_2=\frac{\sigma_1^2-\sigma_2^2}{\sigma_2^2}\{P_2+(a^2+b^2+c^2)\,\sigma_1^2\sigma_2^2-\sigma_1^4\sigma_2^2-\sigma_1^2\sigma_2^4\}$$

$$=\sigma_1^2(\sigma_1^2-\sigma_2^2)(a^2+b^2+c^2-\sigma_1^2-\sigma_2^2),$$

and the second condition gives that

$$\sigma_2^4(a^2+b^2+c^2-\sigma_1^2-\sigma_2^2)^2=a^2b^2c^2(a^2+b^2+c^2-2\sigma_1^2),$$

whence

$$a^2+b^2+c^2-\sigma_1^2-\sigma_2^2=\frac{a^2b^2c^2}{\sigma_2^4}\left\{1\pm\sqrt{1-\sigma_2^4\frac{a^2+b^2+c^2-2\sigma_2^2}{a^2b^2c^2}}\right\}.$$

Now if $P_2=0$, σ_2^2 can have either of the three values a^2, b^2, c^2:

(*a*) if $\sigma_2^2=a^2$, then

$$a^2+b^2+c^2-\sigma_1^2-\sigma_2^2=b^2+c^2-\sigma_1^2>a^2b^2c^2/a^4;$$

hence the upper sign must be taken and

$$\sigma_1^2=b^2+c^2-\frac{b^2c^2}{a^2}\left\{1+\sqrt{1-a^2\frac{b^2+c^2-a^2}{b^2c^2}}\right\}$$

$$=\frac{1}{a^2}\{a^2b^2+c\sqrt{a^2-b^2}\,(c\sqrt{a^2-b^2}-b\sqrt{a^2-c^2})\};$$

(*b*) if $\sigma_2^2=b^2$, then

$$a^2b^2c^2-\sigma_2^4(a^2+b^2+c^2-2\sigma_2^2)=b^2(a^2-b^2)(c^2-b^2)<0,$$

and σ_1^2 is imaginary;

(*c*) if $\sigma_2^2=c^2$, then

$$a^2+b^2+c^2-\sigma_1^2-\sigma_2^2=a^2+b^2-\sigma_1^2<a^2b^2c^2/c^4;$$

hence the lower sign must be taken and

$$\sigma_1^2=a^2+b^2-\frac{a^2b^2}{c^2}\left\{1-\sqrt{1-c^2\frac{a^2+b^2-c^2}{a^2b^2}}\right\}$$

$$=\frac{1}{c^2}\{b^2c^2-a\sqrt{b^2-c^2}\,(a\sqrt{b^2-c^2}-b\sqrt{a^2-c^2})\}.$$

Neither of the values thus determined makes the coefficient of $(\rho-\rho')^2$ vanish, and thus there are real umbilics when and only when

$$\left.\begin{aligned}&\sigma_1^2=\frac{1}{a^2}\{a^2b^2+c\sqrt{a^2-b^2}\,(c\sqrt{a^2-b^2}-b\sqrt{a^2-c^2})\},\ \sigma_2^2=a^2\\ \text{and}\quad&\sigma_1^2=\frac{1}{c^2}\{b^2c^2-a\sqrt{b^2-c^2}\,(a\sqrt{b^2-c^2}-b\sqrt{a^2-c^2})\},\ \sigma_2^2=c^2\end{aligned}\right\}\ldots\ldots(6),$$

that is, the real umbilics are in the elliptic sections of the wave-surface made by the planes of yz and xy respectively.

Since at an umbilic $\quad eg' - e'g = 0, \quad f' = 0,$

we have

$$\frac{g'}{g} = \frac{e'}{e} = X = \frac{(D + a^2b^2c^2)(D + P_1) - \sigma_1^4 (a^2 + b^2 + c^2 - 2\sigma_1^2) D}{\sigma_1 (D + P_1)},$$

and the radius of curvature at an umbilic is

$$\rho = \tfrac{1}{2} \frac{\sqrt{eg}\,(e'g + eg')}{e'g'} = \frac{\sqrt{eg}}{X}$$

$$= \frac{\sigma_1 (D + P_1) \sqrt{D (D + P_1)}}{(D + a^2b^2c^2)(D + P_1) - \sigma_1^4 (a^2 + b^2 + c^2 - 2\sigma_1^2) D}.$$

But since $P_2 = 0$ we have

$$D + P_1 = \sigma_1^2 (\sigma_1^2 - \sigma_2^2)(a^2 + b^2 + c^2 - \sigma_1^2 - \sigma_2^2),$$

$$(D + a^2b^2c^2)(D + P_1) - \sigma_1^4 (a^2 + b^2 + c^2 - 2\sigma_1^2) D$$

$$= a^2b^2c^2 \frac{\sigma_1^2}{\sigma_2^2} \sigma_1^2 (\sigma_1^2 - \sigma_2^2)(a^2 + b^2 + c^2 - \sigma_1^2 - \sigma_2^2)$$

$$- \sigma_1^4 a^2b^2c^2 \frac{\sigma_1^2 - \sigma_2^2}{\sigma_2^2} (a^2 + b^2 + c^2 - 2\sigma_1^2)$$

$$= a^2b^2c^2 \frac{\sigma_1^4}{\sigma_2^2} (\sigma_1^2 - \sigma_2^2)^2.$$

Hence $$\rho = \frac{\sigma_2}{abc} (a^2 + b^2 + c^2 - \sigma_1^2 - \sigma_2^2)^{\frac{3}{2}} \quad \text{.....................(7)}.$$

7. Let ω_1 and ω_2 be the radii-vectores from the centre to the surface of wave-quickness in the direction given by the cosines l, m, n; then from the equation to the surface

$$(b^2 - \omega_1^2)(c^2 - \omega_1^2) l^2 + (c^2 - \omega_1^2)(a^2 - \omega_1^2) m^2 + (a^2 - \omega_1^2)(b^2 - \omega_1^2) n^2 = 0,$$

$$(b^2 - \omega_2^2)(c^2 - \omega_2^2) l^2 + (c^2 - \omega_2^2)(a^2 - \omega_2^2) m^2 + (a^2 - \omega_2^2)(b^2 - \omega_2^2) n^2 = 0,$$

whence since $l^2 + m^2 + n^2 = 1$, we find

$$\left.\begin{array}{c} l^2 = -\dfrac{b^2 - c^2}{Q} (a^2 - \omega_1^2)(a^2 - \omega_2^2), \quad m^2 = -\dfrac{c^2 - a^2}{Q} (b^2 - \omega_1^2)(b^2 - \omega_2^2) \\ n^2 = -\dfrac{a^2 - b^2}{Q} (c^2 - \omega_1^2)(c^2 - \omega_2^2) \end{array}\right\} \text{...(8)}.$$

Writing now

$$\frac{1}{F^2} = \frac{l^2}{(a^2 - \omega_1^2)^2} + \frac{m^2}{(b^2 - \omega_1^2)^2} + \frac{n^2}{(c^2 - \omega_1^2)^2} = -\frac{1}{Q} \Sigma (b^2 - c^2) \frac{a^2 - \omega_2^2}{a^2 - \omega_1^2}$$

$$= \frac{\omega_1^2 - \omega_2^2}{p_1} \quad \text{.................(9)},$$

where $$p_1 = (a^2 - \omega_1^2)(b^2 - \omega_1^2)(c^2 - \omega_1^2) \text{.....................(10)},$$

we have for the coordinates (x, y, z) of the point at which the plane

$$lx + my + nz = \omega_1$$

touches the wave-surface

$$x - l\omega_1 = -\frac{F^2}{\omega_1}\frac{l}{a^2-\omega_1^2}, \quad y - m\omega_1 = -\frac{F^2}{\omega_1}\frac{m}{b^2-\omega_1^2}, \quad z - n\omega_1 = -\frac{F^2}{\omega_1}\frac{n}{c^2-\omega_1^2},$$

whence
$$\sigma_1^2 = \omega_1^2 + F^2/\omega_1^2 \quad \text{.................................(11)}.$$

Now from the equation of the wave-surface

$$\sigma_1^2\sigma_2^2 = \frac{a^2b^2c^2}{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2},$$

$$\therefore \quad \frac{a^2b^2c^2}{\sigma_2^2} = \{(a^2-\omega_1^2)\lambda^2 + (b^2-\omega_1^2)\mu^2 + (c^2-\omega_1^2)\nu^2 + \omega_1^2\}\sigma_1^2;$$

but
$$\lambda\sigma_1 = x = l\omega_1\frac{a^2-\sigma_1^2}{a^2-\omega_1^2}, \text{ and two similar equations,}$$

whence

$$\lambda^2\sigma_1^2 = -\omega_1^2(a^2-\sigma_1^2)^2\frac{b^2-c^2}{Q}\frac{a^2-\omega_2^2}{a^2-\omega_1^2}, \text{ and two similar equations,}$$

$$\therefore \quad \frac{a^2b^2c^2}{\sigma_2^2} = \omega_1^2\left\{\sigma_1^2 - \frac{1}{Q}\Sigma(b^2-c^2)(a^2-\omega_2^2)(a^2-\sigma_1^2)\right\}$$

$$= \omega_1^2(a^2+b^2+c^2-\sigma_1^2-\omega_2^2),$$

and we obtain after some transformations

$$a^2b^2c^2\frac{\sigma_1^2}{\sigma_2^2} = a^2b^2c^2 + \frac{\omega_1^4}{F^2}P_1,$$

or
$$D = \frac{\omega_1^4}{F^2}P_1 \quad \text{.....................................(12)}.$$

Again we have

$$-b^2c^2(b^2-c^2)(a^2-\sigma_1^2)(a^2-\sigma_2^2) = Q\sigma_2^2\lambda^2\sigma_1^2$$

$$= -\sigma_2^2\omega_1^2(b^2-c^2)\frac{a^2-\omega_2^2}{a^2-\omega_1^2}(a^2-\sigma_1^2)^2,$$

$$\therefore \quad b^2c^2\frac{a^2-\sigma_2^2}{a^2-\sigma_1^2} = \sigma_2^2\omega_1^2\frac{a^2-\omega_2^2}{a^2-\omega_1^2},$$

and two similar equations; whence

$$a^4b^4c^4\frac{P_2}{P_1} = \sigma_2^6\omega_1^6\frac{p_2}{p_1} \quad \text{..............................(13)},$$

where
$$p_2 = (a^2-\omega_2^2)(b^2-\omega_2^2)(c^2-\omega_2^2) \quad \text{.......................(14)}.$$

8. Writing then
$$k = \frac{\omega_1^4}{F^2} = \frac{\omega_1^4(\omega_1^2-\omega_2^2)}{p_1},$$

we have $$D = kP_1, \quad \sigma_1^2 = \left(1 + \frac{1}{k}\right)\omega_1^2,$$

$$D + a^2(b^2 - \sigma_1^2)(c^2 - \sigma_1^2) = D + \frac{a^2 P_1}{a^2 - \sigma_1^2} = (1+k) P_1 \frac{a^2 - \omega_1^2}{a^2 - \sigma_1^2}$$

$$= (1+k) P_1 \frac{a^2 - \omega_1^2}{a^2 - \omega_1^2 - \omega_1^2/k},$$

$$D + b^2(c^2 - \sigma_1^2)(a^2 - \sigma_1^2) = (1+k) P_1 \frac{b^2 - \omega_1^2}{b^2 - \sigma_1^2} = (1+k) P_1 \frac{b^2 - \omega_1^2}{b^2 - \omega_1^2 - \omega_1^2/k},$$

$$D + c^2(a^2 - \sigma_1^2)(b^2 - \sigma_1^2) = (1+k) P_1 \frac{c^2 - \omega_1^2}{c^2 - \sigma_1^2} = (1+k) P_1 \frac{c^2 - \omega_1^2}{c^2 - \omega_1^2 - \omega_1^2/k},$$

and substituting these expressions in (4), we obtain

$$\rho^2 - \left\{\frac{a^2 - \sigma_1^2}{a^2 - \omega_1^2} + \frac{b^2 - \sigma_1^2}{b^2 - \omega_1^2} + \frac{c^2 - \sigma_1^2}{c^2 - \omega_1^2} - \frac{1+k}{k}\right\}\omega_1 \rho + \frac{P_1}{p_1}\omega_1^2 = 0,$$

or

$$\rho^2 - \left\{2 - \frac{p_1}{\omega_1^4(\omega_1^2 - \omega_2^2)}\left(1 + \frac{\omega_1^2}{a^2 - \omega_1^2} + \frac{\omega_1^2}{b^2 - \omega_1^2} + \frac{\omega_1^2}{c^2 - \omega_1^2}\right)\right\}\omega_1 \rho + \frac{P_1}{p_1}\omega_1^2 = 0 \quad \text{......(15).}$$

We thus find

$$\omega_1^3(\omega_1^2 - \omega_2^2)(\rho + \rho') = \omega_1^4(a^2 + b^2 + c^2 - 2\omega_2^2) - a^2b^2c^2 \quad \text{......(16),}$$

and from (5) we obtain

$$\omega_1^6(\omega_1^2 - \omega_2^2)^4(\rho - \rho')^2 = [(\omega_1^2 - \omega_2^2)\{a^2b^2c^2 - \omega_1^4(a^2 + b^2 + c^2 - 2\omega_1^2)\} + 2p_1\omega_1^2]^2$$
$$- 4\omega_1^4 p_1 p_2 \quad \text{............(17).}$$

9. The conditions for an umbilic are

$$\{a^2b^2c^2 - \omega_1^4(a^2 + b^2 + c^2 - 2\omega_1^2)\}(\omega_1^2 - \omega_2^2) + 2p_1\omega_1^2 = 0,$$

and $$p_1 p_2 = 0;$$

whence proceeding as in § 6, we find that the real umbilics are given by

$$\left.\begin{aligned} \omega_1^2\{bc + \sqrt{(a^2 - b^2)(a^2 - c^2)}\} &= a^2bc, \quad \omega_2^2 = a^2 \\ \omega_1^2\{ab - \sqrt{(a^2 - c^2)(b^2 - c^2)}\} &= abc^2, \quad \omega_2^2 = c^2 \end{aligned}\right\} \quad \text{............(18).}$$

Since $p_2 = 0$ the first condition for an umbilic gives

$$\{a^2b^2c^2 - \omega_1^4(a^2 + b^2 + c^2 - 2\omega_1^2)\}(\omega_1^2 - \omega_2^2) + 2p_1\omega_1^2$$

$$= (\omega_1^2 - \omega_2^2)\left\{\omega_1^4(a^2 + b^2 + c^2 - 2\omega_2^2) - a^2b^2c^2 - 2a^2b^2c^2\frac{\omega_1^2 - \omega_2^2}{\omega_2^2}\right\} = 0,$$

whence the radius of curvature at an umbilic is

$$\rho = \frac{a^2b^2c^2}{\omega_1^3\omega_2^2} \quad \text{...................................(19).}$$

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